Sets, Categories and Structuralism *

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Abstract

First we introduce some basic theoretical issues that set the stage for subsequent accounts. Secondly, we touch upon some important issues: Set-theory vs. category theory, various conceptions of sets, the problem of universals, combining set-theory and category theory, structuralism, and finally category theory as an application tool. We argue that in the present time, the categorical holistic way of structuring is much needed. Also we would like to see a kind of unification, not only to mathematics but to structuralism as well.

1 Introduction

This work is a continuation of [31]. Here we consider the importance of set-theory and category theory, trying to understand this well-known opposition through the dialectics of analytic-elementwise vs. holistic strucutral (A-E vs. H-S). In fact, this dialectic principle plays a cohesive and illuminating rôle throughout the paper, and is applied to set-theory vs. category theory, through universals, foundations, and structuralism. One may conceive [31] as a case study of a possible journey from the classical and static meaning of an 'element' to that of generalized versions that we meet in category theory. It is therefore a journey from iterative membership set-theory to category and toposes. In the Section 2, we examine some basic theoretical principles that underline the account in this

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paper. These principles are of independent interest: *Analytic-elementwise vs. Holistic-structural*, the primordial notion of '*collection*' and its importance for human thinking, as implied by an etymological analysis of the corresponding Greek words, then we examine the important dialectical scheme '*constant vs. variable*', the dispute Hilbert vs. Frege, and finally the meaning of the term 'foundations'. In Section 3, we list some new proposals of set-like concepts and analyze the concepts of abstract and iterative set concept. Next, we examine the structuralizing methods of the iterative set concept and the categorical one. Also we give a concrete example of the way to regard an ordinary object as a category, supporting the in re reasoning. In the sequel we comment on the meaning of 'universal', and its implication for structuralism. In the next subsection we summarize the important development of Algebraic Set Theory and we also suggest a mixing of superstructures and category theory. As for the structuralism, being a broad subject, we choose to indicate and comment towards the direction we propose. Finally, some possible applications of category theory are indicated although not complete.

2 Basic Theoretical Issues

2.1 Analytic-elementwise vs. Holistic-structural

In this paper we shall use again the Cohen-Myhill principle and the A-E vs. H-S dialectic principle. To illustrate the last principle, let us take a set $A \subseteq U$. If we analyze U to its elements, i.e. the simplest and ultimate constituents, then we may check which of these elements belong to A, and by the extensionality axiom we determine A. This is essentially the analytic-elementwise method.

The holistic-structural, is possible if we know the behavior of A with respect to the other subsets of U without mention of elements. Thus, we are forced to consider a higher order type object, which will include all the subsets of U, specifically $\mathscr{P}(U)$, together with the structure of a boolean algebra $\langle \mathscr{P}(U), \cap, \cup, {}^c, \emptyset, U \rangle$. Then for every $A, B \in \mathscr{P}(U)$, if there exists $X \in \mathscr{P}(U)$ such that $A \cup X = B \cup X$ and $A \cap X = B \cap X$ then $A = B$.

E.g.:

$$\begin{aligned} A &= A \cap (A \cup X) = A \cap (B \cup X) \\ &= (A \cap B) \cup (A \cap X) \\ &= (A \cap B) \cup (B \cap X) \\ &= B \cap (A \cup X) \\ &= B \cap (B \cup X) = B. \end{aligned}$$

In other words, if there exists a set $X \in \mathscr{P}(U)$ for which the $A, B \in \mathscr{P}(U)$, have the same behavior with respect to the operations of union and intersection, then the sets A, B are equal. There are several such statements.

Sometimes mathematicians speak of 'bottom-up' and 'top-down' methods. Usually these methods give equivalent results, but there are cases where the holistic-structural gives more than the sum of the parts. For example, the meaning of the psychological term 'gestalt' describes accurately the concept of holistic method: 'gestalt'$\sim$ shape, form, organized whole: The parts do not exist prior to the whole, but derive their character from the structure of the whole. Holism is a wide variety of theses that basically affirm that the 'whole' of some system has a greater or equal reality or explanatory strength than the parts of the system. In this way the whole has some properties that its parts might lack.

In some cases, if we generalize the meaning of 'element', we may find that the two methods give equivalent results. As in [31], if 'element' means 'morphism', then the analytic-elementwise and the holistic-structural give equivalent descriptions. In fact in [31], we "walked through" from the static classical notion of point to that of generalized point or generalized property.

Thus, the analytic method, consists of a decomposition of the object into its simplest and ultimate constituents, together with the logical analyses of them, hopping to capture again the substance of the whole. On the other hand, holistic-structural is the perception of the object as a gestalt, without reference to elements. Further we must remark that the 'constant' is more accessible to analytic quantitative characterizations, whereas the 'variable' needs holistic and qualitative methods.

Another important example of analytic method is the analytic geometry of Descartes. Using coordinates everything is reduced to equations. Especially in algebraic geometry, where we have polynomial equations, it is possible to have 'variable coordinatizations'. If for example, we would like to study the 'Heracleitean flame', an extremely variable object, then by choosing suitable 'variable coordinates' we can manage to *stop variability* and study the flame as a constant object. Again, one may start here, from analytical-elementwise case and end up with a holistic-structural, which is tantamount the generalized analytical-elementwise. All these comments are very much related to the meaning of category which we shall give later.

A final remark concerns the dialectic scheme of this section as seen in the historical process. It seems that the prehellenic societies can be termed 'holistic' since the power was concentrated in the 'one', the king and the clergy. These societies were 'static' and they developed a kind of 'river-culture'. On the contrary, Greek societies were dynamic and their culture was essentially a kind of sea-culture. The 'whole power' was divided into the citizens, and democracy, the first analytical state, was created. Then we observe that this 'analytic society' was followed by a holistic one, the Middle Ages, where the study of 'universals' was basic, and the whole power was given again to the kings and the clergy. Subsequently, this was followed by an analytic society, the Renaissance and the modern period, where we have democracy again, development of science, arts etc. In this period, we have an analytic approach to almost everything. In the production process, we have the Taylor (Taylor, Frederick W. 1856-1956) system for maintaining high level of productivity at optimum cost, by analyzing the production process into a line of basic operations. In philosophy, we have the domination of Analytic Philosophy. This last analytic society is followed from 1980 on, by a holistic information society, which uses computers instead of the God, in trying to integrate and make a new meaningful synthesis out of the huge mass of incomprehensible and thus useless analytical knowledge. Holistic methods, like the Categorical one, are now more in focus. Notice, however, that in the periphery of the 'metropolis' the societies are characterized by religious fanaticism, as if they were in some new "Middle Age"!

2.2 Collections and Human Thinking

Here we would like to indicate some etymological remarks that seem to reveal and shed light to some contemporary disputes. It is true that set-theory in general (and not any of its particular axiomatic versions) is a very popular foundational approach amongst mathematicians.

There is also a fanatic attachment to set-theoretic thinking by most mathematicians. The situation is nicely described by the review of [61] by M. Berg in the `http://www.maa.org/reviews/setbooks3.html`: "this book ought to serve quite well as a major player in the early education of a mathematician who sees, say, algebraic geometry in his future, or certain obvious parts of algebra or topology".

Many people can hardly notice the unifying power of the categorical approach. It does not matter if one will see its future in algebraic geometry or algebraic topology. The fact is that the future mathematician needs a unified picture of mathematics, and this picture can be obtained through 'a unity and identity of opposites' between category and topos theory and the iterative membership set-theory. Algebraic set theory might be the missing synthesis.

