Structures, Points and Levels of Reality

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Abstract. First a survey of some basic ingredients, like cognition and intelligence and split brain research is presented. Then, some proposals on which this paper is based are exposed. We proceed with an exposition of the main points in the development of the concepts of 'structure' and 'points', while efforts shall be made to delineate the concept of 'structure' and the Protean nature and the dialectics of the concept of 'point' in various characteristic cases. On the basis of the above the concept of 'level of reality' is examined in connection with the concept of 'completeness' and its importance in mathematics in general. Finally some conclusions for mathematics are drawn and a survey of the present state of the philosophy of mathematics is presented, pointing out also some omissions, that we usually find in the current literature.

1 Introduction

In this paper we shall develop an account of the nature of mathematical objects, which renders them as a 'synthesis' of the dialectics of 'the knowing circle', see Figure 1. Under this synthesis, mathematical objects inherit both, 'properties' that we usually attribute to material objects, and 'properties' that the schemes or structures of mind reflect on them.

This view gives our basic philosophical stance, stated in Subsection 2.3, which also constitutes a foundation for mathematical intuition.

The next important thesis is that 'points have structure', and thus structures are conceived as having 'levels of reality' expressed in mathematics either with the help of nonstandard mathematics, or Category and Topos Theory. The concept of 'level of reality' is connected with the general concept of 'completeness' and its importance in mathematics in general. Connected with the level of reality is the 'horizon of observation' as it is introduced by Husserl, see [28, 73, 74]. Levels of reality give 'extensional frameworks' whereas 'horizons of observation' give 'intensional frameworks', for these distinctions with respect to nonstandard theories, see [16].

For levels in general, together with the relevant views of N. Hartmann, see [54].

There are some opinions which seems to run contrary to the existence of levels of reality:

> "Thus emerged the view that the essence of a mathematical structure is to be sought not in its internal constitution, but rather in the nature of its relationships with other structures of the same kind, as manifested through the network of transformations." Bell [4]

The above common view, actually refers to a single mathematical framework based on a single level of reality. For example category theory defines objects holistic-structurally, using 'universal properties', rather than by analytical-element-wise methods, based on the internal constitution of the objects. This however does not preclude the definition of other categories, linked with functors, that express levels of reality connected with the internal constitution of objects.

The next important issue is variability and vagueness. Variability in general is associated with the dual concepts of 'vagueness' and 'fuzziness', see Subsection 5.2. The distrust on variability in philosophy have been expressed, for the entire western thought, by Plato in the following passages:

Plato's dialogue Kratylos is considered as one of the first documents of European philosophy of language. The following passage is the well known Heracleitean stance:

> "*Socrates*: Heracleitos is supposed to say that "all things are in motion and nothing at rest"; he compares them to the stream of a river, and says that "you cannot go into the same water twice"

Later on he states the following passage, which explains the preference of classical mathematics for the constant and motionless, and at the same time marks the whole development of western thought:

> " *Socrates*: Nor can we reasonably say, Kratylos, that there is knowledge at all, if everything is in a state of transition and there is nothing abiding; for knowledge too cannot continue to be knowledge unless continuing always to abide and exist. But if the very nature of knowledge changes, at the time when the change occurs there will be no knowledge; and if the transition is always going on, there will always be no knowledge, and according to this view, there will be no one to know and nothing to be known. But if the knowing subject and that which is known exist ever, and the beautiful and the good and every other thing also exist, then I do not think that they can resemble a process or flux, as we were just now supposing. Whether there is this eternal nature in things, or whether the truth is what Heracleitos and his followers and many others say, is a question hard to determine; and no man of sense will like to put himself or the education of his mind in the power of names: neither will he so far trust names or the givers of names as to be confident in any knowledge which condemns himself and other existences to an unhealthy state of unreality; he will not believe that all things leak like a pot, or imagine that the world is a man who has a running at the nose. This may be true, Kratylos, but is also very likely to be untrue; and therefore I

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> would not have you be too easily persuaded of it. Reflect well and like a man, and do not easily accept such a doctrine; for you are young and of an age to learn. And when you have found the truth, come and tell me."

According to the fifth-century AD neoplatonist Proclos, the problem of "how we can introduce motion into immovable geometric objects" occupied many of the best minds at Plato's Academy for generations afterwards, see [69, p. 56].

In this paper, by presenting the Protean nature of 'elements' and 'structures' we would like to put forward the following objectives:

(i) Philosophy of mathematics should try to preserve all of the mathematical heritage and comprehend 'paradoxes', 'independent axioms' and the problems possed by mathematical practice, by forging ahead with a dialectical synthesis of the best of philosophical stances. Non-classical mathematics should be reasonable extensions of classical one by adding variability and vagueness, and 'observers' and methods of observation, developing at the same time a net of frameworks, that cover these various alternatives.

Such unified philosophical trends have already been expressed e.g by Lambek [36]. We can view e.g. 'intuitionism' as a generalization, by recognizing that there are other 'objects' beyond the classical $0-1$ ones, for which excluded middle does not hold.

(ii) This dialectical synthesis should be based on the acceptance that there is a 'unique' world around us, and at the same time there is a *pluralism in mathematics*, which is rather due to the existence of various 'levels of reality' and different methods of observation of these realities. This pluralism, along with other important ideas, have been nicely expressed by J. Bell in [3].

Finally we want to comment on the issue of 'Set Theory' and 'Category' and 'Topos Theory'. We should consider Category and Topos theory as well as all non-Cantorian frameworks of classical set theory as extensions towards incorporating variability and vagueness into mathematics. So the thesis is that, there are beyond classical $0-1$ objects, other non-classical objects which require non-classical logics and non-classical frameworks to be studied naturally. Actually in classical set theory, with the iterative conception of set, there are no 'abstract sets'. All sets belong to some position in the hierarchy and gets because of that a specific structure. A radically different concept of set is that of the category of sets, see [42]. The elements of the 'horizontal' conception of set are really abstract! There is a pressing question. Why we should extend the classical hierarchy of sets, given the plurality of forms of the hierarchy (a welcome property and not a defect!), and the way of representing any non-classical objects, like 'variable entities' 'fuzzy sets', 'random sets', Boolean-valued sets, and the like. For example $f : A \to [0,1]$ is a perfect crisp set of ordered pairs. The *interpretation* of this however is a non-classical object. Variable sets can be represented in classical set theory, by moving up to the hierarchy, so that the variable set has been 'freezed'.

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Now comes the other important notion of *naturality.* Although one can represent in set theory as much as in a topos theory, the topos has also the advantage of a natural framework, in which representation and interpretation coincide! This naturality has been nicely expressed by Grothendieck, see [49]:

> "This is the really deep simplification Grothendieck proposed. The way to understand a mathematical problem is to express it in the mathematical world natural to it-that is, in the topos natural to it".

This naturality issue deserves much attention and study, since it is the basic reason for introducing all these mathematical worlds, beyond set theory.

In the remaining of the paper we shall present, cognition and intelligence and split brain fundamentals, that will help in pointing out the correct way towards a dialectical synthesis. Next we state some proposals and examine briefly the concept of 'imaginary' elements. In Sections 3 and 4 we present the basic ideas about structures and 'points' and their developments. All these lead to some conclusions, among which is a short presentation of the present state of philosophy, see Subsection 5.3.

