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Abstracts of Contributed Papers

Abstract

Amalgamation in Basic algebra and some of its natural extensions

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I will talk about the Lindenbaum algebra of Basic Propositional Calculus, **BPC**, and some of its natural extensions. The associated Lindenbaum algebra of **BPC** is called Basic algebra, **Ba** (see [1], [2] and [3]. In [4], it is called **V**-algebra). A **Ba** is a distributive lattice with top, 1, bottom, 0, and a binary operation "$\to$" with the following equalities and inequalities:

$$\begin{gathered} a \to a = 1, \\ a \leq 1 \to a, \\ (a \to b) \wedge (b \to c) \leq a \to c, \\ a \to b \wedge c = (a \to b) \wedge (a \to c), \\ b \vee c \to a = (b \to a) \wedge (c \to a). \end{gathered}$$

Every Heyting algebra, **Ha** is a **Ba**, but a **Ba** is a **Ha** if and only if $1 \to a = a$, for all a. The relation "$\leq$" in **Ba**'s is defined by $a \leq b$ iff $a \wedge b = a$ (or iff $a \vee b = b$). This relation "impose" a *restricted* implication on any **Ha** by

$$a \wedge b \leq c \ \text{ iff } \ a \leq b \to c.$$

In **Ba**, we have only one way, i.e.,

$$\text{if } \ a \wedge b \leq c \ \text{ then } \ a \leq b \to c.$$

So contrary to **Ha**'s, where $a \leq b$ iff $a \to b = 1$, in **Ba**'s we have one way, if $a \leq b$, then $a \to b = 1$.

We survey some results in our studying of **Ba**, and two of its natural extensions: *faithful* and *linear* **Ba**'s. Among other things, we show that the class of faithful basic algebras and linear basic algebras have the *amalgamation* property.

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Intensional semantics of fuzzy logics

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Classical intensional semantics plays an important rôle both in logical analysis of natural language and modal logic. Based on the notions of logical space and possible world, it provides sound and complete semantics of classical propositional or predicate calculus and can serve as the basis for all kinds of modal logics (alethic, epistemic, temporal, dynamic, erotetic, deontic, etc.).

In classical intensional semantics, the semantic value of a formula is a subset (called a *proposition*) of some basic set (called a *logical space*). In fuzzy logic, we allow propositions to be *fuzzy* subsets of a logical space (which can itself be a fuzzy set). The notions of validity, tautologicity and entailment are defined analogically to the classical case and their properties are studied. In order to be able to prove formally the laws governing fuzzy entailment and the properties of fuzzy propositions, the apparatus of fuzzy intensional semantics is formalized in the framework of fuzzy class theory FCT developed in [3]. (FCT is a first-order theory over any of a certain class of residuated fuzzy logics, capturing formally the notion of fuzzy set of individuals.) Within FCT, the completeness theorem for intensional fuzzy semantics and several properties of the notion of entailment (e.g., transitivity under certain conditions) are proved, both for propositional and predicate fuzzy logic.

A wide class of residuated fuzzy logics (including schematic extensions of $\mathrm{BL}\Delta$ and logics definable in $\text{Ł}\Pi\frac{1}{2}$, see [1], [2]) is thus equipped with formal intensional semantics, which makes a prospect of further developments of various kinds of modal fuzzy logics.

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Weakly Residuated Lattices

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Abstract: We modify the concept of adjoitness property and of residuated lattice to obtain a weak adjoitness and a weakly residuated lattice. A weakly residuated lattice need not be distributive, an axample of a weakly residuated lattice which is not residuated is e.g. the five-element non-modular lattice (the pentagon). Our results:

1. If L is a weakly residuated lattice then L is sectionally pseudocomplemented.

2. If L is an algebnraic lattice then L is weakly residuated if and only if L is meet-semidistributive

Universal fuzzy logic

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ABSTRACT

The mathematical fuzzy logic has been widely developed in the last decade. Many logics were defined and their properties were examined. Also natural many-valued interpretation was introduced for many already-known logics. The goal of this paper is to systematically study these logic from an abstract point of view.

As a starting point we examine Hájek's concept of a "logic of comparative degrees of truth". Using this concept we introduce the minimal fuzzy logic and then develop a theory of its extensions. So, the universal fuzzy logic is a theory of extensions of this minimal fuzzy logic. Following Hájek's approach, the central logical concepts of our theory are Hilbert style calculi and algebraic semantics. We develop our theory for both propositional and predicate logics.

Our theory is connected to the Abstract algebraic logic, especially in the propositional part. However, we have a different starting point and we deal with a very special (narrow) class of logics (from an AAL point of view) and we classify logics in quite different way using concepts from the fuzzy logics.

The variety of residuated lattices and its subquasi-varieties are important algebraic structures. The connection between these quasi-varieties and *propositional* logics is well known (using AAL terminology we would say that they are *algebraic semantics* for these logics). As a consequence of our theory we make a first step in establishing connection between these quasi-varieties and *predicate* logics. We present an uniform way of defining first-order calculus over some quasi-variety and prove its completeness, Skolem terms introduction, and some other logical properties.

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On Rational Gödel and Nilpotent Minimum Logics

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Abstract

As Pavelka pointed out in [9], it seems natural to introduce truth values in the language in order to be able to deal with *partial truth*. With this aim, he built a many-valued logical system over Lukasiewicz logic by adding into the language as many truth constants as truth values, i.e. a truth constant $\overline{r}$ for each real $r \in [0,1]$, and a number of additional axioms. Although this Lukasiewicz logic extended with truth-constants, PL, is not strong complete (like Lukasiewicz logic), Pavelka proved that it is complete in a weaker sense. Indeed, he introduced a weaker notion of strong completeness based on the degrees of provability and truth of a formula φ in an arbitrary theory T. The truth degree of φ in T is defined as

$$\| \varphi \|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of φ in T as

$$\mid \varphi \mid_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\}.$$

Pavelka proved that these degrees coincide. This kind of completeness, which strongly relies in the continuity of Lukasiewicz logic truth functions, is usually known as Pavelka-style completeness. Moreover he also proved that Pavelka-style completeness is preserved if and only if the language is extended with any connective whose corresponding truth-function on the real unit interval is a continuous (real) function.

Later, Hájek [8] proved that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Namely, Hájek's system is an extension of Lukasiewicz logic by only a countable number of truth-constants, $\overline{r}$ for each *rational* $r \in [0,1]$, and by two additional axiom schemata to deal with the truth-constants, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are the t-norm of Lukasiewicz and its residuum respectively. He denoted this new system as RPL, for Rational Pavelka Logic, and proved the same results that Pavelka proved for his system with continuously many truth-constants. Moreover, in [8] it is proved that RPL is strong complete for finite theories. Remark that the semantics of RPL is kept on the *real* unit interval [0, 1].

Similar *rational* extensions for other popular fuzzy logics can be obviously defined, but Pavelka-style completeness cannot be obtained since Lukasiewicz is the only fuzzy logic with continuous truth-functions in the real unit interval [0,

1]. For instance, in [8] Hájek defines an extension of G_Δ, the extension of Gödel logic with Baaz's Delta operator, with a finite number of rational truth-constants. Later, in [3] the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel logic with an involutive negation) and to Π (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation). For the first system, $RGL_\sim$, usual strong completeness is proved for finite theories, while for the second systems, $R\Pi L$ and $R\Pi L_\sim$, it is possible to prove Pavelka-style completeness provided an infinitary inference rule is added to overcome the problem that the residuum of the product t-norm is not continuous at the point $(0,0)$. Finally also notice that in [1] standard completeness of Gödel logic with rational truth-constants is stated. Although the result holds true (see Section 3), the proof given there is not correct.