Now let us try to explain this attachment to set-theoretic thinking of any kind. To get an idea of a 'set' and an 'element of a set' it is necessary to comment on the etymological content of the words 'collection' and 'elementary'. In the Greek language 'συλλογή' means 'collection'. This word comes from 'συν+λέγω', which means 'bring together', 'collect', 'gather' data and present them into the mind, to think about them. Thus, 'συλλογίζομαι' means '*I'm thinking*', and finally the Aristotelian '*syllogism*' comes from the same etymo! As a consequence, *the ability to form collections, to conceive individuals and distinguish those objects from a collection that satisfy some property* is tantamount *to thinking*! We should remark, however, that the ancient Greeks, believing in potential infinity rather than actual, never formed collections like 'the collection of all points of the plane' etc. So 'thinking' was allowed only on finite collections. On the other hand, 'elementary' means something which is related to 'elements'. Thus, e.g. 'point-free' formulations are not 'elementary', but they are 'generalized elementary'!

The above comments reveal the strong relevance of 'forming collections' and 'determining the membership relation' for the human thinking. In other words, set-theory seems to be closer to the nature of human thinking, and not just to the habit of thinking in membership set-theory. Due to the developments of algebraic topology and algebraic geometry, membership set-theory must be generalized, by first generalizing the concept of 'element' and 'membership' to the dynamical case, using e.g. sheaf theory. I believe that the introduction of 'generalized elements' as well as 'generalized properties' [61] or 'generic figures' [82] express the exact nature of the said generalization, and serves enormously the understanding of category and topos theory in a set-like environment.

2.3 Constant vs. Variable

In [31] a possible tour has been described, starting from the classical constant concept of 'element' or 'point', and ending up with Grothendieck's general points (geometric morphisms), which can have internal symmetries, revealing that 'points have structure'; see also [23, §7]. This tour should convince anyone that in order to achieve a unified understanding of mathematics, he/she should proceed to generalize set-theory along these lines, i.e. using the '*dialectical process of replacing the constant by the variable*'.

Actually, the 'point free' formulations are isomorphic with 'generalized point' formulations. For the dialectics of points see [31], and for the constant vs. variable see [16].

We should also like to comment on the constant vs. variable scheme. Variable is regarded as dual to vagueness (see [31]), and it is known that the western cultures are rather hostile to vagueness and variability. It is a common belief that variable environments are not ideal to study and obtain knowledge. There is even a psychological need to return to constancy! Bell in [16] remarks in this connection:

> "Now in certain important cases we can proceed in turn to dialectically negate the "variation" in $\mathbf{E}$ to obtain a new classical framework $\mathbf{S}^*$ in which constancy again prevails. $\mathbf{S}^*$ may be regarded as arising from $\mathbf{S}$ through the dialectical process of negating negation."

Another important way of 'stopping variation' is described in [16]:

> "Consider, for example, the concept "real-valued continuous function on a topological space X" (interpreted in a topos $\mathbf{S}$ of constant sets). Any such function may be regarded as a real number (or quantity) varying continuously over X. Now consider the topos $\mathbf{shv}(X)$ of sheaves over X. Here everything is varying (continuously) over X, so shifting to $\mathbf{shv}(X)$ from $\mathbf{S}$ essentially amounts to placing oneself in a framework which is, so to speak, itself "moving along" with the variation over X of the given variable real numbers. This causes the variation of any variable real number not to be "noticed" in $\mathbf{shv}(X)$; it is accordingly there regarded as being a constant real number. In this way the concept "real-valued continuous function on X" is transformed into the concept "real number" when interpreted in $\mathbf{shv}(X)$.

A similar phenomenon is common in modern algebraic geometry, where variable coordinatizations are used to study variable objects; see for example the 'Heracleitean flame' mentioned above.

Finally, the abundance and the importance of Hom-functors in category theory is a compelling witness of the need of returning to the 'paradise' of constancy.

2.4 Axioms vs. Noetic Principles: Hilbert vs. Frege

> About the Hilbert vs. Frege dispute we quote from [84] "Frege's point is a simple one. If the terms in the proposed 'axioms' do not have a meaning beforehand, then the statements cannot be true (or false), and thus they cannot be axioms. If they do have a meaning beforehand, then the 'axioms' cannot be definitions."
>
> "Hilbert replied on December 29, rejecting Frege's suggestion that the meanings of the words 'point', 'line', and 'plane' are 'not given, but are assumed to be known in advance': I do not want to assume anything as known in advance. I regard my explanation . . . as the definition of the concepts point, line, plane . . . If one is looking for other definitions of a 'point', e.g. through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there; and everything gets lost and becomes vague and tangled and degenerates into a game of hide and seek.
>
> This talk of paraphrase is an allusion to 'definitions' like Euclid's 'a point is that which has no parts'. Such 'definitions' play no role in the mathematical development, and are thus irrelevant.

Let us recall the Euclid's definition together with Cantor's famous one:

> "A point is a place with neither length nor breadth."

> "By a 'set' we mean any collection M into a whole of definite, distinct objects m (called the 'elements' of M) of our intuition or our thought."

Manin in [68, p. 29] remarked on these definitions as follows:

> Cantor's definition of a set, which is given above, has with some irony been called "naive" by comparing it with Euclid's definition of a

> point [above]. This criticism is testimony to the lack of understanding of the fact that the fundamental concepts of mathematics, which are not reducible in a given system to more elementary concepts, must necessarily be introduced in two ways: the concrete ("naive") way and the formal way. The purpose of the concrete definition is to create a prototypical, not yet completely formed shape, to tune different individual intellects to one scale, as by a tuning fork. But the formal definition does not really introduce a concept, but rather a term; it does not introduce the idea of a "set" into the structure of the mind, but rather introduces the word "set" into the structure of admissible texts about sets.

From the above comments it is clear that we should pay attention to both 'formal definitions' together with 'naive, intuitive' or 'conceptual' enlightenments, insights and motivations. Mathematics includes not only the formal but also the 'conceptual' side. We may recall, at this point, Kant's famous slogan: "Concepts without intuitions are empty, and intuitions without concepts are blind". We could paraphrase this as: Axioms without noetic principles are empty, and noetic principles without being concentrated into axioms are blind. Finally, category theory, properly understood, is in fact conceptual!

Concerning the Hilbert-Frege dispute, we can see from the above discussions that a pressing question arises: Is it possible to have both Hilbert and Frege? Hilbert describes the formal axiomatic side, whereas Frege exemplifies the noetic principles that support the axioms. Compare, e.g., an axiom with the Cohen-Myhill principle in [31]. In this way the seemingly contradictory nature of Hilbert-Frege issue takes the form of a dialectical synthesis: unity and identity of opposites. I believe that it is possible, with a significant ammount of difficulties to 'walk through' the poles of a dialectic scheme. Actually, [31] is such a 'tour' from the static classical concept of 'point' to the dynamical one expressed by a morphism; or in other words, it is a walk from classical set-theory to category theory. By adapting the one or the other absolute view, one can destroy the 'naturality' of the framework. For example, in Hilbert's case:

"Hilbert's *Grundlagen der Geometrie* [1899] represents the culmination of this development, delivering a death blow to a role for intuition or perception in the practice of geometry. Although intuition or observation may be the source of axioms, it plays no role in the actual pursuit of the subject." [84].