2 Surveys and Proposals.

2.1 Cognition and Intelligence.

There are new theories on cognition and mathematics, which are really interesting. In Lakoff and Núñez [34] a *cognitive science of mathematics* is launched. Fauconnier and Turner [23] develop a method of 'blending' conceptual spaces. Goguen in [25, 279-85], gives a categorical treatment of the phenomenology of blending. Another line of research is the one taken by Macnamara, suggested by the Montreal categorists: The mathematization of the relationship between perception and cognition by means of functors, between *the category of gestalts* (the domain of perception) and *the category of kinds* (the domain of cognition), see [48, 47].

Since the alternative to analysis and the elementwise is the holistic-structural and dialectical, it seems that the concept of adjoint functors is crucial in all further developments, see also [37]. All of the above new ideas have to be synthesized with older views by Piaget and others.

For the purpose of this paper we find that the views of Piaget, and especially their dialectic nature are more closely related to the paper's objective. So we will summarize some basic principles on which Piaget's theory is based, following closely [67].

Piaget's theory is a theory on creation, formation and development of intelligence, and has important consequences for Mathematics, Psychology, Pedagogy and Epistemology. Some of the salient features of Piaget's theory are:

(i) Knowledge is regarded as a dynamic rather than a static matter. "For Piaget, the direct internal representation of external reality is but one aspect

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of thought-the figurative aspect. The transformation is referred to as the operative aspect." Thus knowledge always has a sensory-motor or activistic part.

(ii) "The organism inherits a genetic program that gradually (through a process called 'maturation') provides the biological equipment necessary for constructing a stable internal structure out of its experiences with environment." These structures or schemes together with the functions of adaptation and organization form the basis of intelligence.

(iii) Another principle of Piaget's theory, which has a holistic categorical flavor as well, is the following: ***"Everything is related to everything else."***

These relationships and interdependencies reduce to the corresponding relationships and interdependencies of 'functions' and 'structures'. Let us briefly list the basic ingredients of the theory. Functions are biologically inherited modes of interacting with environment. There are two basic such functions: *adaptation* and *organization*; organization beeing the ability to form compound, higher order structures out of simpler, lower ones, while adaptation consists of a pair of dialectically opposing functions: *assimilation* and *accommodation.* Assimilation is the process to fit the 'objects' or 'events' to the existing 'structures' or 'schemes'[1]. Accommodation is the opposite process, that is, the attempt to adjust the existing structures to objects and events. The dialectic nature of adaptation is better understood through the following diagram [43, p. 174],

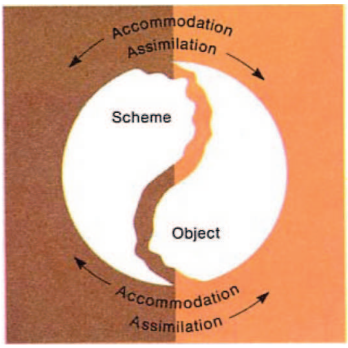

Figure 1: The Knowing Circle

[1]Skemp, in [61, p. 39] best sums up structures (schemes, schemata) as follows:
Scheme or schema is a "general psychological term for a mental structure. The term includes not only complex conceptual structures of Mathematics, but relatively simple structures which co-ordinate sensory-motor activity"..."The study of structures themselves is an important part of Mathematics; and the study of the ways in which they are built up and function, is at the very core of the psychology of learning mathematics"..."A schema has two main functions. It integrates existing knowledge, and it is a mental tool for the acquisition of new knowledge."

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Assimilation and accommodation are reciprocal processes that continue working until a fit is achieved between a ‘mental scheme’ and an ‘object’.

When accommodation and assimilation are in balance, we say that we have an equilibration. This balance is between the assimilation of objects to structures and the accommodation of structures to objects.

However, equilibration is always dynamic and therefore carries with it the seeds of its own destruction. According to Piaget, intelligence is constructed through the dialectics of adaptation (assimilation vs. accommodation) and it is developed through the changes and variability of structures, towards equilibration.

2.2 Split Brain Research.

For split brain research there are a number of excellent books and papers[2], [23]. Although the results on the asymmetry of the two halves of the brain permits many interpretations, there is a unanimous agreement that the functions of the two hemispheres are not symmetric.The two hemispheres are essentially two separate brains, connected with the corpus callosum. Every hemisphere is specialized, in a dynamic sense, in a definite type of processes. In normal brains, the left side of the body is controlled by the right hemisphere and the right side of the body by the left hemisphere.The following picture is adapted from [43],

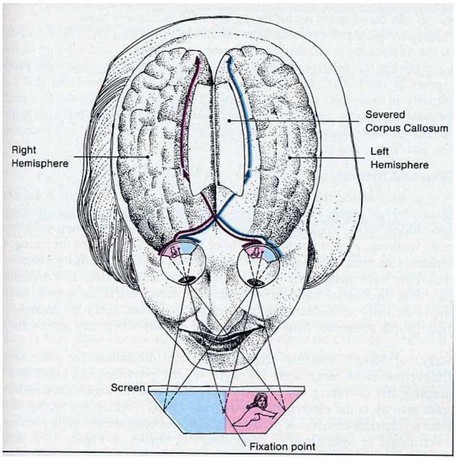

Figure 2: Split Brain

For the kind of specializations, various dichotomies have been proposed, for example:

Left Hemisphere	$\leftrightarrows$	**Right Hemisphere**
Analytic	$\leftrightarrows$	Holistic
Verbal	$\leftrightarrows$	Non-verbal, visuo-spatial
Rational-Logical	$\leftrightarrows$	Intuitive-euristic
Intellectual	$\leftrightarrows$	Sensuous
Argumentative	$\leftrightarrows$	Activistic, Empirical
Focal	$\leftrightarrows$	Diffuse
Time	$\leftrightarrows$	Space
Sequential	$\leftrightarrows$	Simultaneous
Serial processing	$\leftrightarrows$	Parallel processing
Western thought	$\leftrightarrows$	Eastern thought

It has been said that the left hemisphere is something like a digital computer, whereas the right like an analog one. From a philosophical point of view one can say that analytic philosophy is more focused on the left brain, whereas continental philosophy is rather related to the right brain's holistic-structural processes. We would like to conceive philosophy as a synthesis of the two.

2.3 Proposals

Basic philosophical stance for mathematics.

By considering again the knowing circle, see Figure 1, and restricting ourselves in perceiving the real physical world around us, we can make the following hypothesis:

> The final product of the dialectics of perception and cognition, or assimilation and accommodation in Piaget terms, incorporates both: *Properties that mind imposes through the existing schemes* and *properties that the material objects reflect* on the final product.

Consequently every mathematical object has a dual nuture: A mental and an empirical one. For example if we consider a line segment on a computer monitor, then imagination can infinitely bisect it, whereas empirically there are 'levels of reality', 'pixels' 'glass particles' and 'subparticles' and so on. From this point of view, mathematical objects are material-like objects, and so we can always search for such 'material-like properties' in all mathematical objects. This program follows an inverse road to 'mathematization of science': We would rather like to see a 'scientification of mathematics', introducing 'levels of reality, 'observers' 'observation methods', etc. These scientific concepts already exist in mathematics in an implicit way: For example,

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(i) Between the qualitative structures (structures that satisfy idempotent law), *Boolean algebra*, is a kind of structure that refers to a given level of reality, presumably macroscopic rigid subbodies, and its properties prohibit the mixings of levels of reality. In Lawvere's words, [40]:

> "The theory of parts of a body...naturally concentrates on the subbodies (which might with luck form a Boolean algebra) but also must take account of boundaries (which are not sub-bodies)".