In this paper we will introduce the extensions of Gödel and Nilpotent Minimum logics (See [2] for a description of Nilpotent minimum Logic NM) by adding rational truth-values as truth constants in the language and by adding corresponding book-keeping axioms for the truth-constants. Weak and strong standard completeness of these logics are studied in general and when we restrict ourselves to formulas of the kind $\overline{r} \to \varphi$, where r is a rational in [0, 1] and φ is a formula without rational truth-constants which are the type of formulas used in most of the fuzzy systems.

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Notes on special Hilbert algebras

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Several authors have accepted as evident that the class of Hilbert algebras in which each pair of elements has infimum, which initially in 1963 H. Porta called Hertz algebras, are the same as the class of implicative semilattices or Brouwerian semilattices. But this assertion is false as it is known.

In this communication, firstly we describe how Hertz algebras arose and we also analyse the conclusions obtained by J. Cirulis in relation to some results indicated by D. Busneag and M. Kondo.

Next, we define and investigate a special class of Hilbert algebras with infimum which is a subclass of order algebras introduced by I. Chajda and R. Halaš in 2002 and which we named order algebras with infimum. Among others results we determined the congruences and the subdirectly irreducible algebras.

The main references we based this communication on are cited below for further corroboration.

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On the variety of Ockham–Nelson algebras

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We iniciate an investigation into an equational class of algebras which we called Ockham–Nelson algebras. An Ockham–Nelson algebra is an algebra $\langle L, \wedge, \vee, \rightarrow, f, 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ where the reduct $\langle L, \wedge, \vee, f, 0, 1\rangle$ is an Ockham algebra and $\rightarrow$ fulfills the following identities:

(B1) $x \rightarrow x = 1$,

(B2) $(x \rightarrow y) \wedge (f(x) \vee y) = f(x) \vee y$,

(B3) $x \wedge (x \rightarrow y) = x \wedge (f(x) \vee y)$,

(B4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,

(B5) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$.

This variety is an extension of both Ockham algebras introduced by J. Berman in 1977 and generalized $N-$ lattices introduced by A. V. Figallo in 1990. Our main result is the duality theory for these algebras which extends that obtained by A. Urquhuart for Ockham algebras. This duality enables us to determine the lattice congruences for these algebras.

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Non-associative substructural logic: algebraization, cut elimination and separation.

Nikolaos Galatos, joint work with Hiroakira Ono.

Full Lambek calculus (**FL**), is a prominent example of a substructural logic. The structural rules of exchange, weakening and contraction are not assumed, but associativity is stipulated. **FL** enjoys the cut elimination property and is algebraizable in the sence of Blok and Pigozzi. Its algebraic semantics, the variety $\mathcal{RL}$ of residutated lattices, has recently received a lot of attention. In particular, the decidability of $\mathcal{RL}$ (Jipsen and Tsinakis) and the cut elimination property for **FL** and related systems (Bernadinelli, Jipsen and Ono) are proven in an algebraic way; a Hilbert system that has the separation propery is shown to have the variety of commutative residuated lattices as equivalent algebraic semantics and an explicit axiomatization of the classes of subreducts is given (Raftery and van Alten); and an embedding of a partial subalgebra of a residuated lattice into new a residutated lattice is constructed that in certain cases preserves finiteness, thus yielding the FEP (Blok and van Alten).

The underlying idea behind the constructions in these results, as well as in various other papers, is common. We provide a general context that includes the above results and illuminates the main idea; our study is further extended to the non-associative case. In particular, we present a non-associative version **GL** of the Gentzen system **FL**, we show the cut elimination property for **GL** in an algebraic way, we provide an equivalent Hilbert system that has the separation property and is algebraizable, we show that the equivalent algebraic semantics is given by the variety of residuated ℓ-groupoids with unit, we give explicit axiomatizations for the classes of subreducts, we characterize the congruence blocks of the unit and their negative cones and we provide an embedding of a partial subalgebra into a residuated ℓ-groupoids with unit that preserves finiteness in certain cases.

Some Axiomatic Extensions of the Involutive Monoidal T-norm Logic

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Abstract

The Involutive Monoidal t-norm Logic (IMTL for short) is obtained from the Monoidal T-norm Logic (MTL for short) by adding the involutive condition $(\neg\neg\varphi \to \varphi)$. IMTL can also be seen as a weaker logic of Łukasiewicz infinite valued calculus, failing the satisfy the divisibility condition of the strong conjunction, the same way as MTL can be obtained from Hajek's Basic Fuzzy Logic by dropping this divisibility condition.

In this communication we deal with two specific IMTL -logics: Nilpotent Minimum Logic (NML for short) and IMTL3, and their associated algebras. Nilpotent Minimum t-norms (NM t-norms) were introduced by Fodor in order to give examples of left continuous t-norms which are not continuous and NML is defined to be complete with respect NM t-norms. IMTL3 is the logic obtained from IMTL by adding the axiom $\sigma_3 = \neg(\varphi\&\varphi\&\varphi) \vee \varphi$.

After studying their associated algebras, we obtain a characterization, classification and axiomatization of all axiomatic extensions of NM and all axiomatic extensions of IMTL3. Although both logics are in many ways different their lattices of axiomatic extensions are isomorphic and have the same invariant conditions. Every axiomatic extension is complete with respect to a class of NM(IMTL3)-chains and given a family of NM(IMTL3)-chains the number of elements of the largest odd finite subalgebra in the family and the number of elements of the largest even finite subalgebra in the family turns out to be a complete classifier.

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Representations of monadic MV-algebras

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Abstract

Representations of monadic MV-algebra, with internal characteristic of them, is given. The characterization of locally finite monadic MV-algebras, with axiomatization of them, is also given. The characterization of finitely generated projective MV-algebras is given.

1 Introduction

The finitely-valued propositional calculi, which have been described by J. Lukasiewicz and A. Tarski in [6], are extended to the corresponding predicate calculi. The predicate Lukasiewicz (infinitely-valued) logic $Q\mathcal{L}$ is defined in the following standard way. For some universe and some complete MV-algebra (or Chang algebra by Rutledge's terminology), which in most cases is a chain - particularly it is the real unit interval, it is defined existential (universal) quantifier as supremum (infimum). Then the predicate calculus is defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by J. D. Rutledge in [7]. B. Scarpellini in [9] has proved that the set of valid formulas is not recursively

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enumerable. We also refer a reader to the papers [10], [11] and [4] concerning to the Lukasiewicz predicate calculus.

Monadic MV-algebras (monadic Chang algebras by Rutledge's terminology) were introduced and studied by J. D. Rutledge in [7] as an algebraic model for the (monadic) predicate calculus $q\mathcal{L}$ of Lukasiewicz infinite-valued logic, in which only a single individual variable occurs. J. D. Rutledge followed P. R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [7], showing the completeness of the monadic predicate calculus, has been a great interest.