However, this is exactly the severe criticism against formalism that delivers a death blow to the rôle of intuition or perception in the practice of geometry. Finally, if one discovers that the axioms of geometry that implicitly define “points, straight lines, and planes”, also define implicitly “tables, chairs, and beer mugs”, then he should be very reluctant accepting the axioms, as axioms for ‘geometry’.

For further remarks about the Hilbert-Frege controversy see the remarks after the axioms for category theory.

2.5 Foundations of Mathematics

Examining again the dialectic scheme A-E vs. H-S, we are forced to consider two types of foundations, which constitute the poles of the dialectical scheme.

An analytic approach to foundations presupposes a disintegration of the holistic character of mathematics, in Cavaillès’ sense say. In [28] we find:

> “Mathematics constitutes an organic whole of concepts, methods and theories crossing each other and interdependent in such a way that any attempt to isolate one branch, even for the sake of providing a foundation for the whole body, would be in vain; nor would it make sense to break this organic solidarity by looking for an external foundation (be it natural science or logic or some psychological reality). As a consequence, a point made by Hilbert is reaffirmed: logic does not precede mathematics but is a (very elementary) part of it, isolated artificially for the sake of a tractable translation of the theories and ‘having the same authority as arithmetic or analysis with respect to richer theories’ (Cavaillès 1938, p. 180).”

After this disintegration we see a linear, extremely reductive, approach to foundations. It searches for a basis on which all other mathematics can be built. The most important such basis is the Cantorian iterative membership set-theory. Set-theory together with classical logic constitute a major attempt to apply analytic methodologies into the whole of mathematics. In this way the holistic-synthetic geometric form is analysed into an amorphus mass of points, whose logical analyses along with the climbing up to the hierarchy, would reproduce hopefully the synthetically given geometric form! The analytic-elementwise reduction of structures to sets (set-theoretic reductionism) is expressed by Bell, [13]:

> "Thus, although the notions of structure and operations on structure had come to play a fundamental role in most mathematical disciplines, these notions are *not* taken as primitive, but were themselves explicated by *reduction* to the more fundamental notions of set and membership".

Related to this analytic approach to foundations is the following point of view: Roger Penrose contends that the foundations of mathematics cannot be understood absent the Platonic view that "mathematical truth is absolute, external, and eternal, and not based on man-made criteria ... mathematical objects have a timeless existence of their own...." (see http://en.wikipedia.org/wiki/Problem_of_universals).

Another famous Platonist is A. Connes. In [26], he expresses his view about the 'primordial mathematical world':

> From my view, the power of a proposition lies simply in the implications that it has to the primordial mathematical world, which I distinguish from the instrument of appreciation that is constructed gradually and that, of course, never has more than limited precision... "Primordial" is a deliberately term, referring to a prior existence, which somehow precedes axiomatization.

The other version of foundations, actually, is not a foundation at all. Its main characteristic is that it reconstitutes the holistic character of mathematics, and attains again their unification that the analytic approach had broken down into pieces. This is evident, for example, in the case of topos theory, where the mathematician has at his disposal a mixture of geometrical methods, raging from simple geometry to algebraic geometry and topology, together with the logical calculus of depended type theory of higher-order logic. Furthermore in [33], there is a remark about the 'grand unification' attained with Grothendieck notions:

> The theory of schemes is the foundation for algebraic geometry formulated by Alexandre Grothendieck and his many coworkers. *It is the basis for a grand unification of number theory and algebraic geometry, dreamt of by number theorists and geometers for over a century.* (Italics added)

Connected with the above meaning of foundations are the following opinions of Lawvere:

> "Concentrate the essence of practice and in turn use the result to guide practice" [59]
>
> [a] foundation makes explicit the essential features, ingredients, and operations of a science as well as *its origins and general laws of development*. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A 'pure' foundation that forgets this purpose and pursues a speculative 'foundation' for its own sake is clearly a nonfoundation. ([61, p. 235], italics added)

A similar opinion is expressed by Manin:

> I will understand 'foundations' neither as the para-philosophical preoccupation with the nature, accessibility, and reliability of mathematical truth, nor as a set of normative prescriptions like those advocated by finitists or formalists. I will use this word in a loose sense as a general term for the historically variable conglomerate of rules and principles used to organize the already existing and always being created anew body of mathematical knowledge of the relevant epoch. At times, it becomes codified in the form of an authoritative mathematical text as exemplified by Euclid's Elements. In another epoch, it is better expressed by the nervous self-questioning about the meaning of infinitesimals or the precise relationship between real numbers and points of the Euclidean line, or else, the nature of algorithms. In all cases, foundations in this wide sense is something which is relevant to a working mathematician, which refers to some basic principles of his/her trade, but which does not constitute the essence of his/her work. ([69])

Manin continues by discussing the foundational rôle of Cantorian sets, before proceeding to explain how this rôle was taken up first by categories, and now by higher-dimensional categories.

Corfield's position is also related to this version of foundations of mathematics. In [27], he tries to "reorient the philosophy of mathematics away from what is often called the foundations of mathematics and towards the 'real mathematics' of the working mathematician." This view is in harmony with the opinions of Lawvere and Manin.

From a set-theoretic perspective, we also conceive a kind of non-Cantorian foundations of mathematics, which are referring to frameworks where the AC and certain principles of classical logic do not hold. For example, Intuitionistic Set Theory, Constructive Set Theory, Algebraic Set Theory and topoi, Alternative Set Theory (Vopěnka), etc.

In summary, the true notion of foundations is always an appropriate synthesis of the two poles, which facilitates the needs of each particular society.

3 Sets and Categories

3.1 Various Conceptions of Sets

Although it seems that Cantorian set-theory has prevailed in the entire field of mathematics, there are some "singularities" where the invention of new set-like concepts is inevitable. A partial list of such instances is the following: The mathematics of intuitionists, the algebraic geometry of the Italian School, the Cohen's method of forcing, and its equivalent formulation of Boolean-valued models of Scott-Solovay-Vopěnka, the nonstandard analysis, the theory of topoi, and the Vopěnka's Alternative Set Theory.

From a different perspective, we have: The theory of Probabilistic Metric Space (Menger, Schweizer and Sklar), and Zadeh's Theory of Fuzzy Sets. In chronological order, we have:

(i) Sheaf Theory (Leray(1945), Cartan(1948,49)).

(ii) Grothendieck Topoi (Grothendieck, Giraud (1957,1961)).

(iii) Probabilistic Metric Spaces (Menger(1942,51), Schweizer and Sklar (1961)).

(iv) Nonstandard Analysis (Robinson (1961)).

(v) Forcing (Cohen (1963)).

(vi) Boolean-valued models (Scott, Solovay, Vopěnka,(1965)).

(vii) Kripke models (Kripke (1965)).

(iix) Fuzzy Sets (Zadeh (1965).

(ix) Elementary Topoi (Lawvere and Tierney (1969)).

(x) Internal Set Theory (Nelson (1977)).

(xi) Alternative Set Theory (Vopěnka (1979)).

(xii) Non-well-founded sets or Hypersets, Generalized Set theory, Constructive Set Theory (Aczel, but see also Appendix A in [2]).

(xiii) Algebraic Set Theory (Joyal-Mordijk (1995)).