(ii) In topological spaces '*Hausdorffness*' expresses a kind of macroscopic level and prohibits mixing of levels. For example referring to Section 4.4, if we have two different points in the monad of zero of $^*\mathbb{R}$, one cannot find two different neighborhoods in $\mathbb{R}$ separating the two given points. If this is always possible then we essentially refer to single level of reality.

(iii) On the quantitative side we have *group theory* which expresses, the rigidness, of the macroscopic level of reality, especially in the form of isometries of the plane.

(iv) Axiom of Choice (AC), Continuum Hypothesis (CH), Löwenheim-Skolem Theorems and other similar statements, are strongly connected with an 'observer' of the mathematical reality, and are explained and comprehended using 'levels of reality', and the crispness of the mathematical framework. One can use the truth value of such independent statements in order to characterize corresponding appropriate mathematical frameworks.

The validity of the proposal about 'levels of reality' and 'observers' in mathematics, should be judged on the basis of their exegetical power.

Fundamental principles of mathematical intuition.

Since we have supposed that any mathematical entity is a dialectical product of both empirical and mental properties, the mental properties give the rationalistic side and the empirical-material side is the ground on which mathematical intuition is based. This also parallels the unity of opposites: 'Geometry vs. Logic'.

The basic principle for intuition, is nicely expressed by the following:

> "...all our intuition comes from our belief in a natural, almost physical, model of mathematical universe."
>
> P. Cohen [8, p.107].

Expressed otherwise we may *"treat mathematical objects as material-like objects"*. We call this *Cohen-Myhill principle*[2].

[2] In the summer of 1986 the late John Myhill participated in the International Conference "On Music and Microcomputers" held in Patras, Greece. I had a discussion with Myhill about mathematics, and he told me that he met with P. Cohen, when he was a graduate student in Chigago, where he expressed to him similar ideas. The same ideas he expressed to me: "mathematical objects should be treated like beeing 'meterial' objects". Doing justice to John, I called this principle Cohen-Myhill.

One may describe the rationalistic side of the mathematical entities using the well known 'axiom systems'. Similarly we may introduce some guiding for the intuition 'principles' which are based essentially on the Cohen-Myhill principle:

1 Geometry vs. Logic, [38]

- Extention−Intention
- Analytic-Logical−Holistic-Structural
- Non-clssical geometrical objects−Many-Valued Logics
- Constant−variable

2 Local vs. Absolute [3]

- Dynamic−Completed infinity
- internal & local point of view−external & global one
- standard & Cantorian−non-standard & non-Cantorian
- fuzzy or vague−crisp or definitive.

3 Level of Reality (extensional), & Horizon of Reference (intentional) [74]

- discrete−continuous
- Distinguishable−intistinguishable and non-standard
- Dynamic infinity≡ Cantorian finiteness+indefiniteness and vagueness

3 Qualitative vs. Quantitative

- Boolean algebras−MV-algebras
- Boolean & Heyting valued models−Residuated lattices-valued models
- Topos theory−Monoidal closed categories

Let us illustrate the internal-external point of view with a simple example. Let us consider two functions f, g as candidates for variable real numbers:

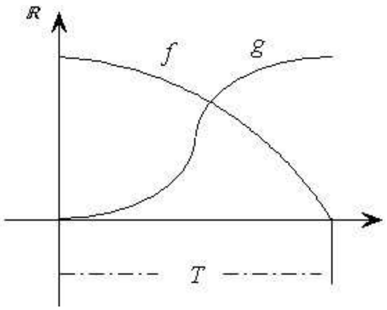

Figure 3: External vs. internal point of view

Let us ask if the two variable real $f, g : T \to \mathbb{R}$ satisfy the trichotomy law. From the external-global point of view f, g are not even comparable, and from this point of view the law seems to break down. Suppose now that we imagine two observers embodied in the T-axis and $\mathbb{R} - axis$ correspondingly. Suppose also that the two observers communicate say with telephone. We examine if the trichotomy law holds pointwise for each $t \in T$. Since for each $t \in T$ for the real numbers $f(t), g(t)$, the $\mathbb{R} - axis$-observer reports that the law holds, we conclude that the law holds for each $t \in T$ and thus the truth value of the law takes the value $T \equiv 1$.

Similarly, if we observe the following from the global-external point of view,

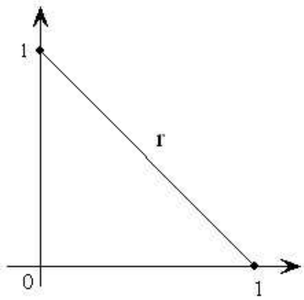

Figure 4: External-actual vs. internal-dynamic infinity.

we do not have any foundational problem with the points $(0, 1), (1, 0)$ and similarly with the line segment r. The external-global observer is then characterized by the actual infinity. On the other hand if we imagine an observer embodied in the line segment r that can measure the distance of the line segment r using rational approximations, then this observer is characterized with dynamic infinity. In conclusion changing the point of view, changes the meaning of infinity. One can find similar examples for each one of the above oppositions.

We summarize in the Figure 5, the basic dialectical system, which express mathematics as a quintessence.

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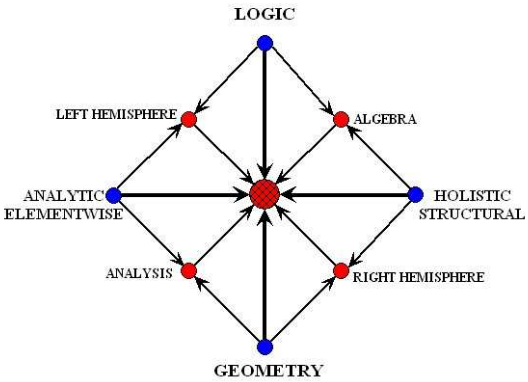

Figure 5: The Dialectical System of Quintessence of Mathematics

2.4 Imaginary Elements

Utilizing the Cohen-Myhill principle we may claim that every object carries with it elements that belong to other 'levels of reality'. Thus vector spaces has a dual space, a Boolean algebra has a Stone space, Real numbers has other structures associated with it, which capture ideal or imaginary elements in another level of reality, see Section 4.4, and finally a ring has its spectrum.

In [21, p. 33], a view by M. Stiefel is mentioned:

> "Just as an infinite number is no number, so an irrational number is not a true number, because it is so to speak conceled under the fog of infinity"

In general there are a lot of historical moments where criticisms for imaginary elements are expressed. It is worth to mention however the case of Husserl. Da Silva in [10], states:

> "In 1901 Husserl addressed the Mathematical Society in Göttingen in order to present his views on a problem that had occupied him, in one form or the other, since the beginning of 1890: the problem of imaginary entities in mathematics. In the Göttingen lectures this problem is twofold: (1) when is an object 'imaginary' from the perspective of a formal axiomatic system (the ontological problem)? (2) how to justify the use of 'imaginary' elements in mathematics (the epistemological problem)?"