Adapting for the propositional case the axiomatization of monadic MV-algebras given by J. D. Rutledge in [7], we can define modal Lukasiewicz propositional calculus $MLPC$ as a logic which contains Lukasiewicz propositional calculus Luk, the formulas as the axioms schemes

$M1.\ \alpha \to \exists\alpha$
$M2.\ \exists(\alpha \vee \beta) \equiv \exists\alpha \vee \exists\beta$
$M3.\ \exists(\neg\exists\alpha) \equiv \neg\exists\alpha$
$M4.\ \exists(\exists\alpha + \exists\beta) \equiv \exists\alpha + \exists\beta$
$M5.\ \exists(\alpha + \alpha) \equiv \exists\alpha + \exists\alpha$
$M6.\ \exists(\alpha \cdot \alpha) \equiv \exists\alpha \cdot \exists\alpha$

and closed under modus ponens and necessitation ($\alpha/\forall\alpha$, where $\forall\alpha = \neg\exists\neg\alpha$).

Let L denote a first-order language based on $\cdot, +, \to, \neg, \exists$ and L_m denotes monadic propositional language based on $\cdot, +, \to, \neg, \exists$, and $Form(L)$ and $Form(L_m)$ - the set of all formulas of L and L_m respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $F_p(x)$ in L and define by induction a translation $\Psi : Form(L_m) \to Form(L)$ by putting

- $\Psi(p) = F_p(x)$ if p is propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$,
- $\Psi(\exists\alpha) = \exists x \Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x. $\Psi(MLPC) \subseteq Q\mathcal{L}$.

2 Preliminaries and basic facts

The definition and investigation of monadic MV-algebras (monadic Chang algebras in other terminology) is given by J. D. Rutledge in [7]. MV-algebras were introduced by C. C. Chang in [1] as algebraic models for infinitely-valued Lukasiewicz logic.

An algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ is said to be *monadic MV-algebra* (for short *MMV-algebra*) if $(A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities :

$E1.$ $x \leq \exists x$
$E2.$ $\exists(x \vee y) = \exists x \vee \exists y$
$E3.$ $\exists(\exists x)^* = (\exists x)^*$
$E4.$ $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$
$E5.$ $\exists(x \odot x) = \exists x \odot \exists x$
$E6.$ $\exists(x \oplus x) = \exists x \oplus \exists x$

This definition belongs to J. D. Rutledge (see [7]). Sometimes we shall denote a monadic MV-algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity.

We can define a unary operation $\forall x = (\exists x^*)^*$ corresponding to the universal quantifier. Then in any monadic MV-algebra hold the identities which are dual to $E1 - E6$:

$A1.$ $x \geq \forall x$
$A2.$ $\forall(x \wedge y) = \forall x \wedge \forall y$
$A3.$ $\forall(\forall x)^* = (\forall x)^*$
$A4.$ $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$
$A5.$ $\forall(x \odot x) = \forall x \odot \forall x$
$A6.$ $\forall(x \oplus x) = \forall x \oplus \forall x$

Let $\exists A = \{x \in A : x = \exists x\}$. Then it holds:

Definition 2.1. *[3]A subalgebra A_0 of an MV-algebra A is said to be relatively complete (or rc-subalgebra) in A if, for every $a \in A$, the set $\{b \in A_0 : a \leq b\}$ has the least element, which is denoted by*

$$inf\{b \in A_0 : a \leq b\} \quad or \quad \bigwedge_{a \leq b \in A_0} b.$$

In order to characterize monadic operators on MV-algebras, we give the following:

Definition 2.2. *(see[3])A subalgebra A_0 of an MV-algebra A is said to be m-relatively complete (or mrc-subalgebra) in A, if A_0 is relatively complete in A and the following additional conditions hold:*

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \odot a \Rightarrow v \geq a \ \& \ v \odot v \leq x)$,

(##) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \ \& \ v \oplus v \leq x)$.

The subalgebra $(\{0,1\}, \oplus, \odot, ^*, 0, 1)$ is not m-relatively complete. Indeed, let $a = 1/2$ and $x = 0$. Then $x \geq a \odot a$. But the only element $v \in \{0,1\}$ such that $v \geq a$ is 1, and $v \odot v \nleq x$.

Proposition 2.3. *[3] Let $(A, \oplus, \odot, ^*, \exists, 0, 1)$ be a monadic MV-algebra. Then the MV-subalgebra $\exists A$ of MV-algebra $(A, \oplus, \odot, ^*, 0, 1)$ is m-relatively complete.*

Definition 2.4. *Let A_0 and A be arbitrary MV-algebras and $h : A_0 \to A$ a function. The function $\exists_h : A \to A_0$ is called left adjoint to h, if $\exists_h(b) \leq a \Leftrightarrow b \leq h(a)$ for any $a \in A_0$ and $b \in A$. If in addition $\exists_h(x \odot x) = \exists_h(x) \odot \exists_h(x)$, $\exists_h(x \oplus x) = \exists_h(x) \oplus \exists_h(x)$, then $\exists_h$ is called left m-adjoint.*

Proposition 2.5. *[3] There exist one-to-one correspondences between :*
1)the class of monadic MV-algebras $(A, \exists)$;
2)the class of the pairs (A, A_0), where A_0 is m-relatively complete subalgebra of A;
3)the class of the pairs (A, A_0), where A_0 is a subalgebra of A and the canonical embedding $h : A_0 \hookrightarrow A$ has left m-adjoint function.

Denote by $\mathbf{MV^2}$ the category whose objects are pairs (A, A_0) of MV-algebras, where every injective MV-algebra homomorphism $h : A_0 \hookrightarrow A$ has left m-adjoint function $\exists_h$, and whose morphisms are pairs of functions $(f, f_0) : (A, A_0) \to (A', A'_0)$ such that the following conditions are satisfied :

(1) $f : A \to A'$ is MV-algebra homomorphism and $f_0 : A_0 \to A'_0$ is a function.

(2) For every injective homomorphism $h : A_0 \hookrightarrow A$, there exists an injective homomorphism $h' : A'_0 \hookrightarrow A'$ such that $f \circ h = h' \circ f_0$.

(3) $f_0 \circ \exists_h = \exists_{h'} \circ f$, where $\exists_h$ and $\exists_{h'}$ are the left m-adjoint function to h and h' respectively.

(1) and (2) imply that f_0 is also an MV-algebra homomorphism.

Proposition 2.6. *[3] The category* **MMV** *is equivalent to the category* $\mathbf{MV^2}$.

3 Representations of monadic MV-algebras

In this section we investigate the properties of monadic MV-algebras. In particular, we characterize congruences of a given monadic MV-algebra and prove that **MMV** is *congruence distributive* and has *congruence extension property.* We characterize *subdirectly irreducible* monadic MV-algebras and prove a monadic analogous of Chang's representation theorem for MV-algebras. We also shall give conditions thanks to that it is possible to define monadic operator on an MV-algebra.