Careful examination of all of these new set-like concepts, reveals that the common notion inherent in them are *variability and vagueness*. The variable point of view is manifested in sheaf theory, whereas the vagueness approach is realized in the Boolean and Heyting-valued models and in general in the mathematical theory of fibrations. Fuzzy sets is a naive approach to vagueness. The two points of view (variable vs. vague) are dual and variability coincides with the extensional aspect of entities, whereas regarding vagueness with the corresponding intentional aspect, see [31].

Two of the basic conceptions of sets are the *iterative notion of set* (Cantor, Zermelo) as presented, e.g., in [86], and the abstract notion of set, as presented in [61]. We shall call the iterative conception of set 'vertical', and the abstract conception of set 'horizontal'. Let us recall the notion of abstract set from [61]:

> "An **abstract set** is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for number of elements."

According to the above definition, the elements of an abstract set are similar to urelements. Urelements are some given objects which are supposed not to have any set-theoretic structure, and their existence is independent of any set-theoretic constructions. Thus if $V(U)$ is a set universe with urelements U, then the classic membership relation '$\in$' is modified to '$\in_U$,' and is defined as follows:

$$(\forall u \in U)(\forall x \in V(U))[x \notin_U u] \quad \mathit{or} \quad u \neq \emptyset, \quad \& \quad u \cap V(U) = \emptyset.$$

For the sets $V(U) \setminus U$ of the universe $V(U)$ the relation '$\in$' is unchanged.

The notion of 'urelement' is not absolute, but it is rather a convenient convention, which at the same time implicitly incorporates a notion of 'level of reality', where the urelements play a rôle of 'macromolecules'. The existence of urelements is in harmony with 'Occam's razor': "Entia non sunt multiplicanda praeter necessitatem".

In ZF, it is possible to eliminate urelements from our theory, due to axioms of 'Powerset' and 'Substitution' or 'Replacement'. There are, however, weaker set-theoretic axiomatic systems, which allow urelements; see e.g. the Kripke-Platek system [11, 70].

In defining these two types of sets, the iterative and the abstract notion of set, we think that directed graphs can play the ultimate foundation, both for set-theory (trees) and abstracts sets (pointed directed graphs).

Let us give some examples of abstract and structured sets.

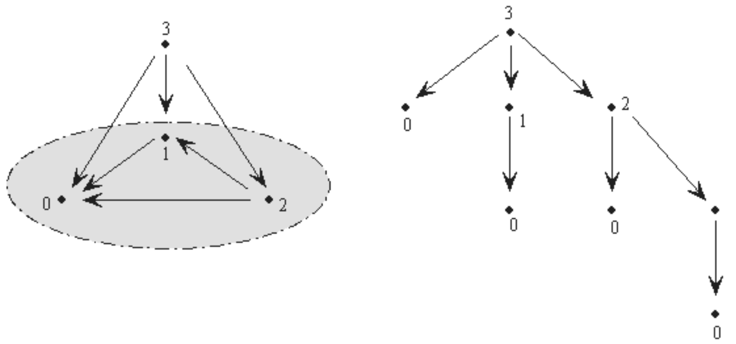


Figure 1: The number 'three' structured as an oriented graph or as a tree.

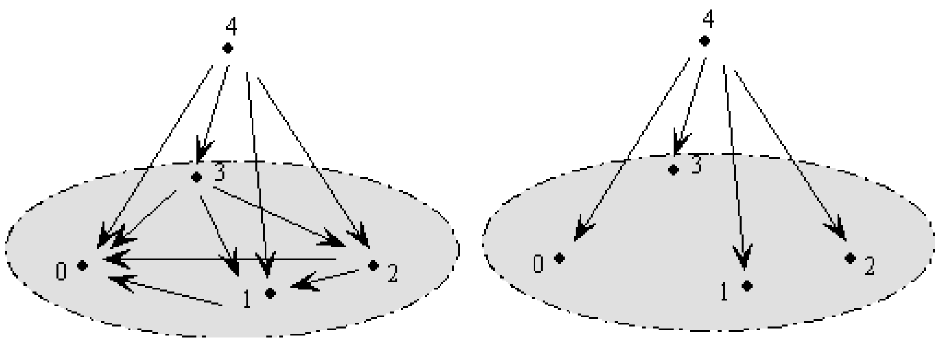


Figure 2: The number 'four' as structured and as abstract set.

One way of getting the class of abstract sets from the class of pointed graphs is as follows:

Let $\mathscr{G}$ be the class of oriented pointed graphs, i.e. graphs of the form, $\langle G, n_G \rangle \equiv \langle G_0, G_1, (\mathrm{dom}, \mathrm{cod}); n_G \rangle$, where n_G is the point or top vertex of the graph. Then the class of abstract sets is taken by keeping all the external arrows, $\{\alpha \in G_1 \,|\, \mathrm{dom}(\alpha) = n_G \quad \text{and} \quad \mathrm{cod}(\alpha) \in G_0 - \{n_G\}\}$ of the graph together with the class of nodes G_0. Actually this is a forgetful operation, similar to the forgetful functor. Now taking as objects the abstract sets and as arrows their functions, we are getting essentially the category of **Set**.

3.2 Sets and Categories

Every conception of set must incorporate two processes:

- The process of forming a '*collection into a whole,* and
- *a process of reconstituting the collections* as structures.

In the iterative motion of a set both processes are formed in 'stages'. We start at zero stage with urelements and later form collections using the powerset operation, which at the same time are getting structured by the membership relation; see the examples above. Thus, the location in the universe $V(U)$ determines the kind of structure of the 'set', which usually takes the form $\langle X; \mathscr{F}, \mathscr{R}, \mathscr{C}\rangle$, where X is the carrier set, and $\{\mathscr{F}, \mathscr{R}, \mathscr{C}\}$ is the signature of the structure, consisting of a set of 'operations', 'relations' and 'constants' as usual. In this way, we get the von Neumann universe,

$$\begin{aligned}
V_0(U) &:= U \quad (U \text{ is the set of urelements}) \\
V_{n+1}(U) &:= V_n \cup \mathscr{P}(V_n(U)), \quad n \in \mathbb{N} \\
V_\alpha(U) &:= \bigcup_{\beta < \alpha} V_\beta(U) \quad (\text{if } \alpha \text{ is a limit ordinal}) \text{ and} \\
V(U) &:= \bigcup_{\alpha \in \mathrm{ON}} V_\alpha(U) \quad \text{where ON is the class of all ordinals.}
\end{aligned}$$

One of the areas of logic, where set-theoretic structures are studied is model theory. In recent years, model theory is seen to converge to algebraic geometry, where categorical methods are indispensable. The situation is nicely expressed by Macintyre [63]:

> I see model theory as becoming increasingly detached from set-theory, and the Tarskian notion of set-theoretic model being no longer central to model theory. In much of modern mathematics, the set theoretic component is of minor interest, and basic notions are geometric or category-theoretic. In algebraic geometry, schemes or algebraic spaces are the basic notions, with the older "sets of points

in affine or projective space" no more than restrictive special cases. The basic notions may be given sheaf-theoretically, or functorially. To understand in depth the historically important affine cases, one does best to work with more general schemes. The resulting relativization and "transfer of structure" is incomparably more flexible and powerful than anything yet known in "set-theoretic model theory"...

There are various hints in the literature as to categorical foundations for model-theory [56]. The type spaces seem fundamental ... , the models much less so. Now is perhaps the time to give new foundations, with the flexibility of those of algebraic geometry. It now seems to me natural to have distinguished quantifiers for various particularly significant kinds of morphism (proper, étale, flat, finite, etc), thus giving more suggestive quantifier-eliminations. The traditional emphasis on logical generality generally obscures geometrically significant features ...