Today imaginary elements is a common tool in mathematics, and are introduced through some kind of quotient object. In [27] the following definition is stated:

> " Let L be a first-order structure and A an L-structure. An equivalence is a formula $\varphi(\bar{x}, \bar{y})$ of L, without parameters, such that the relation $\{(\bar{a}, \bar{b}) : A \models \varphi(\bar{a}, \bar{b})\}$ is a non-empty equivalence relation E_φ. We write ∂_φ for the set $\{\bar{a} : A \models \varphi(\bar{a}, \bar{a})\}$. We write $\bar{a}/\varphi$ for the E_φ-equivalence class of the tuple $\bar{a} \in \partial_\varphi$. Items of the form $\bar{a}/\varphi$, where φ is an equivalence formula and $\bar{a}$ a tuple, are known as **imaginary elements** of A"

'Imaginary elements' arise everywhere in mathematics: In Model Theory, where there is also the method of elimination of imaginaries, see [27, p. 117], in Topology, where we have the quotient or identification topologies, in Algebraic Topology (homology and co-homology groups), in Algebraic Geometry and Commutative Algebra, in non-standard mathematics, etc.

3 Structures

> [I]f there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither 'number' nor 'size', but always form. And among the thousand-and one faces whereby form chooses to reveal itself to us, the one that fascinates me more than any other and continues to fascinate me, is the structure hidden in mathematical things.
>
> —— Grothendieck: Récoltes et Semailles, page P27

Felix Klein in his famous Erlanger Program (1872) proposed that the geometric forms, studied by geometry are essentially those that remain invariant under the group of automorphisms of the given 'space'. Essentially 'geometric form' and 'mathematical structure' are two different faces of the same thing. Thus Klein's program also defines what is structure.

In [60], a generalization of the Klein's program is developed:

$$\begin{tikzcd} \text{Euclidean Geometry} \arrow[r, "generalization"] \arrow[d, "generalization"'] & \text{Klein Geometries} \arrow[d, "generalization"] \\ \text{Riemannian Geometry} \arrow[r, "generalization"'] & \text{Cartan Geometries} \end{tikzcd}$$

In general the concept of structure has something to do with *external form of the objects* and their codification. The internal structure is not directly relevant to its external form. The internal algebraic structure exits only as a mental generalization of the external form. On the other hand the scientific experience with microscopic levels of reality, imposes other kind of views of the internal micostructure of the objects.

Lawvere essentially endorses the internal microstructure and level of reality of the objects [41]:

> "Certain considerations would require that even the one-dimensional segment be recognized to have an infinite-dimensional microstructure."

Geometric forms and geometric structures are of course prior to sets. However after the discovery of non-Euclidean geometries, everything has to be reduced to

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abstract sets, together with some structuring steps to regain the originally given structure. The importance of Category Theory for the foundations of mathematics is that re-establishes the order and at the same time introduces variability in the freezed world of set theory. Lawvere in [38] best sums up this as follows:"...a 'set theory' for geometry should apply not only to abstract sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. For such sets, usually logic is 'intuitionistic' (in its formal properties) usually the axiom of choice is false, and usually a set is not determined by its points defined over 1 only". See also, [42].

The concept of 'structure' seems to be very broad. In van Hiele's words, [71]:

> ...brought me into contact with the works of Gestalt psychologists, from which I learned that insight might be understood as the result of perception of a structure...structure is what structure does. First see how structures work, and afterward you will understand what structures are.

In the above book we have a broad concept of structure, mainly that of Gestalt Psychology, and one can even find such chapters as: "How Do We Meet a Structure". I think seeing structure to work in category theory, and remembering the slogan of Macnamara that, "calculus is to physics as category theory is to psychology" and his efforts towards the mathematization of the relationship between perception and cognition by means of functors, between *the category of gestalts* (the domain of perception) and *the category of kinds* (the domain of cognition), one should be convinced that the 'variable structures' of category theory [39], can capture all the various conceptions of a structure.

Finally we would like to comment on the duality of 'variability' and 'vagueness' or 'fuzziness'. Let us take a simple example for brevity. Let us consider a function $f : T \to A$ considered as a 'variable element' over the time domain set T of the set A. This function can be expressed dually as a function,

$$f^{\lhd} : A \to \mathscr{P}(T) \;/\!\!/\; a \mapsto T_a := \{t \in T \mid f(t) = a\}$$

where, $T_a \cap T_{a'} = \emptyset,\;\; a \neq a'$ and $\bigcup_{a \in A} T_a = T$. From the above one can see that the T-variable element f is dual to the $\mathscr{P}(T)$-fuzzy element of A. Introducing the co-graph of f we can summarize the situation using the following diagram:

$$\begin{tikzcd}
A \arrow[dr, hook, "i_A"] \arrow[drr, bend left, "f"] & & \\
& A+B \arrow[r, two heads, "G_f^*"] & B \\
B \arrow[ur, hook', "i_B"] \arrow[urr, bend right, "1_B"'] & &
\end{tikzcd}$$

i.e. $G_f^* \equiv f1_B, \quad G_f^* \circ i_A = f \quad \& \quad G_f^* \circ i_B = 1_B,$ see also, [42]. Finally, the function $f = (T, G_f, A)$ can be expressed as (A, G_f^*, T) which belongs to the opposite category $\mathbf{Set}^{op}$, thus the opposite category converts generalized elements

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to generalized properties and conversely, and in this way hides ‘fuzzy elements’ and present them as ‘generalized properties’. Now if we generalize the above situation to general Boolean-valued models, and then to Heyting-valued models, then we are almost to topos theory. On the other hand structures that satisfy the idempotent law, like Boolean and Heyting algebras, are called here ‘qualitative’, whereas algberas that do not satisfy the idempotent law, like residuated lattices (or residuated integral, lattice-ordered monoids) are termed ‘quantitative’. These structures categorically lead to ‘monoidal closed categories’.

4 The Protean Nature and the Dialectics of Points

In this section various important examples will be described, that show the Protean nature of the concept of ‘point’. The concept of ‘point’ is fundamental in the development of a true intuition and a kind of synthetic language for mathematics.

4.1 Stone Duality

First we examine the case of probability spaces:

4.2 The Negation of Points: Probability Algebras

Let $(\Omega, \mathscr{A}, P)$ be a point-probability space. This is the classical probability setting. We may negate the ‘points’ and obtain a kind of point-free probability setting proposed by Carathéodory and Kappos, see e.g. [32]. We define on the σ-algebra $\mathscr{A}$ an equivalence relation by,

$$A_1 \approx A_2 \quad \text{iff} \quad P(A_1 \vartriangle A_2) = 0, \quad A_1, A_2 \in \mathscr{A}.$$

Then we define the probability algebra $(\mathbb{B}, p)$ induced by $(\Omega, \mathscr{A}, P)$ as follows: $\mathbb{B} := \mathscr{A}/\approx$ and $p(a) := P(h^{-1}(a)), \quad a \in \mathbb{B}$, and $h : \mathscr{A} \to \mathbb{B}$ is the canonical map. See [32] for a development of point-free probability theory. Although this approach have difficulties in developing e.g. stochastic processes, it is very relevant e.g. in developing a kind of Boolean nonstandard analysis as e.g. in [17, 19]. In these papers although a probability algebra does not have points we may represent point discrete random variables on $(\Omega, \mathscr{A}, P)$ in a point-free way using the corresponding partitions of unity in $\mathbb{B}$, and then we can have arbitrary random variables by completing the elementary stochastic space, see [17].