Definition 3.1. *(see [7]) An ideal M of an algebra $(A, \exists) \in$* **MMV** *is called monadic ideal , if M is an ideal of MV-algebra A and for every $a \in A$ we have $a \in M \Rightarrow \exists a \in M$.*

Theorem 3.2. *(a) There exists a lattice isomorphism between the lattice of congruence relations of $(A, \exists)$ and the lattice of all congruence relations of $A_0(= \exists A)$.*
*(b) The variety **MMV** is congruence distributive.*
*(c) The variety **MMV** has the congruence extension property.*

Proposition 3.3. *[7]. Any monadic MV-algebra $(A, \exists)$ is isomorphic to a subdirect product of monadic MV-algebras $(A_i, \exists_i)$ such that $\exists_i A_i$ is totally ordered.*

Theorem 3.4. *[3] If $(A, \exists)$ is a finite monadic MV-algebra with totally ordered $\exists A$, then MV-algebra A is isomorphic to a product of totally ordered MV-algebras A_i, $i \in I$, such that $A_i \cong \exists A$ and $\exists A$ is isomorphic to the diagonal subalgebra of the product.*

Theorem 3.5. *Let $(A, \exists)$ be an MMV-algebra and $A_0(= \exists A)$ a complete MMV-algebra. Then, for every subset $\{a_i, i \in I\}$ of A_0, $\exists(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \exists a_i$.*

The question now arises that if A is a complete MMV-algebra, is A_0 complete? The positive answer is given in the following

Theorem 3.6. *If MMV-algebra A is complete, then $A_0(= \exists A)$ is complete.*

Now we give some constructions of monadic operators.

Theorem 3.7. *Let L be a linearly ordered MV-algebra. Let X be a finite set. Let D be the algebra of constant functions in L^X. If $A \subseteq L^X$ is a subalgebra such that $D \subseteq A$, then there is an operator $\exists$ on A such that $(A,\ \exists)$ is an MMV-algebra with $D = \exists A \cong L$.*

Corollary 3.8. *There exists an MMV-algebra $(A,\ \exists)$ such that A is a perfect MV-algebra and $A_0 = \exists A$ is linearly ordered.*

4 A General Construction

Let S be any MV-algebra, I a non-empty set. Let F be the ideal in S^I defined by $F = \{< a_i > \mid \text{support of } < a_i > \text{is finite}\}$ and D the set of all constant functions of S^I. Let $A = [F \cup D]$ be the subalgebra generated by F and D. It returns that A is the set of all the functions, which are constant up to a finite set. Define on A, $\exists < a_i >=< c_i >$ where all $c_i = max\{a_i | i \in I\}$. Proceeding as in the Theorem 3.7, we obtain an MMV-algebra A with $A_0 = S$.

Notice that in the construction above that S becomes relatively complete even though it need not be complete.

Observe also that if I is infinite, and S finite, we have an example of an MMV-algebra $(A,\ \exists)$ with A infinite and $\exists A$ finite.

We recall that an involution on a non-empty set X is a function $\eta : X \to X$ such that $\eta(\eta(x)) = x$ for every $x \in X$.

Theorem 4.1. *Let X be a non-empty set with an involution η, L be any MV-algebra and $A = L^X$. Then the following map $\exists : A \to A$ for $f \in A$, defined by $(\exists f)(x) = f(x) \vee f(\eta(x))$, for every $x \in X$, is a monadic operator on A.*

It is easy to see that in the above construction $A_0 = \exists A = \{h \in A |\ h(x) = h(\eta(x)),\ \text{for all } x \in A\}$.

Example. Let L be any MV-algebra, $X = \{0,\ 1,\ 2, \ldots\}$. Define on X, $\eta : X \to X$ as follows: $\eta(2x + 1) = 2x + 2;\ \eta(2x + 2) = 2x + 1;\ \eta(0) = 0$. Then η is an involution on X, and $A = L^X$ is an MMV-algebra.

Using this method we can in fact construct infinitely many MMV-algebras on L^X. As above, let X be an infinite set. Let $N_1,\ N_2$ be two disjoint infinite equipotent subsets of X with $N_1 \cup N_2 = X$. Let $\alpha : N_1 \to N_2$ be a one-one onto map. Let $\eta : X \to X$ be defined by: if $x \in N_1$, $\eta(x) = \alpha(x)$; if $x \in N_2$ set $\eta(x) = \alpha^{-1}(x)$. Then η is an involution on X and we have an MMV-algebra defined in the above manner.

There's no reason in the above construction that requires X to be infinite. For suppose X has $2k$ elements. Again divide X into two disjoint subsets of the same size, and take α to be a one-one onto map between the subsets. If X has $2k+1$ elements, write $X = X' \cup \{x_0\},\ x_0 \notin X'$. Divide X' into two equal-sizes disjoint subsets $X_1',\ X_2'$ and let α be a one-one, onto map from X_1' to X_2'. Extend α to all of X by: $\alpha(x_0) = x_0$. Again define $\eta : X \to X$ by $\eta(x) = \alpha(x)$ if $x \in X_1'$; $\eta(x) = \alpha^{-1}(x)$ is $x \in X_2'$, and set $\eta(x_0) = x_0$. Then η is an involution on X and an MMV-algebra is defined. This procedure allows us to construct a finite number of different MMV-algebras on L^X.

Theorem 4.2. *Let A be an MV-algebra and $\phi : A \to A$ an isomorphism such that $\phi^2 = 1_A$. Then the map: $a \in A \to \exists a$, defined by $\exists a = a \vee \phi(a)$, for every $a \in X$, is a monadic operator on A.*

We observe that the above theorem generalizes Theorem 4.1. In fact, if $\eta : X \to X$ is an involution, then the mapping $\phi : A \to A$, given by $\phi(f) = f \circ \eta$, satisfies the following conditions:

1) $\phi(f \oplus g) = (f \oplus g) \circ \eta = (f \circ \eta) \oplus (g \circ \eta)$.

2) $\phi(f^*)(x) = (f^* \circ \eta)(x) = f^*(\eta(x)) = (f(\eta(x)))^* = (f \circ \eta)^*(x)$, for every $x \in X$, that is, $\phi(f^*) = (\phi(f))^*$.

Hence we see that ϕ is a homomorphism from $A \to A$. Moreover we have $\phi(\phi(f)) = \phi(f \circ \eta) = f \circ (\eta \circ \eta) = f$. It follows that $\phi^2 = 1_A$.

The monadic operator defined here is $\exists f = f \vee \phi(f)$ which, when evaluated at x, gives $(\exists f)(x) = f(x) \vee (\phi(f))(x) = f(x) \vee f(\eta(x))$.

Theorem 4.3. *Suppose that $A = L^X \times A_1$ where $L,\ A_1$ are arbitrary MV-algebras. If X has more than one element, then A admits a non-trivial monadic operator.*

5 Some representations for MMV-algebras

Let $A_i,\ i \in I$ be MV-algebras, and let $\hat{A} = \Pi_{i\in I} A_i$. Let F_I be the ideal in $\hat{A}$ consisting of all elements with finite support over I.

Lemma 5.1. *Suppose that $(A,\ \exists)$ is an MMV-algebra such that:*
1) A is a subdirect subalgebra of $\hat{A}$,
2) $\pi_i A_0$ is an rc-subalgebra of A_i, where $A_0 = \exists A$,
3) $F_I \subseteq A$.
Then, for each $i \in I$, we have $\pi_i A_0$ is an mrc-subalgebra of A_i.