In the above reference, we may add, e.g. [67, 4]. On the other hand, the category-theoretic way of structuralization also involves two processes: Abstract sets are collected into a class $\mathbf{Set_0}$. This finishes the first process.

To structure the members of $\mathbf{Set_0}$, we consider the functions between the abstract sets, the class $\mathbf{Set_1}$, as the basic structuralizing tool. The interactions of the members in $\mathbf{Set_0}$, through the net of functions in $\mathbf{Set_1}$, give to each abstract set a structure or form in a universal way. For examples see [61, 82]

The following passage from McLarty is related to the above view of structuralizing:

"The main point of categorical thinking is to let arrows reveal structure. Categorical foundations depend on using arrows to *define* structures. But this approach only determines *structure*, that is it only defines objects up to isomorphism. Set-theory as practiced today is unique among branches of modern mathematics in not generally defining its objects up to isomorphism. It is nearly unique in focusing on structure that is *not* preserved by its arrows. It is (and this is nearly a definition of set-theory from the categorical point of view) the branch of mathematics whose objects have the least structure preserved or revealed by their arrows. So set-theory as practiced today is a uniquely bad example for category theory.

Of course set-theory might not always be practiced as it is today. Lawvere points out that the major questions in set-theory deal with

> isomorphism invariant properties and are easily stated in categorical set-theory: choice, the continuum hypothesis, various large cardinals. Since the membership relation is unnecessary in stating these problems we might wonder how far it will help in settling them. But for now virtually all research in set-theory is membership theoretic." [72, p. 366]

Let us give a more concrete example. Consider the following surface:

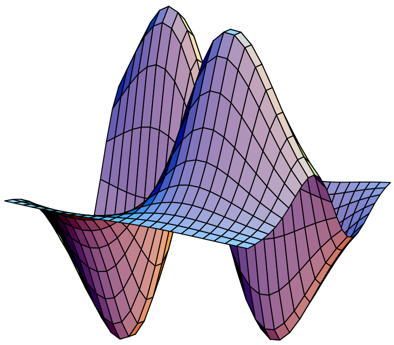


Figure 3: The morphisms $f : T \to S$ reveal the geometrical form or structure of the surface S.

We may even consider the above surface as a category. Suppose that the surface is made up by glass, say. Then each molecule has a chemical structure, which is usually expressed by a graph. We may then consider the class of molecules of the surface as the objects of a category. We may also take as morphisms any curve on the surface which connects molecules. Obviously these morphisms preserves the structure of the molecules, and essentially coincide with the fundamental group of the surface. This is a kind of *in re* extraction of the concept of category.

The power of looking first at morphisms is nicely described in the following passage [29, p.175, footnote # 5]:

> This way of defining a structure is fairly common in mathematics. For instance, on the set $\mathbb{C}$ of complex numbers one may consider the following increasingly general sets of functions: linear, affine, polynomial, analytic, differentiable, continuous, measurable, and - lastly

- arbitrary. Then $\mathbb{C}$ will be, respectively, an object of linear algebra, of affine geometry, of algebraic or analytic geometry, a smooth manifold, a topological or a measurable space and simply a continuum.

Thus to describe axiomatically the two processes, the formation and the structuralizing process, we have to take as axioms essentially those which implicitly grasp the concept of 'function', the basic strucutralizung tools:

- the abstract sets form a collection, usually called a class, and
- the elements of $\mathbf{Set_1}$ satisfy the usual axioms for functions:

(i) $(h \circ g) \circ f = h \circ (g \circ f)$, whenever the two sides are defined, or with a diagram,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] \arrow[drr, dashed] & B \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

Note that the notation $f;g \equiv g \circ f$ is used in theoretical computer science.

(ii) $f \circ 1_A = 1_B \circ f = f$, for every function $f : A \longrightarrow B$, or in a diagram,

$$\begin{tikzcd}
A \arrow[d, "1_A"'] \arrow[dr, dashed, "f=f \circ 1_A"] & \\
A \arrow[r, "f"] \arrow[dr, dashed, "1_B \circ f=f"'] & B \arrow[d, "1_B"] \\
& B
\end{tikzcd}$$

(iii) furthermore we demand the following:

(a) $\mathrm{dom}(g \circ f) = \mathrm{dom}(f), \quad \mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

(b) $\mathrm{dom}(1_A) = \mathrm{cod}(1_A)$.

In the above specification, we may take the notion of collection or class as a primitive naive notion without any further axiomatic specification. In any of the existing set-theories, the notion of class acquires a structure through the iteration process, which we may not wish to have.

However, revealing the structure is not enough to do mathematics. We must have a way to operate on them. For this reason, we introduce a kind of morphism between categories, which will represent various mathematical activities: mathematical constructions, interpretations or models, picturing one category into the other, etc. All these elements are incorporated in the concept of a *'functor'*. On the other hand there is a kind of 'categorical sliding' of one construction into another, which is a morphism between functors, and thus preserve the construction represented by the particular functor. In this way we arive at the important concept of *'natural transformation'*. These basic concepts together with *limits* and *colimits* conclude the fundamental tools of category theory. In addition, from these basic concepts springs out another notion, that of *'adjunction'*, which is the the most important concept in category theory.

The above axioms are intended to describe axiomatically the structuralizing notion of function. However, as is the case for any first-order structure, there are unintended models, which actually catch the general notion of morphism, thus leading to axioms of category theory. In [41], Hellman attributes to the axioms of category theory a schematic Hilbertian sense:

> ... the tendency of modern mathematics toward a structuralist conception has been marked by the rise and proliferation of Hilbertian axiom systems (practically necessitated by the rise of non-Euclidean geometries), with relegation of Fregean axioms to a set-theoretic background usually only mentioned in passing in introductory remarks. Category theory surely has contributed to this trend; we now even have explorations of "Zermelo-Fraenkel algebras" (Joyal and Moerdijk [1995]).

There is, however, a difference. In the Hilbertian case the axioms are schematic even in cases when they should not be. There are nevertheless cases that the axioms describe a 'type' of objects and not the actual 'tokens' that exemplify the type. In this case, the axioms must by necessity be schematic, but not in a Hilbert sense. As Hellman pointed out "the rise and proliferation of Hilbertian axiom systems (practically necessitated by the rise of non-Euclidean geometries)". Furthermore in contemporary mathematics, this phenomenon is common. One axiomatizes, in a Fregean sense, a given well conceived intended reality, and then one discovers that there are also unintended nonstandard models. This situation makes any Fregean axiomatic system to look like a Hilbertian one. In our view, we may still preserve the Fregean approach to axiomatics, even in the case that we have nonstandard unintended models. In [31], we gave meaning to these nonstandard models by introducing the concept of level of reality.

3.3 Universals

The theory of universal has a long history. It starts with Plato and Aristotle, goes through the Middle ages, where it becomes the main subject of philosophico-theological study, and continuous on up to the present day. One may divide the various positions about universal into main four classes: The metaphysical Platonic *ante rem realists*, according to which universals exist prior to and independent of any items that instantiate them, the Aristotelian *in re realists* that argue that universals does not exist before things and so they are depending on their instances; the nominalists who maintained that the only universals are words, since they can be applied to many instances; and the *post res conceptualists* for whom universals arose a posteriori in the mind as concepts, abstracting from particulars. Using some neuro-cognitive argumentation, one may combine the 'in re' realism of Aristotle and the 'post res' conceptualism into a coherent empirical position. Later on we also comment about nominalism as well. Thus in the following we shall deal with two kinds of universals: *the abstract* and *the concrete*. In particular we will confine ourselves to relationships of universals to mathematics.