4.3 The Negation of Negation of Points: Stone Probability Spaces

We shall not admit the one element Boolean algebra $\mathbb{1} := (\mathscr{P}(\emptyset), \dots, 0, 1)$ with $0 = 1$, since in that case the category of Boolean algebras would have two non-isomorphic initial elements. Since the category **Boole** have an initial object,

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$\mathbb{2}$ the two element Boolean algebra we may consider Boolean properties, i.e. for every Boolean algebra $\mathbb{B} \in \mathbf{Boole}_0$ we consider outgoing arrows or Boolean homomorphisms $\mathbb{B} \to \mathbb{2}$, which are actually ultrafilters. These Boolean properties correspond through the Stone Duality,

$$\mathbf{Stone} \cong \mathbf{Boole}^{\mathrm{op}} \tag{SD}$$

to global points $\mathbb{1} \to S(\mathbb{B})$, where $\mathbb{1}$ is the trivial Boolean space, recapturing points in the Stone space $S(\mathbb{B})$ of the boolean algebra $\mathbb{B}$. The Development of probability theory on Stone spaces, has been not pursued. There are two interesting papers towards this, see [6, 45].

We encounter the same dialectic phenomenon in the case of the Lindenbaum-Tarski Algebras. If we consider formulas as 'points' then we have the same dialectics:

$$\begin{array}{lcl} \text{Propositional Logic} & \xrightarrow{\neg\ points} & \text{Lindenbaum-Tarski algebra} \\ & \xrightarrow{\neg\neg\ points} & \text{Stone Space.} \end{array}$$

and in the case of topological spaces:

$$\begin{array}{lcl} \text{Point topology} & \xrightarrow{\neg\ points} & \text{point-free topology (locales and frames)} \\ & \xrightarrow{\neg\neg\ points} & \text{spectral spaces} \end{array}$$

All we have in the Stone duality is a bridge linking spaces and points to algebra and logic. Schematicaly we have:

$$\frac{\text{topology}}{\text{geometry}} \quad \text{vs.} \quad \frac{\text{algebra}}{\text{logic}} \tag{$(BR)_1$}$$

We can regard the Stone duality from two points of view:

(i) The Intensional or logical point of view: Properties (elements of the lattice Clop$X$), where X is a topological space, are taken as primary, whereas 'points' are constructed as sets of properties (prime filters).

(ii) The extensional or geometrical point of view: We take the spacial side of the Stone duality as primary, i.e. 'points' in

$$\mathrm{Spec}(\mathbb{B}) := \{h \in 2^{\mathbb{B}} \mid h \text{ is a Boolean homomorphism}\}$$

and then construct 'properties' as open sets of 'points':

$$U_b := \{x \in \mathrm{Spec}(\mathbb{B}) \mid b \in x\}, \quad b \in \mathbb{B}$$

If we combine the above two points of view with the various instances of the (SD), we essentially get a kind of 'soundness and completeness' relationship. For more details, see [1, 31, 72].

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4.4 Nonstandard Mathematics.

> "In the 1930's Banach spaces were sneered at as abstract,
> later it was the turn of locally convex spaces,
> and now it is the turn of nonstandard analysis"
> J. L. Doob [14].

We may divide the existing theories of nonstandard analysis (NSA) into two categories: The extensional and the intentional one. The former is Robinson's NSA, which is based on some extensions of the usual absolute Cantorian objects, whereas the later, Nelson's Internal Set Theory (IST), especially in the form of Vopěnka's AST, is a rather non-Cantorian one, see also [15, 16].

In the following we shall explain briefly the Robinson's NSA.

In order to get the idea behind nonstandard mathematics, it is enough to consider the case of real numbers.

Splitting the Atoms of $\mathbb{R}$.

We consider the real numbers $\mathbb{R}$ as a kind of 'macroscopic' abstraction formed through a dialectic synthesis of properties of 'material' lines and properties of mental perception and cognition. As such it is highly motivating to look at the real line both as a kind of 'material' line and as an abstract mental form.

Considering the atoms of $\mathbb{R}$ as 'material molecules' we try to split them in order to get infinitesimals as a kind of 'particles' or imaginary or ideal elements.

We shall follow the suggestion of Bell [3], for a dialectic construction of $^*\mathbb{R}$; see [18] for more details.

We consider $\mathbb{R}$ as a given structure with constant elements. We negate constancy by considering 'variable reals' in discrete time $\mathbb{N}$, i.e. by considering the ring $\mathbb{R}^{\mathbb{N}}$. Next we are going to negate the negation of constancy, by constructing out of $\mathbb{R}^{\mathbb{N}}$, some quotient set $\mathbb{R}^{\mathbb{N}}/\!\approx$, in such a way that the imaginary elements $r/\!\approx\, \equiv [r]$ are now constant elements. Thus the main problem is how to construct the equivalent relation $\approx$.

To do this we consider the old equivalent class of Cauchy sequences converging to 0, $[0]_{\text{Old}}$. In order to split this class we try to impose some discriminating properties:

(i) Speed of convergence, and

(ii) Asymptotic behavior.

For example, $(\frac{1}{n})$ and $(\frac{1}{n^2})$ have different speed of convergence, whereas $(\frac{1}{n})$ and $(\frac{2}{n})$ have the same speed of convergence but different asymptotic behavior. We now define,

$$[0]_{\text{New}} := \{(a_n) \in \mathbb{R}^{\mathbb{N}} \mid (\exists n_0 \in \mathbb{N})(\forall n > n_0)[a_n = 0]\}.$$

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The following sequences,

$$0, \ldots, \left(\frac{1}{e^{e^{n^2}}}\right), \left(\frac{1}{e^{e^n}}\right), \ldots, \quad \left(\frac{1}{e^{n^2}}\right), \left(\frac{1}{e^n}\right), \ldots,$$
$$\left(\frac{1}{n^2}\right), \left(\frac{1}{n}\right), \ldots, \quad \left(\frac{1}{\ln n}\right), \left(\frac{1}{\ln \ln n}\right), \ldots, \tag{1}$$

which can become as dense as we wish, all belong to $[0]_{\text{Old}} - [0]_{\text{New}}$ and give new ideal elements: the infinitesimals.

The two discriminating properties (speed of convergence and asymptotic behavior) depend on the tails of natural numbers. Let,

$$\mathscr{T} := \{T \subseteq \mathbb{N} \mid \mathbb{N} - T, \text{ is a finite set } \}.$$

The above set $\mathscr{T}$ is the Fréchet filter.

To solve also problems such as 'divisors of zero' and e.g. the trichotomy law, we have to extend the set of tails of natural numbers to an ultrafilter, getting finally the ultrapower ${}^*\mathbb{R}$ of $\mathbb{R}$, as the product of the above dialectics; see for details [18].

The True Mataphysics of Complex Numbers

In a letter to Hansen, Gauss (1825) (see [30, p. 112]) wrote:

> "...Those investigations penetrate deeply into many others, I may even say into mataphysics of the theory of space, and only with difficulty can I tear myself away from results arising there from, as, for example, the true metaphysics of negative and imaginary quantities. The true meaning of $\sqrt{-1}$ stands very vividly before my soul, but it will be very difficult to put it into words, which can only give but a vague fleeting image."

Regarding again the 'macroscopic' real line in a physical, material way, we may attribute to it various 'levels of reality' with various 'microscopic details'. All these are various assortments of the real line, and refer to one dimensional 'material-abstract' object! Completeness of the real line is rather connected to the macroscopic (where numbers are rather stable) nature of the real line, and since it is second order property, it is not preserved in elementary extensions. On the other hand, using various other levels of reality (where we have rather variable elements) we can introduce a lot of other 'microscopic real numbers'.