Corollary 5.2. *Suppose that $(A,\ \exists)$ is as in Lemma 5.1. Then for each $i \in I$, for which A_i is linearly ordered we have $\pi_i A_0 = A_i$.*

Corollary 5.3. *Suppose that $(A,\ \exists)$ is as in Lemma 5.1, with A_0 locally finite. Then for each $i \in I$, for which A_i is linearly ordered, we have $A_0 \cong A_i$.*

Corollary 5.4. *Suppose that $(A,\ \exists)$ is as in Lemma 5.1, with A_0 finite, locally finite, and for each $i \in I$, A_i is linearly ordered. Then $A \subseteq A_0^I$.*

Let A be an MV-algebra. Call A *directly full over I* if for some set $I \subseteq \mathrm{Spec}A$ with $\bigcap I = 0$ we have $F_I \subseteq \hat{A} \subseteq \Pi_{P\in I} A/P$, where $\hat{A} \cong A$.

Restating the above results yields,

Theorem 5.5. *Suppose that $(A,\ \exists)$ is an MMV and A is directly full over I. Suppose also that each $\pi_P A_0$ is an rc subalgebra of $\pi_P A$. Then:*

i) *$\pi_P A_0$ is an mrc-subalgebra of A/P;*

ii) *If A_0 is locally finite then $\pi_P : A_0 \to A/P$ is an epimorphism;*

iii) *If A_0 is finite, locally finite, then $A \subseteq A_0^I$*

Corollary 5.6. *If $(A,\ \exists)$ is a directly full, finite MMV-algebra with A_0 linearly ordered, then $A \cong A_0^I$ for some finite set I.*

Corollary 5.7. *If A is a finite MMV-algebra with A_0 linearly ordered, then $A \cong A_0^I$ for some finite set I.*

Suppose $\pi : (A,\ A_0) \to (A',\ A_0')$ is an epimorphism where $A_0,\ A_0'$ are *mrc*-subalgebras and $\pi A_0 = A_0'$. Let $\exists,\ \ \exists'$ be the respective induced monadic operators. Then for each $x \in A$ we have $\exists' \pi x \leq \pi(\exists x)$.

6 Locally finite varieties of MMV-algebra

Let U be some class of algebras. By $\mathcal{V}(U)$ we denote the variety generated by the class U.

From the variety **MMV** select the subvariety $\mathbf{K_n}$ for $1 \leq n < \omega$, which is defined by the following equation :

$(K_n) \quad x^n = x^{n+1}$,

that is $\mathbf{K_n} = \mathbf{MMV} + (K_n)$. From the variety $\mathbf{K_n}$ we select the subvariety $\mathbf{MMV_n}$ which is defined in the following way:

$\mathbf{MMV_n} = \mathbf{K_n} + (L_n)$,

where $(L_n) = \{n(x^j \oplus (x^* \oplus (x^{j-1})^*) = 1 : \ 1 < j < n \, \text{and} \, j \text{ does not divide} \, n\}$.

Let us note, that if $A \in \mathbf{K_n}$, then $A \in \mathbf{MV_m}$ for some $m \leq n$, where $\mathbf{MV_m}$ is the subvariety of the variety of all MV-algebras studied in [5]. More precisely $\mathbf{MV_m} = \mathbf{MV} + (K_m) + (L_m)$. $\mathbf{MV_m}$ is generated by the linearly ordered simple MV-algebra $S_m = (S_m, \oplus, \odot, *, 0, 1)$, which in turn is the subalgebra of MV-algebra of the real unit interval $[0, 1]$, where $S_m = \{0, 1/m, \ldots, m-1/m, 1\}$ $(0 \neq m \in \omega)$.

Note that we have an increasing sequence of subvarieties : $\mathbf{K_1} \subset \mathbf{K_2} \subset \mathbf{K_3} \subset \ldots \mathbf{MMV}$.

Theorem 6.1.

$$\mathbf{MMV} = \mathcal{V}(\bigcup_{n\in\omega} \mathbf{MMV_n}) = \mathcal{V}(\bigcup_{n\in\omega} \mathbf{K_n}).$$

6.1 Monadic Operators on finite MV-Algebras

In this subsection we characterize all monadic operators over an arbitrary finite MV-algebra. In other words we define all monadic operators which can be exist on a finite MV-algebra.

One of the the characterization immediately follows from Theorem 3.11. Another characterization of the ones is given below.

Let $A = S_{n_1} \times \ldots \times S_{n_k}$ be a finite MV-algebra which is a direct product, k times, of the same finite MV-chain S_n. That means A is the MV-algebra of all function from the set $\mathbb{K} = \{1, \ldots, k\}$ to S_n. Assume that $(A, \exists)$ is a monadic MV-algebra (MMV-algebra). Then over the set $\mathbb{K}$ we can define the following relation: for every $i, j \in \mathbb{K}$

$$i \approx j \Leftrightarrow h(i) = h(j)$$

for every $h \in \exists A$.

Of course $\approx$ is an equivalence over $\mathbb{K}$ and the restriction of h over any equivalence class, $\pi(i)$ is a constant function.

Claim 6.2. *For every $f \in A$ $(\exists f)(j) = sup_{\pi(i)} f(j)$, for every $i \in \mathbb{K}$, is a monadic operator.*

So, summarizing, we get: given a monadic MV-algebra $(A, \exists)$ such that A is a finite power $A = S_n \times \ldots \times S_n$, k-times, of a finite MV-chain, then there exists a partition $\pi = \{\pi(1), \ldots, \pi(k)\}$ of $\mathbb{K}$, such that $(\exists)_{\pi(i)}(f_{\pi(i)})(j) = \bigvee_{\pi(i)}(f_{\pi(i)})(j)$.

And conversely, given $A = S_n \times \ldots \times S_n$ and a partition $\pi = \{\pi(1), \ldots, \pi(k)\}$ of $\mathbb{K}$, the map $(\exists f)(j)$ defined as above makes A a monadic MV-algebra.

Now we generalize the above result to an arbitrary finite MV-algebra.

Let $A = S_{n_1} \times \ldots \times S_{n_k}$ and $\pi = \{\pi(1), \ldots, \pi(k)\}$ be a partition of $\mathbb{K}$. Then we say that π is *homogeneous* iff for every block $\pi(i)$ of π, $S_{n_j} = S_{n_i}$ for every $j \in \pi(i)$.

6.2 Subvarieties of $\mathbf{K}_n$

In this section we will describe all finite subdirectly irreducible monadic MV-algebras from the variety $\mathbf{K}_n$.

Since every finite totally ordered MV-algebra is simple, then any monadic MV-algebra $(A, \exists)$ with totally ordered $\exists A$ is simple and, hence, subdirectly irreducible. Let us denote by $S_n^{(m)}$ the monadic MV-algebra $(A, \exists)$ such that $A = S_n^m$ and $\exists A$ is the diagonal subalgebra of S_n^m, which is isomorphic to S_n. So we arrived to

Theorem 6.3. *The only (up to isomorphism) finite subdirectly irreducible monadic MV-algebras from $\mathbf{K}_n$ are $S_n^{(m)}$, $m, n \in \omega$ and $m, n \geq 1$.*

It is easy to prove the following

Lemma 6.4. **(a)** *$S_n^{(i)}$ is a subalgebra of $S_n^{(j)}$ for $i \leq j$.*

(b) *$S_i^{(m)}$ is a subalgebra of $S_j^{(m)}$ iff S_i is a subalgebra of S_j.*

Let $\mathbf{K}_n^{(m)}$ be the subvariety of $\mathbf{K}_n$ generated by $\{S_1^{(m)}, \ldots, S_n^{(m)}\}$.