There is a slogan according to which:

> "Category Theory is the study of universal properties and universal constructions."

To clear up the issue as to what kind of universals category theory is referring to, one has to consult Ellerman who has conducted the most decisive analysis on this issue in [34]. We should like, however, to make a few comments that might clear some additional points on the issue.

An alternative to *ante rem universals* are the Aristotle's *in re universals*,

> "...for Aristotle things in the physical world have Forms, but there is no separate world to house these Forms. Forms exist in the individual objects." [83].

Actually, the explicit notion of a *concrete universal* is attributed to Hegel. Construed in this way, a concrete universal is essentially 'a becoming', 'a motion' and at the same time a real entity, which characterizes the living beings as well. In this way the abstract universals are the ideal types from which the entities derive their existence, whereas concrete universals are formed from a mind act, which frees the common elements from various things and express them as a 'concept'. In Gadamer [47], there is a kind of dialectic between universals and particulars:

> "This situation, as we shall see, perfectly mirrors the dialectical resolution that occurs in Gadamer's thought concerning both the relation between universals and particulars and the closely associated relation between the notions of atemporality and temporality."...
>
> "Application is precisely the bringing of the universal and particular into an equable relation, and it is in this way, and this way only, claims Gadamer, that understanding is achieved. The important point here, epistemologically, is that the abstract universal is in fact meaningless without the particular cases it is applied to, and likewise the particular is meaningful only in light of the universal applied to it. Understanding, therefore, is achieved only in the dialectic resolution or fusion of these two poles. One can recognize here a hermeneutic circle of whole and parts, formulated in terms of a harmony of universal and particular."...
>
> "Understanding, we must remember, is a fusion of horizons, and as the horizon of the reader develops so the meaning of a work will undergo modification."
>
> "Experience, to become 'universal', must be shared – must become a shared, dialogical understanding. Or, in Hegelian terms, the "I" must become a "we"."

Then from the above passages it becomes clear that a concrete universal is essentially 'a becoming', 'a motion', and at the same time a real entity. This 'becoming' refers essentially to the changing of Husserlian 'horizons', which corresponds to the changing of categorical frameworks.

The sets of categorical set-theory are themselves abstract structures in exactly this sense. An element $x \in S$ in categorical set-theory has no properties except that it is an element of S, and is distinct from any other elements of S. This is discussed historically and philosophically in Lawvere [59].

Now let as make some remarks on Ellerman's treatment of concrete universals. The classification of universals to abstract or concrete depends on the property of 'participation'. "A universal u_F is said to be abstract if it does not participate in itself, i.e., $\neg(u_F \, \mu \, u_F)$. Alternatively, a universal u_F is concrete if it is selfparticipating, i.e., $u_F \, \mu \, u_F$. He claims that

Figure 4: Abstract vs. concrete universals

category theory is a theory of concrete universals: "Is there a precise mathematical theory of concrete universals? Is there a theory that is to concrete universals as set theory is to abstract universals? Our claim is that category theory is precisely that theory."

We agree with Ellerman's claim. There is, however, a small problem with the defining property for concrete universals: 'selfparticipation'. Although classical set theory is a theory of abstract universals and does not satisfy 'selfparticipation', there is however the non-well founded sets which satisfies selfparticipation without causing any paradoxes. Is then non-well founded sets another theory of concrete universals? My claim from the beginning was that non-well founded sets lead to category of sets and from there to category theory and to algebraic set theory, which might be regarded as a dialectical product of motionless abstract universals with the 'becoming' theory of concrete universals. This is very much related to Gadamer's dialectics of atemporality vs. temporality.

A final remark concerns nominalism. Aczel [2] has introduced an elegant theory of algebras and co-algebras in order to model syntax as an initial algebra (a universal object), and the interpretation of the syntactic theory, the models of the theory are all the other objects of the category. This can be considered as an attempt to encompass nominalism in a categorical framework. Words in the language of this initial algebra are 'universals', because they can be interpreted to all other objects.

Finally, the distinction between abstract and concrete universals will determine the relationship of Category Theory to structuralism as well.

3.4 Combining Set-theory and Category theory

3.4.1 Algebraic Set Theory

In algebraic Set Theory a synthesis of three basic foundational ideas is dialectically given: *Set-theory*, *Type Theory* and *Category Theory*. The last topic is acting as an organizational principle of the other two. This synthesis is based on the ideas introduced by Joyal and Moerdijk [44]. The replacement of their ZF-algebra $\langle A, s\rangle$ where A is a complete sup-lattice and $s : A \to A$ a successor operator, by a simpler one of an Φ-algebra in the sense of Aczel [2], i.e. $\langle A, \alpha\rangle$, with $\alpha : \Phi A \to A \in \mathscr{C}_1$ and $\Phi : \mathscr{C} \to \mathscr{C}$ in an endofunctor of the category $\mathscr{C}$; see [8]. Another crucial idea is the shifting of focus from the category of sets to the idea of *category with class*. This shifting parallels the one from ZF set-theory to NGB set-theory with classes. These developments are very much connected with the development of Intuitionistic and in general constructive set-theories; see [3].

It seems that one way of synthezing the paths exposed above might be "Algebraic Set Theory". Joyal and Moerdijk in [44], see also [8], have developed a theory that extends both topos theory and (intuitionistic) set-theory. Furthermore with their theory provide a uniform description of various constructions of the cumulative hierarchy of sets, forcing models, sheaf models and realisability models.

Moerdijk and Palmgren [76], comment in this connection:

> Using an auxiliary notion of 'small map', it is possible to extend the axioms for a topos, and provide a general theory for building models of set-theory out of toposes ("Algebraic Set Theory")
>
> ...the study of predicatively constructive logical systems, such as Martin-Löf type theory, Aczel–Myhill set-theory (CZF) and Fefer-

> man's systems for explicit mathematics, has shown (or indicates) that large parts of mathematics can be formalised without the use of an "impredicative" notion of power set. Instead, these systems use generalized inductive definitions. These predicative systems are closely related to each other. For example, Aczel showed how CZF can be interpreted in Martin-Löf type theory...
>
> Thus the question naturally arises as to what would be a useful notion of "predicative topos".
>
> ...given such a notion of "predicative topos", the question arises whether the algebraic set-theory based on small maps in a topos can be adapted to predicative toposes so as to give sheaf and other categorical models for CZF.

Some results of AST are listed in [8]:

> The main results regarding this notion of a class category and elementary set-theory are the following:
>
> 1. In every class category, the universe U is a model of the intuitionistic, elementary set-theory BIST.
> 2. The elementary set-theory BIST is logically complete with respect to such class category models.
> 3. The category of sets in any such model is an elementary topos.
> 4. Every topos occurs as the sets in a class category.
> 5. Every class category embeds into the ideal completion of a topos.

The notion of class category solves in a categorical way the foundations of categories and topoi. Although from a foundational point of view AST is very satisfying, it remains to see what the impact on applications to mathematics and to other sciences will be.