Let us consider the following two constructions,

$$\begin{array}{rcl} {}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/\mathscr{F}_M & \Leftrightarrow & \mathbb{C} := \mathbb{R}[x]/(x^2+1) \\ \text{ring } \mathbb{R}^{\mathbb{N}} & \Leftrightarrow & \text{ring } \mathbb{R}[x] \\ \text{ultrafilter } \mathscr{F}_M & \Leftrightarrow & \text{maximal ideal } (x^2+1) \\ \text{The imaginary axis} & & \text{The imaginary axis} \\ \text{is a ring} & \Leftrightarrow & \text{is a field} \end{array}$$

Here $\mathbb{R}[x]$ is the ring of polynomials of one variable, with coefficients in $\mathbb{R}$, which in our case can be identified with real polynomial functions of one variable. We can also regard the maximal ideal (x^2+1) as all polynomial implications of the equation $x^2+1=0$.

Comparing the above two constructions, we can conclude that *the field of complex numbers, consists of the 'macroscopic' real line* $\mathbb{R}$, *together with 'microscopic' or imaginary details which are posited in an appropriate level of reality and which imaginary details solve the equation* $x^2+1=0$. For more on this see [18, pp. 224-236].

A similar idea is expressed by Hodges in [27, p. 113]:

> "Suppose a structure B is interpretable in a structure A. Then we can think of the elements of B as 'implicit' elements of A. The reduction theorem supports this intuition–it tells us that we can talk about the elements of B by making statements about elements of A. Probably the most familiar example is the complex numbers: we can think of them as pairs of real numbers, so that complex numbers are 'implicit' in the field of real numbers."

These field extensions, that add more (ideal or imaginary) points in such a way that some equations become solvable, can be generalized, and even the *Fundamental Theorem of Algebra*, can be generalized to *Hilbert Nullstellensatz*, see e.g. [33]. From complex numbers one can step into a completely new branch of mathematics, including elliptic functions and Riemann surfaces. These extensions can only be possible through different levels of mathematical reality.

Boolean Models: Extending the Scalars

The development of the ultrapower ${}^*\mathbb{R}$ can be interpreted as equivalent classes of random variables defined over the finitely additive, zero-one probability space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu_{\mathscr{U}})$, where $\mathscr{U}$ is an ultrafilter generated from Fréchet filter, and

$$\mu_{\mathscr{U}}(X) := \begin{cases} 1 & \text{if } X \in \mathscr{U} \\ 0 & \text{otherwise.} \end{cases}$$

The question now arises: Can we extend the situation by considering variable elements (random variables) over an arbitrary probability space $(\Omega, \mathscr{A}, P)$? If we consider elementary random variables in the sense of Kappos, see [32], then we get the elementary stochastic space which can be seen to be isomorphic to the Boolean power $\mathbb{R}[\mathbb{B}]$, where $(\mathbb{B}, p)$ is the probability algebra induced by the given probability space. The Boolean power $\mathbb{R}[\mathbb{B}]$, is a model of the reals, which together with 'macroscopic' reals $\mathbb{R}$, includes randomly variable elements, posited in a 'microscopic' level, see [17, 19] for details. In [20], there is a first attempt to generalize the above results to the quantitative structure of an MV-algebra.

If we take $\mathbb{B}$ to be a projection Boolean algebra in a Hilbert space, then from Takeuti's work [63], we can conclude that the Boolean power $\mathbb{R}[\mathbb{B}]$ is a Boolean model of $\mathbb{R}$ and its nonstandard elements, are self-adjoint operators.

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The next question then arises: Can one develop Quantum Mechanics, as a non-standard Classical Mechanics, using this kind of non-standard reals?

4.5 Algebraic Geometry

Another extremely important field where the Protean nature of points takes a lot of extreme forms is Algebraic Geometry. This is one of the main reasons that contemporary Model Theory (a theory that focus on the nature of points that can satisfy formulas.) is rather connected to Algebraic Geometry or that Algebraic Geometry can be considered as a general theory of models minus the syntactic part of the theory of models.

Algebraic Geometry studies, using algebraic methods, the geometric objects called varieties, which are curves and surfaces and higher dimensional objects, defined by polynomial equations. In other words studies the geometric spaces, where polynomial equations take their solutions.

But what is 'algebra'? In [59] the opinion of H. Weyl, is stated; *Algebra* results from the various ways of coordinatization of space. In [59] a lot of coordinatization methods are presented along with the following thesis:

> "**Thesis.** Anything which is the object of mathematical study (curves and surfases, maps, symmetries, crystals, quantum mechanical quantities and so on) can be 'coordinatised' or 'measured'. However, for such a coordinatisation the 'ordinary' numbers are by no means adequate.
>
> Conversely, when we meet a new type of object, we are forced to construct (or to discover) new types of 'quantities' to coordinatise them. The construction and the study of the quantities arising in this way is what characterises the place of algebra in mathematics (of course, very approximately)."

We should like here to note that we will understand 'algebraic' as refereing to some kind of coordinatisation of space, whereas we shall use the term 'holistic-structural' for any 'top-down' algebraic approach.

The road to algebraic geometry is described in [33], using the following generalizations, where $\mathbb{P}^n(\mathbb{C})$ indicates the n-complex projective space:

$$\begin{array}{c}
\text{'analytic' geometry} \\
\text{in } \mathbb{R}^2, \ldots, \mathbb{R}^n \\
\Big\downarrow \\
\text{'analytic' geometry} \\
\text{in } \mathbb{C}^2, \ldots, \mathbb{C}^n \\
\Big\downarrow \\
\text{'analytic' geometry} \\
\text{in } \mathbb{P}^2(\mathbb{C}), \ldots, \mathbb{P}^n(\mathbb{C})
\end{array}$$

The important thing is that through all these different coordinatisations, the Fundamental Theorem of Algebra takes various forms. We may state the Funda-

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mental Theorem of Algebra in $\mathbb{C}[x]$:"*every nonconstant polynomial in* $\mathbb{C}[x]$ *has a zero in* $\mathbb{C}$". This can be translated to the language of ideals: "*Every proper ideal of* $\mathbb{C}[x]$ *has a zero.*" The last statement generalizes directly to $\mathbb{C}[x_1, \ldots, x_n]$ and gives the celebrated,

Hilbert's Nullstellensatz: Every proper ideal of $\mathbb{C}[x_1, \ldots, x_n]$ has a zero.

Also Bézout's Theorem, already an important generalization of the Fundamental Theorem of Algebra, can be generalized to higher dimentions, see [33].

It seems that projective geometry and n-dimensional geometry paved the way for the modern concepts of 'abstract' varieties and schemes. With algebraic Geometry we have another important bridge between algebra and geometry.

In [62] it is stated that: "The purpose of this section is to convince you that commutative algebras are really spaces seen from the other side of your brain" One may add that since space is perceived by the right hemisphere then commutative algebras are perceived by the other side of our brain, the left! He also states that: "For general **k**-algebras A and B it is suggestive to call a morphism $f : A \to B$ a B-point of A."

This bridge between algebra and geometry is nicely expressed in [56]: "The purpose of this course is to build one of the bridges between algebra and geometry. Not the Erlangen program (linking geometries via transformation groups with abstract group theory) but a quite different bridge linking rings A and geometric objects X; The basic idea is that it is often possible to view a ring A as a certain ring of functions on a space X, to recover X as the set of maximal or prime ideals of A, and to derive pleasure and profit from the two-way traffic between the different world on each side." The metaporphoses of the concept of point through out the development of Algebraic Geometry and especially in the Grothendieck phase, is admirably described in [7], see also [22, Ch. VI: Schemes and Functors] and [46, Ch. VII, §5.], where there is a treatment of points in topoi as special geometric morphisms.