As we see $\mathbf{K}_1$ is a variety of all monadic Boolean algebras. It is well known that the lattice of subvarieties of $\mathbf{K}_1$ forms a chain of type $\omega : \mathbf{K}_1^{(1)} \subset \mathbf{K}_1^{(2)} \subset \ldots \mathbf{K}_1$.

A little more complicated situation we have for subvarieties of $\mathbf{K}_n$ for $n \geq 2$.

According to beforehand theorem and lemma it holds

Theorem 6.5. **(a)** *$\mathbf{K}_n^{(1)} \subset \mathbf{K}_n^{(2)} \subset \ldots \mathbf{K}_n$, $2 \leq n \leq \omega$.*

(b) *$\mathbf{K}_n^{(i)}$ is a subvariety of $\mathbf{K}_n^{(j)}$ iff $i \leq j$.*

(c) *$\mathbf{K}_n^{(i)}$ is a subvariety of $\mathbf{K}_m^{(i)}$ iff $n \leq m$.*

Let us consider the variety $\mathbf{K}_1$ of monadic Boolean algebras. Algebras from the variety $\mathbf{K}_1$ are algebraic models of well-known modal system $S5$ which have been studied by many authors. K. Segerberg in [8] have given a formula

$$Alt_m = \forall x_1 \vee \forall(x_1 \to x_2) \vee \cdots \vee \forall(x_1 \wedge x_2 \wedge \cdots \wedge x_m \to x_{m+1})$$

which holds in subdirectly irreducible closure algebra iff it contains no more than 2^m elements. We adapt the formula for our case for axiomatization of subvarieties of the variety $\mathbf{K}_n$ for $n \geq 2$. Taking into account that $S_1^{(k)}$ is a subalgebra of $S_n^{(k)}$ for every $k \geq 2$, the formula

$$Alt_m^n = \forall x_1^n \vee \forall(x_1^n \to x_2^n) \vee \ldots \forall(x_1^n \wedge x_2^n \wedge \cdots \wedge x_m^n \to x_{m+1}^n)$$

axiomatizes the variety $\mathbf{K}_n^{(m)}$ for $m \geq 2$ inside of the variety $\mathbf{K}_n$, i.e $\mathbf{K}_n^{(m)} = \mathbf{K}_n + Alt_m^n$.

6.3 Monadic operator on finitely generated free MV-algebra

As well known n-generated free MV-algebra $F(n)$ is isomorphic to a subalgebra of inverse limit of inverse system $\{F_i(n)\}_{i \in Z^+}$, where $F_i(n)$ is a free

n-generated algebra over locally finite subvariety $\mathbf{MV} + (K_i)$ of the variety $\mathbf{MV}$ of all MV-algebras. $F_i(n) \cong S_1^{v_n(1)} \times \ldots \times S_i^{v_n(i)}$ [2], where $v_m(x)$ is the function defined on the positive integers $\mathbb{Z}^+$ as follows: $v_n(1) = 2^n$, $v_n(2) = 3^n - 2^n$, $\ldots$, $v_n(i) = (i+1)^n - (v_n(i_1) + \ldots + v_n(i_{k-1}))$, where $i_1 (= 1), \ldots, i_{k-1}$ are all the divisors of i distinct from $i (= i_k)$. Let $g_1, \ldots, g_n \in F(n)$ ($g_1^{(i)}, \ldots, g_n^{(i)} \in F_i(n)$, respectively) be the generators of $F(n)$ ($F_i(n)$, respectively). To represent the generators more clear notice that the generator of $S_i^{v_1(i)}$ $(i > 1)$ consists of all fractions with co-prime nominators and denominators. For example the generator of $S_3^{v_1(3)}$ is $(1/3, 2/3)$, for $S_4^{v_1(4)}$ is $(1/4, 3/4)$, for $S_5^{v_1(5)}$ is $(1/5, 2/5, 3/5, 4/5)$ and so on.

Theorem 6.6. *There exists a monadic operator $\exists$ on n-generated free MV-algebra $F(n)$ converting the one into a monadic MV-algebra $(F(n), \exists)$.*

Now we are ready to give a characterization of projective Heyting algebras.

Theorem 6.7. *n-generated subalgebra A of n-generated free MV-algebra is projective if and only if A is mrc-subalgebra.*

Now the question arises : is the monadic operator unique non-trivial on n-generated free MV-algebra $F(n)$?

Conjecture. *The monadic operator $\exists$ on 1-generated free MV-algebra $F(1)$ is unique non-trivial operator.*

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An Implication in Orthologic

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Abstract: We involve a certain propositional logic based on an ortholattice. We characterize the implication reduct of such a logic and show that its algebraic counterpart is the so-called orthosemilattice. Congruence properties of these algebras will be described.

Structure of Commutative Cancellative Residuated l-monoids in $[0, 1]$

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It turns out that for many logical calculi (e.g. fuzzy logics or substructural logics) the corresponding algebras of truth values form residuated l-monoids, i.e., lattice ordered monoids endowed with a residuum. In this talk we will concentrate on the structure of commutative cancellative residuated l-monoids in the real unit interval $[0, 1]$ because of the following two reasons.

In the paper [1], Esteva and Godo introduced so-called monoidal t-norm based logic (MTL). Properties of this logic and its schematic extensions were recently studied (see [2, 5, 6, 7]). One of the possible schematics extensions is so-called ΠMTL introduced by Hájek in [5] and studied in [2, 6]. The corresponding algebras of truth values for ΠMTL are so-called ΠMTL-algebras and they are exactly bounded commutative cancellative residuated integral l-monoids. Moreover, it was shown in [6] that ΠMTL satisfies standard completeness theorem, i.e., the variety of ΠMTL-algebras is generated by ΠMTL-algebras in $[0, 1]$. Thus ΠMTL-algebras in $[0, 1]$ as the generators of the whole variety are the most important and the knowledge of their structure is more desired.

The second reason is that the monoid operation of a ΠMTL-algebra in $[0, 1]$ is a left-continuous t-norms (for the definition of a t-norm see [8]). While the class of continuous t-norms is completely characterized, the class of left-continuous t-norms not at all. So far only construction methods for left-continuous t-norms were published. In this talk we partially contribute to this characterization task and give a characterization of the class of cancellative left-continuous t-norms.

We will mainly concentrate on the structure of subdirectly irreducible ΠMTL-algebras in $[0, 1]$, since they are more important as the generators of the variety of ΠMTL-algebras. However, we will also deal with the rest. Since a ΠMTL-algebra is a cancellative l-monoid, it is possible to embed it to an l-group

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by forming fractions in the same way as we can construct the positive rational numbers from positive integers. Then we can use Hahn's embedding theorem (see [3, 4]) and embed this l-group to a full Hahn group. The full Hahn group is a group of functions from the set of convex subgroups to reals under addition. Moreover, the supports of the functions (the regions where the functions are not zero) are inversely well ordered (i.e., each subset has a maximum) w.r.t. the order induced by inclusion of the convex subgroups. Thus it is possible to make the full Hahn group totally ordered by lexicographic order. In the talk, we will show how to select an arbitrary ΠMTL-algebra from the full Hahn group which is order-isomorphic to $[0,1]$. In this way, we obtain a characterization of the structure of ΠMTL-algebras in $[0,1]$ and left-continuous cancellative t-norms up to an isomorphism.