Summarizing the above discussion, instead of arguing on the set-theoretic foundations of category theory or the categorical foundations of set-theory, we propose to use as a foundation for both set-theory and categorical set-theory the notion of directed graphs, which can be formed without mention of any set-theory except maybe the primordial concept of 'collection'. Going from category of sets to algebraic set-theory, one may introduce as many category theory as needed (functors, natural transformations, adjunctions, functors, abelian categories, e.t.c.). This is essentially the method indicated both by [61] and [82].

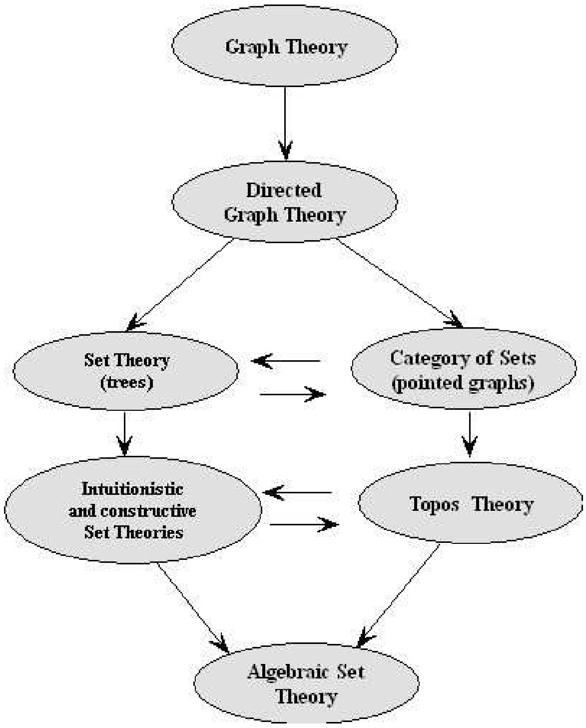


Figure 5: A Summary of Foundations

3.4.2 Superstructures Instead of Structures

Another way of synthesizing category theory and set-theory could be the building of superstrcutres over the category of sets. We shall examine here a hierarchy of sets with urelements A, in symbols $V(A)$. $V(A)$ does not contain 'abstract' sets beyond A. All the entities of $V(A)$ are structured sets. As we proceed up to the hierarchy $V(A)$, we meet topological spaces on A, measurable spaces on A, algebraic structures on A etc. In this way if we choose $A \in \mathbf{Set}_0$, then $V(A)$ contains all possible structures defined on A in finite time. Varying $A \in \mathbf{Set}_0$ and utilizing the functorial properties of the covariant and contravariant powerset functors and the fact that set-theory does not include a replacement axiom, we can iterate the powerset functor to obtain any category at certain level of the hierarchy n. This manipulation with the hierarchy levels can frieze any variable set, and present it as a constant set of ordered pairs. Thus although sets are sheaves over a singleton, and therefore, are constant sets, from this point of view, by utilising the levels of hierarchy set-theory can code any variable set as a constant set of ordered pairs, residing some levels above the level that dom and cod reside.

Superstructures are very common in Robinsonian nonstandard analysis; see [24, §4.4. Nonstandard Universes]. See also [78, 9, 49, 46, 88] for different approaches to nonstadard mathematics. The possibility is suggesting itself: One may consider instead of structures and structures preserving maps, superstructures and superstructure preserving maps. In fact a superstructure is defined as follows; see, [24].

Given a set A, the *superstructure over* A, denoted by $V(A)$, is obtained from A by taking the power set countably many times. For each $n \in \mathbb{N}$, we define the set $V_n(A)$ recursively by

$$V_0 \equiv V_0(A) := A, \ldots, V_{n+1} \equiv V_{n+1}(A) := V_n \cup \mathscr{P}(V_n), \ldots$$

Also we define the sets of the hierarchy as follows:

$$\mathscr{S} := V(A) - A.$$

Finally the superstrucutre on A is defined as:

$$V \equiv V(A) := \bigcup_{n=0}^{\infty} V_n,$$

and is denoted as a structure by $\langle V(A), =_A, \in_A \rangle$. This structure incorporates all that can be said about A in finite time. We consider A as a set of individuals or urelements, so that for every $a \in A$, $x \neq \emptyset$, but $a \cap V(X) = \emptyset$. Another description closer to algebraic set-theory is that $V(A)$ is the smallest transitive set that contains A, and is closed under binary unions and power sets $\mathscr{P}(A)$.

Suppose that we want to consider instead of just structures, superstructures, $\langle V(A), =_A, \in_A \rangle$, and as morphism those functions $h : V(A) \to V(B)$ that preserve the structure, i.e. $x =_A y \Rightarrow h(x) =_B h(y)$ and $x \in_A y \Rightarrow h(x) \in_B h(y)$.

Lemma [24, p. 264] *For each natural number n,*

$$V_{n+1}(A) \quad = \quad A \cup \mathscr{P}(V_n(A)),$$

and

$$V_{n+1}(A) \setminus A \quad = \quad \mathscr{P}(V_n(A)).$$

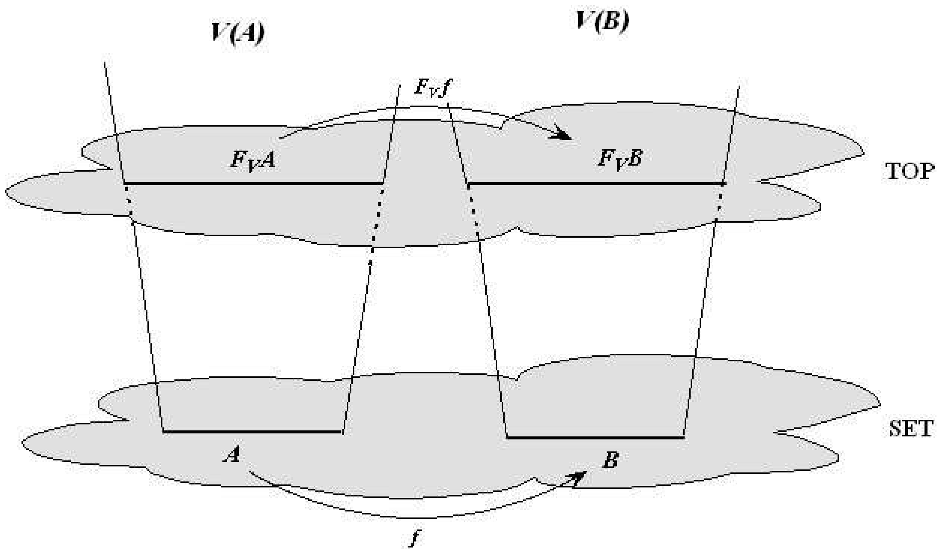
A topology on A is a subset of $\mathscr{P}(A)$, hence a subset of $V_1(A)$, and so it belongs to $V_2(A)$. Thus, a topological space $\langle A, \tau_A \rangle : \{\{A\}, \{A, \tau_A\}\}$, and all topological spaces on A belong to $V_3(A)$. Notice also that the contravariant power set functor $\mathscr{P}^{-1} : \mathbf{Set} \to \mathbf{Set}$ is both representable and adjoint, whereas the covariant set functor is neither. Finally each function $f : A \to B$ induces a function

$$\mathscr{P}^3(f) : \mathscr{P}^3(A) \to \mathscr{P}^3(B),$$

which is continuous since,

$$(\mathscr{P}^{-1})^3(f) : (\mathscr{P}^{-1})^3(B) \to (\mathscr{P}^{-1})^3(A).$$

Then taking as a base the category **Set** and by building the superstructure $V(A)$ over A, for each $A \in \mathbf{Set}_0$ we have a picture as the following one:



As a conclusion, using the classical superstructures or superstructures in the sense of Algebraic Set Theory, we may combine the two ways of structuring an abstract set.