The following extract from [29, part II:p. 1211], is characteristic of the relevance of Grothendieck's work to our theme:

> Grothendieck's work also has parallels with another great twentieth century advance, that of quantum mechanics, which turned conventional notions upside down by replacing 'point particles' by 'probability clouds'.
>
> "[T]hese 'probability clouds', replacing the reassuring material particles of before, remind me strangely of the elusive 'open neighborhoods' that populate the toposes, like evanescent phantoms, to surround the imaginary 'points'," he wrote (R&S, page P60).

To understand the generalization from varieties to schemes, we consider the following diagram from [22]:

$$\begin{tikzcd}
\textit{affine varieties} \arrow[r, leftrightarrow] \arrow[d] & \textit{finitely generated, nilpotent-free rings over an algebraically closed field } K \arrow[d] \\
\textit{affine schemes} \arrow[r, leftrightarrow] & \textit{commutative rings with identity}
\end{tikzcd}$$

The category of affine schemes is equivalent to the opposite category of rings with identity, see [22, p. 30]. "[Scheme theory] is the basis for a grand unification of number theory and algebraic geometry, dreamt of by number theorists and geometers for over a century." [22].

A very nice classification of various conceptions of 'point' in Algebraic Geometry, is described in [55, §(8.13), p. 121]:

(a) Scheme-theoretic points of a variety,

(b) Field-valued points in scheme theory,

(c) Generic points in Weil constructions and

(d) Points as morphisms in scheme theory.

Thus basic techniques are 'extending the scalars', 'generic points' and 'change of bases' and finally a 'point' of X is a map $\mathrm{Spec}(L) \to X$ where L is a field extension of k see also, [12]

The bottom line in Grothendieck's developments is:

(i) There are several species of points and that points carry Galois group actions.

(ii) The most general type of a 'point' takes the form of arrow in a category, and especially the form of a 'fiber functor' [7, p. 404].

(iii) Such general points can have internal symmetries, revealing that 'points have structure', see [7, §7] for details.

(iv) Scheme theory can unify Arithmetic and Geometry.

The concept of 'level of reality' and 'imaginary' or 'ideal points' was always hidden in these developments.

4.6 Categories

> "One of the points which should be emphasized is that with Serre and still more with Chevalley, birational geometry fades out of the picture and the concept of morphism comes to the fore."
> Dieudonné, [12, p. 860]

All the above developments point to the following slogan:

Replace elements by morphisms

We can see that ordinary elements can be captured as global elements: $a : \mathbb{1} \to A$. Thus in any category we can consider generalized or variable elements $T \xrightarrow{x} A$. The dual conception is that of generalized properties or outgoing to A arrows $A \xrightarrow{p} \Omega$.

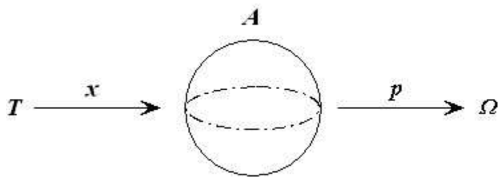

Figure 6: Generalized points x vs. generalized properties p

Any variable element $T \xrightarrow{x} A$ give an extensional aspect of the object of interest A, whereas a generalized property induces on A a process of 'observation' and splits A into a family of indistinguishable to observation objects $(A_\omega)_{\omega \in \Omega}$.

Sometimes we write $x \in_T A$ instead of $T \xrightarrow{x} A$. Then all properties known for sets are generalized for variable elements and generalized properties. For example, if $A \xrightarrow{f} B$, and $x, y \in_T A$, then f is monomorphism iff,

$$f \circ x \equiv f(x) = f(y) \equiv f \circ y \quad \Rightarrow \quad x = y.$$

That is the extensional characteristics of A and B are into one-to-one correspondence. Similarly if $A \xrightarrow{f} B$ is an epimorphism, one cannot construct a generalized property, or intensional characteristic of B that does not have a corresponding through f, property of A.

Elements, Properties and the Yoneda Lemma

Yoneda's Lemma is an important result, which allows us to embed a category $\mathscr{C}$ into a special category of presheaves, that is, of variable sets. This is very similar with the Cayley representation theorem in group theory. Thus instead of studying a small category $\mathscr{C}$, we can study the functorial category $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}} \equiv \widehat{\mathscr{C}}$, of functors from the category $\mathscr{C}^{\mathrm{op}}$ into the category $\mathbf{Set}$. Since the category $\widehat{\mathscr{C}}$ is co-complete, and is made up by functorial variable elements, it reveals the hidden elements of the category $\mathscr{C}$, a similar phenomenon with the embedding of $\mathbb{R}$ into $^*\mathbb{R}$.

Thus one could say that the deep meaning of the Yoneda's Lemma is the 'pathology of the concept of point' or concerns the Protean nature of the concept

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of 'point'. Thus we shall use the variable elements and generalized properties in order to express appropriately the Yoneda's Lemma.

Yoneda's embedding using generalized elements is a full and faithful covariant functor:

$$\begin{array}{rcl} Y^{\triangleright} : \mathscr{C} & \longrightarrow & \mathbf{Set}^{\mathscr{C}^{\text{op}}} \\ A & \mapsto & Y^{\triangleright}(A) := \text{Hom}(-, A) \equiv H_A(-), \\ (f : A \to B) & \mapsto & (f^{\triangleright} : H_A(-) \Rightarrow H_B(-)) \end{array}$$

Yoneda's embedding using generalized properties is a full and faithful contra-covariant functor:

$$\begin{array}{rcl} Y^{\triangleleft} : \mathscr{C} & \longrightarrow & \mathbf{Set}^{\mathscr{C}} \\ A & \mapsto & Y^{\triangleleft}(A) := H^A(-) \equiv \text{Hom}(A, -) \\ (f : A \to B) & \mapsto & (f^{\triangleleft} : H^B(-) \Rightarrow H^A(-)) \end{array}$$

Yoneda's Lemma *Let $\mathscr{C}$ be a small category and let $F : \mathscr{C} \longrightarrow$ **Set** be a functor, and $A \in \mathscr{C}$. Consider also the hom-functor,*

$$Hom_{\mathscr{C}}(A, -) : \mathscr{C} \longrightarrow \textbf{\textit{Set}}$$

Then there is a bijection,

$$\vartheta_{F,A} : Nat(Hom_{\mathscr{C}}(A, -), F) \xrightarrow{\cong} FA$$

which sends each natural transformation $\alpha : Hom_{\mathscr{C}}(A, -) \Rightarrow F$ to $\alpha_A 1_A$ the image of the identity, $1_A : A \to A$.

Summarizing the above we can assert that,

> **Determination of an object of $\mathscr{C}$.** To determine an object $A \in \mathscr{C}_0$ it is sufficient to determine either, all the generalized elements of A that is all ingoing arrows to A, or all generalized properties of A, that is all outgoing arrows to A.

This is a kind of generalization of the extensionality axiom of set theory.

All the above lead to the ultimate generalization of the concept of 'space' and 'point' which is that of topos, see [3] for the impact of the concept of topos on the foundations of mathematics.

Categorification

Under the slogan that '*points have structure*' it is natural to consider that the 'elements' or the points of a structure can be taken as the objects of a category. At the same time the above slogan introduces a level of reality as well.

Thus a function between sets at the 'macroscopic level' becomes a functor at the 'microscopic' one.