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On pseudo-BL algebras

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Abstract

Since a pseudo-BL algebra is a non-commutative residuated lattice satisfying pseudo-divisibility and pseudo-prelinearity conditions, we first make the connection between non-commutative residuated lattices and pseudo-BCK algebras and then we decompose the pseudo-divisibility and pseudo-prelinearity conditions into other conditions. Thus, we obtain new classes of non-commutative residuated lattices and we establish hierarchies between them.

Keywords pseudo-BCK algebra, pseudo-BCK(pP) lattice, non-commutative residuated lattice, pseudo-BL algebra, pseudo-MTL algebra, pseudo-IMTL algebra, pseudo-WNM algebra, pseudo-NM algebra

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On the Structure of Rotation-Invariant Semigroups

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Abstract: Motivated by a geometrical characterization of rotation-invarian semigroups (which is based on the notion of rotation-invariance) two construction methods (called rotation and rotation-annihilation) of rotation-invariant semigroups are introduced [2]. These constructions allow us to construct a wide class of classical residuated lattices. The rotation construction plays a crucial role in the theory of perfect algebras, as shown very recently in [1]. The above-mentioned geometrical characterization is extended to the whole class of residuated lattices by introducing c-factors of a residuated lattice. The notion of involutive elements are introduced, and some related theorems are proved.

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BASIC LOGIC ALGEBRAS AND LATTICE-ORDERED GROUPS AS ALGEBRAS OF BINARY RELATIONS

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Abstract: A *residuated lattices of relations* is a set of binary relations that is closed under union, intersection, composition and residuals, and contains a relation that is an identity element with respect to composition. The class RLR of all algebras isomorphic to some residuated lattice of relations is properly contained in the variety of distributive residuated lattices. RLR is a quasivariety, but it is apparently an open problem whether it is a variety or whether it generates the variety of all distributive residuated lattices.

It is shown that RLR contains the variety of all lattice-ordered groups (ℓ-groups) and all basic logic algebras (BL-algebras). The identity element of an ℓ-group or BL-algebra corresponds to the partial order relation rather than the diagonal relation, and the collection of binary relations is not closed under relation converse. It is interesting to note that, in contrast to the varieties of ℓ-groups and BL-algebras, the class RLR is not finitely axiomatizable. The construction shows that ℓ-groups and BL-algebras are subreducts of representable relation algebras with an additional constant for the (possibly non-diagonal) identity relation.

If we add the union of all binary relations and the empty set to a residuated lattice of relations, we again get a residuated lattice of relations. For ℓ-groups we give a simple equational basis for the variety generated by such bounded extensions and discuss some general connections between bounded and unbounded residuated structures.

We furthermore show that the finite members of the variety of representable generalized BL-algebras are all commutative.

SPLITTINGS REVISITED

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In [1] we showed that the only splitting algebra in the variety of $\mathrm{FL}_{\mathrm{ew}}$-algebras is the two-element Boolean algebra. $\mathrm{FL}_{\mathrm{ew}}$-algebras (called residuated lattices in [1] but the terminology has changed since) can be viewed as algebras $\langle A; \vee, \wedge, \cdot, \rightarrow, 1, 0, \top, \bot \rangle$ such that $\langle A; \vee, \wedge, \top, \bot \rangle$ is a bounded lattice, $\langle A; \cdot, \rightarrow, 1 \rangle$ is a residuated commutative monoid, and moreover the identities $1 = \top$ and $0 = \bot$ hold. The variety of $\mathrm{FL}_{\mathrm{ew}}$-algebras is the class of algebraic models logics without contraction, i.e., logics extending Full Lambek Calculus with exchange and weakening—hence the name. If we do not require that the unit of the monoid conincide with the top of the lattice, nor the zero element with the bottom, we obtain the variety of FL_{e}-algebras—algebraic models of logics extending Full Lambek calculus with exchange. One important member of this class that does not belong to the former one is Linear Logic without exponentials.

Drawing on methods developed in [1] one can obtain the following partial result. Let $\mathcal{V}$ stand for the variety FL_{e}-algebras.

Theorem 1. *If* $\mathbf{A}$ *is a finite si algebra in* $\mathcal{V}$ *such that* 1 *is not an atom, then* $\mathbf{A}$ *is not splitting in* $\mathcal{V}$.

Notice that since $\mathcal{V}$ is congruence distributive and generated by its finite members, the only candidates for splitting algebras in $\mathcal{V}$ are finite si algebras. So, it may seem that splittings are rare. Unfortunately, we also have:

Theorem 2. *There are infinitely many finite simple* FL_{e}*-algebras such that* 1 *is an atom.*

Quite annoyingly, it is not even known whether the two element Boolean algebra is splitting in $\mathcal{V}$.

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Dually Residuated Lattice-Ordered Monoids

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Abstract: Commutative dually residuated lattice-ordered monoids (DRl-monoids for short) were introduced in the sixties as a common abstraction of Abelian l-groups and Brouwerian algebras. It turns out that, e.g., well-known MV-algebras are a special case of bounded commutative DRl-monoids. We deal with non-commutative DRl-monoids which generalize l-groups and non-commutative extensions of MV-algebras (pseudo MV-algebras/GMV-algebras). We define the natural concept of an ideal and concentrate especially on the ideal lattices of DRl-monoids.

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Moisil possibility operators on $(n+1)$-valued Łukasiewcz BCK-algebras

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It is well-known that a $(n+1)$-valued Łukasiewcz BCK-algebra is an algebra of type $(2,0)$ which satisfies the identities: $\bullet\, 1 \to x = x$, $\bullet\, x \to (y \to x) = 1$, $\bullet\, (x \to y) \to ((y \to z) \to (x \to z)) = 1$, $\bullet\, (x \to y) \to y = (y \to x) \to x$, $\bullet\, ((x \to y) \to (y \to x)) \to (y \to x) = 1$, $\bullet\, (x^n \to y) \vee x = 1$, where $x^1 \to y = x \to y$ and $x^{n+1} \to y = x \to (x^n \to y)$, for all integer n, $n \geq 1$ (see [1],[2],[3]).
On the other hand, A. V. Figallo defined the modal $(n+1)$-valued Łukasiewcz BCK-algebras (or mBCK_{n+1}-algebras) as algebras $\langle A, \to, \sigma_1, \sigma_2, \cdots, \sigma_n, 1\rangle$, where $\sigma_1, \ldots, \sigma_n$ ar unary operations which satisfy some additional identities (see [1]).
In this communication we describe a method to build $\sigma_2, \cdots, \sigma_n$ from $\to$ and σ_1. This method is similar to the one indicated by W. Suchoń in [4]

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Residuation on weakly Heyting algebras

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Abstract

The variety of weakly Heyting algebras, or WH-algebras, was introduced in [1]. *A weak Heyting algebra* is an algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\rightarrow$ is a binary operation satisfying the equations:

1. $(x \rightarrow y) \wedge (x \rightarrow z) \approx x \rightarrow (y \wedge z)$,
2. $(x \rightarrow z) \wedge (y \rightarrow z) \approx (x \vee y) \rightarrow z$,
3. $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$,
4. $x \rightarrow x \approx 1$.