For a different view for the future of set theory see [85]. Other references related to set theory are [10, 12, 25, 32, 77, 89].

4 Structuralism

The problem of category theory and structuralism have raised extensive discussions; see [6, 7, 39, 40, 41, 73, 74]. Furthermore, there are interesting review articles about structuralism; see [40, 62, 79] as well as [80].

Here we shall confine ourselves here to frame some of the relevant issues into the perspective of this paper.

There are several meanings to the term 'structuralism' ranging from philosophical, mathematical, psychological (gestalt), to French structuralism which belongs to social and human sciences proper. We would like to see a kind of unification in the meaning of this term. Following our dialectic scheme A-E vs. H-S, we may classify any version of structuralism into two basic types, which are induced by the above dialectics, and in this way, one may obtain the said unification.

In our view, another common misunderstanding, concerning structuralism is the point that in structuralism the specific nature of the structure is not involved. This view is implied by the universality in the definitions structures. We believe that any structure has two sides: *the external geometrical form,* and *the internal structural constitution.* Universal properties determine the geometrical external form, whereas the internal constitution is approached using concepts like 'levels of reality' and non-standard methods; see [31]. An example of this is the notion of 'group'. In various frameworks, become, group, topological group, Lie group, etc. However the internal identity and its internal constitution remains the same, i.e. in all cases we have a 'group'.

In this section, we shall confine ourselves to two types of structuralisms: set-theoretical and categorical. We suppose that all other types are ramifications of these two. We would also like to characterize structuralism according to the philosophical debate of the *ante rem* versus *in re*. This debate concerns structuralism as well as universals. The problem is *whether structures exist independently of the systems* that are structured (*ante rem*), or whether they are inherent in the them (*in re*). The 'eternal', 'static' and unchanged objects of set-theory go with abstract Platonic universals, whereas the 'variable' objects of category theory are in harmony with concrete universals.

The thesis of this paper is that the above debate constitutes a dialectical scheme, and as such there should be a unity and identity of opposites. However presently we feel that the categorical in re structuralism surpasses all other views.

Hellman describes several types of structuralism; see [40]. For Set-theoretical structuralism (STS) he states:

> Although, within mathematics, structuralism has been conceived informally, as in the influential work of Bourbaki, nowadays no doubt mainstream mathematicians would appeal to set-theory, or to model theory as a part of set-theory, for the definitive, precise notions of 'structure' and 'mappings among structures'. Indeed, it is not just for the sake of punning that one may speak of model theory as a model inspiring the other versions of structuralism. Model theory works so well from a purely mathematical point of view that one may reasonably ask why philosophers of mathematics would even bother investigating any alternatives. Some of the motivation emerges when we look at certain central features of STS, standardly conceived.

We can contrast this with the opinion expressed above by Macintyre; (see page 15), where a convergence of model theory to algebraic geometry and to categorical methods is described.

Set-theoretical structuralism uses analytical and reductive methods, and reduces every structure to membership relation. On the contrary categorical structures are defined using universal properties.

Let us see some views about strucutralism; e.g. in [75] we find:

> I begin with a brief, hence necessarily caricatured, summary of Resnik's influential view. According to structuralism, the subject matter of a mathematical theory is a given *pattern*, or *structure*, and the objects of the theory are intrinsically unstructured points, or *positions*, within that pattern. Mathematical objects thus have no identity, and no intrinsic features, outside of the patterns in which they occur. Hence, they cannot be given in isolation but only in their role within an antecedently given pat- tern, and are distinguishable from one another only in virtue of the relations they bear to one another in the pattern (see, e.g., Resnik (1981)).
>
> Grosholz's judgment on Resnik's structuralism is rather severe: "Neither points or positions, nor structures, seem to be the kind of thing about which one could pose an interesting mathematical problem." The reason for this, in the case of points, is simply that they have no intrinsic features; there simply is nothing to say about them per se. But since there is nothing to say about points, she continues, then because there is nothing more to structures than relations between points, there won't be anything interesting to say about structures,

since these are relations among entities about which there is nothing to say.

For criticism of the structuralism of Resnik and Shapiro, see also [62].

If in the structuralism of Resnik and Shapiro, one translates the term 'pattern' as meaning 'graph' then there is a way of seeing these versions as belonging to either set-theoretic (ante-rem) or to categorical structuralism (in re). In fact the underline graph of any category, may be seen as a 'type', but this do not contribute substantially to the development of category theory.

In conclusion, there are two basic types of structuralism, set-theoretic (model theory) and categorical. According to Macintyre [63], model theory converges to algebraic geometry and category theory. For the 21st century categorical structuralism will be the main type.

5 Category Theory as an Application Tool

The applications of category theory to various scientific branches, starts from mathematics itself (algebraic topology, algebraic geometry), and passes over to theoretical computer science, music [71], quantum mechanics, biology and psychology. Applications in the last two fields are at the early stages. For psychology we have, the Macnamara's project: The mathematization of the relationship between perception and cognition by means of functors, between *the category of gestalts* (the domain of perception) and *the category of kinds* (the domain of cognition); see [66, 65], leading to something like:

category theory is to psychology as calculus is to physics

See also [45], for a sheaf theoretic and categorical formulation of consciousness, and [38] for applications to neural networks, knowledge and cognition.

As for quantum physics, the situation is described by the following passage: "To date, one could not say that topos theory has taken the world of physics by storm, although Isham and Butterfield [43] have found uses for it in quantum mechanics, and a search of the Web shows that there are some physicists in Omsk who are using precisely the toposes appropriate for synthetic differential geometry to reformulate general relativity." see also [1].

For biology, there are very interesting applications by the theoretical biologist Robert Rosen (e.g., Life Iteself). Rosen was able to distinguish nonmechanical 'anticipatory' systems from mechanical ones; see also [22].

Category Theory my find applications to teaching theory. If it is true that some day category Theory might be the basic tools in psychology, one might claim that it would also be the basic tool in a 'Theory of Teaching'. If one can express categorically, e.g. the conceptual system which refers to a particular body of knowledge, then we may assume the the teacher's and students conceptual systems constitute categories and the correct teaching could be an appropriate functor, which translates one conceptual system to the other.

In music theory, Mazzola [71] has applied algebraic geometry and topos theory to express the music theory. This work should have tremendous implications for the foundations of mathematics. We all know that in music there is indistinguishability and vagueness. We hear a whole interval of frequencies as just a note. It should be similar if instead of the point real line we have a set of reals, where each real coincide with an interval. It is exactly this kind of indistinguishability and vagueness that needs algebraic geometry and topos theory to treat this non-Cantorian situation. It would be very interesting to apply the same methods to develop non-Cantorian mathematics, i.e. mathematics where the AC does not hold.

Finally, the applications of category theory to theoretical computer science is really taken the subject by storm. Due to lack of space and the unsuitability of the present author, it is difficult even to summarize the field. For applications in artificial intelligence see [5] and references therein.

In conclusion, taking all the above into account, we may suggest that Category and Topos theory as well as Algebraic Topology and Algebraic Geometry might constitute the core tools for the 'applied mathematics' in the 21st century.

should this happens, then, categories and Topoi would be the basic, common methods among scientists and mathematicians, and the unification of mathematics would be a reality.

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