"Categorification is the process of finding category-theoretic analogs of set-theoretic concepts by replacing sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors, which in turn should satisfy certain equations of their own, called 'coherence laws'." More precisely we have the following correspondence, see [2]:

Set Theory	$\leftrightarrows$	**Categories**
elements	$\leftrightarrows$	objects
equations between elements	$\leftrightarrows$	*isomorphisms between objects*
sets	$\leftrightarrows$	categories
functions	$\leftrightarrows$	functors
equations between functions	$\leftrightarrows$	*natural transformations between functors*

Table 1: Analogies between set theory and category theory

> "If one studies categorification one soon discovers an amazing fact: many deep-sounding results in mathematics are just categorifications of facts we learned in high school! There is a good reason for this. All along, we have been unwittingly 'decategorifying' mathematics by pretending that categories are just sets. We 'decategorify' a category by forgetting about the morphisms and pretending that isomorphic objects are equal. We are left with a mere set: the set of isomorphism classes of objects." See [2]

The notions of 'monoidal category', 'groupal category', **FinSet** is just the categorification of a monoid, of a group and of $\mathbb{N}$, correspondingly.

5 Conclusions

5.1 Levels of Reality and Completeness.

If 'points have structure', then a structure may have levels of reality. The archetype of this is the pair $\mathbb{R}, {}^*\mathbb{R}$. Note that $\mathbb{R}$ is a complete field. These kind of completeness refers rather to the 'macroscopic' level, since changing 'level' of reality we immediately have a non-complete filed, ${}^*\mathbb{R}$ with a lot of new places for imaginary elements. On the other hand the way we change from the level of $\mathbb{R}$ to the level of ${}^*\mathbb{R}$ where point have become structures, should have an impact on the consideration of 'first' and 'second order' logics.

The problem of completeness has been recognized as an important problem for foundations of mathematics, see *Problem # 6: The Varieties of Complete-*

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ness and Their Role in the Foundations of Mathematics, in PROBLEMS OF PHILOSOPHY, *Synthese* **114:** 161-162, 1998. From an ontological point of view, completeness has something to do with the 'levels of reality' of mathematical frameworks. However we may have e.g. a completely virtual world, where completeness can have a fictionalistic meaning. Finally all the varieties mentioned in *Synthese* should be examined and possibly give a unified treatment, using the Protean nature of points exposed here.

5.2 Non-Cantorian Theories and Vagueness

We claim that mathematical frameworks which incorporates non-probabilistic variability or dually fuzzines, or vagueness, are non-Cantorian in nature, see also [16]. When we have fuzzines, or vagueness, and the like, then we usually have many-valued logics, or gray truth-values. Then excluded middle (EM) cannot hold in these frameworks. But since by the result of Diaconescu, if an elementary topos $\mathscr{E}$ satisfies the internal AC, then $\mathscr{E}$ is Boolean and so EM holds. Thus we get that $\neg$EM $\Rightarrow$ $\neg$AC. Thus frameworks incorporaing, non-probabilistic type of variability, fuzziness, vagueness, indistinguishability, and the like, are essentially non-Cantorian. This is the meaning we give to the term 'non-Cantorian' which of course is compatible with the one in [9].

5.3 The Present State of Philosophy of Mathematics.

S. Shapiro has written recently a really excellent exposition of the main trends of thought in the area of philosophy of mathematics, see [69]. In this book however, we believe that the author left out quite a lot of important cases, maybe due to space available, or due to his inclination towards the analytic tradition. On the other hand there are books like R. Hersh, [26], Lakoff-Núñez [34], and many others which in a sense are complementary to that of Shapiro. We would like however to add some points in order to complete the picture of the current state of the philosophy of mathematics. These points, in a sense, are consequences of the main points of the present paper.

(i) Analytic vs. Continental Philosophy. The opposition between analytic and continental tradition in philosophy in general and in philosophy of mathematics in particular, has a tendency towards dialectic synthesis, see e.g. [44, 51, 58, 75]. This philosophical opposition is similar, or should be of the same nature with the dialectics of the split brain, see Subsection 2.2.

(ii) Dialectics and Mathematics. Dialectics and mathematics is the least mentioned issue in philosophy of mathematics and mathematics itself. Since 'variability' and 'variable structures' have been introduced in category theory see [39], it is natural for one to expect some kind of dialectics as well. These dialectics are introduced by Lawvere through the concept of adjoint functor. What is often ignored or misunderstood is that the dialectics between 0 and 1, or 'white' and 'black', of classical mathematics give way to a 'gray' truth-value, leading to non-classical many-valued logics, see e.g. [35, 3, 18]. For the last reference we are going to give the basic ideas later on.

(iii) Embodied Mind Theories. These theories express ideas very similar to those expressed by M. Heidegger, M. Merleau-Ponty, M. Foucault and others from the continental tradition, about the importance of the human body, including Poincaré, see [34, 23].

(iv) Phenomenology and Mathematics. Husserl influenced a lot of mathematicians, including H. Poincaré, K. Gödel, G. Carlo Rota, P. Vopěnka and many others. Especially Vopěnka developed his Alternative Set Theory (AST), based directly on Husserl and more precisely on [28], and the concept of horizon, see [73, 74]. AST is a rather non-Cantorian set theory, [9], where the whole of mathematics, is reconstructed, based on the phenomenology of Husserl. Actually this theory can be construed as an intentional, non-Cantorian semantic interpetation of Nelson's Internal Set Theory, whereas Robinson's Nostandard Analysis actually is an extensional theory, developed within Cantorian set theory, see also [15, 16].

Gödel and Wittgenstein, in later years, are also connected with phenomenological thinking, see [66, 70] and [50, 68, 24].

(v) Toposophy. Another important omission from recent books on philosophy of mathematics is Category and Topos Theory and their implications to philosophy of mathematics and psychology.

It is known that the three sources of the theory of topoi, are the NSA of Robinson, Cohen forcing & Boolean-valued models of Scott-Solovay-Vopěnka, and algebraic geometry especially in Grothendieck's tradition.

Category and Topos Theory seems to be connected with the foundations of mathematics in general and with the 'cognitive turn' in particular. Macnamara in [48, p.256], nicely describes the situation with the following slogan:

" calculus is to physics as category theory is to psychology",

see also [47]. For the impact of the Category and Topos Theory on the founations of mathematics, see the insightful article by J. Bell, [3]. Also connected with Topos Theory is the quest for a 'synthetic language' of mathematics, see e.g. [57].

(vi) Cognitive Turn. As we go deeper and deeper into the information society, the focus of science and mathematics will be more on human than on physical studies. Epistemology will tend to be rather a subfield of cognitive psychology. We would like to add the following references towards this end, namely [13, 11, 34, 47, 52].

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(vii) Postmodern Thinking. Finally there is a connection between the philosophy of mathematics and postmodern thinking. See e.g. the heretic but interesting article [25, pp. 288-291], which is rather closer to the continental philosophy, and speaks also about the 'deconstruction' of the concept of mathematical 'proof'. For more about Derrida, and the relashionships of Poincaré, Brouwer and Weyl with post-structuralism and deconstruction see the very insightful paper by Tacić, [65], as well as his book [64].

The above list gives a very short description of the contemporary currents of the philosophy of mathematics, which are beyond those described for instance in Shapiro's book. All these positions together with Shapiro's and Hersh's books, give a rather complete picture of the present state of the philosophy of mathematics.

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