If we consider a modal algebra with $\supset$ being the Boolean implication and define $a \rightarrow b := \Box(a \supset b)$ then what we obtain is a weakly Heyting algebra. And from the Priestley-style duality developed in [1] it is clear that every weakly Heyting algebra is embeddable into one that is obtained from a modal algebra. That is, the variety of weakly Heyting algebras corresponds to the strict implication reduct (also with $\wedge, \vee, 0, 1$) of the modal algebras (see [2] for the logical counterpart).

In the talk we will start giving a purely algebraic proof of the previous fact (cf. [3, pp. 128–130]). Then, we will consider two new varieties in the language enlarged with $\odot$. The variety of *residuated weakly Heyting algebras*, or RWH-algebras, is the one obtained by adding:

5. $x \odot (x \rightarrow y) \leq y$,
6. $x \leq y \rightarrow (y \odot x)$,
7. $x \odot (y \wedge z) \leq x \odot y$.

The members of this variety are exactly the weakly Heyting algebras such that *the law of residuation* holds, i.e., $a \leq b \rightarrow c$ iff $b \odot a \leq c$ for every $a, b, c \in A$. And we also introduce the variety of *Boolean residuated weakly Heyting algebras*, or BRWH-algebras, obtained from all the previous equations by adding:

8. $(x \wedge y) \odot z \approx x \wedge (y \odot z)$.

Then, it is possible to see that **RWH** $\neq$ **BRWH** while both **RWH** and **BRWH** are conservative expansions of **WH**. This is an easy consequence from the finite embeddability property of these varieties. The law of residuation determines univocally the operation $\odot$ but it does not always exist, e.g., it is possible to give a complete WH-algebra where it is not possible to define $\odot$. One of the laws that $\odot$ satisfies over the BRWH-algebras is the monotonicity in both components. However, all the following

equations are not valid: $x \odot y \approx y \odot x$, $x \odot (y \odot z) \approx (x \odot y) \odot z$, $1 \odot 1 \approx 1$, $x \odot y \leq y$ and $x \wedge y \leq x \odot y$.

Finally, I would like to point out that every BRWH-algebra is embeddable into a RWH-algebra that admits Boolean implication, i.e., there is a certain binary operation $\supset$ under which the lattice becomes a Boolean algebra. This justifies the name of the variety. Thus it is easy to see that the equational logic associated with **BRWH** corresponds to the local consequence defined by Kripke models where

$$\mathcal{M}, w \Vdash \varphi \odot \psi \quad \text{iff} \quad \mathcal{M}, w \Vdash \varphi \text{ and exists } u \in R^{-1}[w] \text{ such that } \mathcal{M}, u \Vdash \psi.$$

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Perfect and bipartite IMTL-algebras and disconnected rotations of basic semihoops.

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Abstract: Involutive monoidal t-norm logic (IMTL) was defined by Esteva and Godo and was proved to be the logic of bounded residuated lattices with prelinearity and involution, the so-called IMTL-algebras. It's also the logic of involutive left-continuous t-norms and their residua. In this talk the concepts of perfect, bipartite and local algebra used in the classification of MV-algebras will be generalized to the wider variety of IMTL-algebras. Perfect algebras turn out to be the algebras obtained from a basic semihoop by Jenei's disconnected rotation. We also will prove that the variety generated by all perfect IMTL-algebras is the variety of the IMTL-algebras that are bipartite by every maximal filter and equational axiomatizations for it will be given.

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Points in cm-lattices

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Abstract: An important notion introduced in the context of irreducible representations of C* -algebras is the notion of prime element of a quantale. In 2002, Kruml introduced the notion of a distributive quantale and proved that any algebraic distributive quantale is spatial.

Motivated by this result, by the results of the present author and by the work of Banaschewski and Erne in the context of two-sided quantales, we shall generalize the corresponding ideas concerning prime elements and semiprime ideals for arbitrary quantales and cm-lattices. In fact, it is precisely the lack of two-sidedness that causes some difficulties.

On the interval in the clone lattice that contain a semiprojection

Jovanka Pantovic

Abstract: Semiprojection s is an n-ary operation on a finite set such that $s(x_1, ldots, x_n) = x_1$ if $|x_1, ldots, x_n| < n$, and near-projection l_k is a semiprojection with $l_k(x_1, ldots, x_n) = x_n$ for $|x_1, ldots, x_n| geqn$. There are continuum many clones on a finite set A, $|A| > 3$, that contain a semiprojection s, and infinitely many of them on a three element set. We describe all the clones of operations on a three element set that contain l_3 and show that their number is countably infinite.

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∗-autonomous lattices and fuzzy sets

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Abstract: ∗-autonomous lattices are the algebraic models of subexponential linear logic without additive constants, and at the same time a plausible solution (for the Abelian case) to Garrett Birkhoff's well-known problem suggesting to develop "a common abstraction of Boolean algebras and l-groups". Examples include (term equivalent versions of) commutative Girard quantales, De Morgan monoids, classical residuated lattices, MV-algebras, Abelian l-groups. The subvariety of ∗-autonomous lattices satisfying a divisibility condition turns out to be especially interesting. We shall indeed show: 1) how it is possible to extract MV-algebras out of divisible ∗-autonomous lattices, generalizing Mundici's Gamma functor; 2) how to describe those divisible ∗-autonomous lattices that are direct products of an Abelian l-group and an MV-algebra; 3) that semisimple divisible ∗-autonomous lattices are representable as algebras of real-valued functions, generalizing an analogous result by Chang-Belluce for semisimple MV-algebras. The algebras of functions mentioned under 3), moreover, turn out to have a nice intuitive interpretation. The Chang-Belluce target algebras can be seen as algebras of (characteristic functions of) fuzzy sets. Our target algebras can plausibly be seen as algebras of (characteristic functions of) "really fuzzy sets", where not only degrees of approximate membership are allowed, but also degrees of definite membership.

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Fuzzy concepts as pointwise Galois connections

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Abstract: I propose a flexible way to build formal concepts within fuzzy logic and set theory, by revisiting the very basic Galois connection underlying a conceptual hierarchy. The framework is general enough to capture some important particular cases, with their own independent interpretations, like "antitone" or "isotone" concepts constructed from fuzzy binary relations, but also to allow the two universes (of objects and attributes) to be equipped each with its own truth structure. Perhaps the most important feature of this approach is that one does not commit to any kind of logical connector, covering thus the case of a possibly non-commutative conjunction too.

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Functional representations of MV- and GMV-algebras

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(Joint work with Andrew M. W. Glass (Cambridge, U.K.)
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Łukasiewicz infinite valued propositional logic is one of the important ingredients involved in the theory of fuzzy sets. It has truth values in the real interval $[0, 1]$. This interval is a linearly ordered MV-algebra which is obtained from the additive ordered group $\mathcal{R}$ (with 1 as strong unit) using truncated addition at 1. In general, every MV-algebra can be obtained in this manner from an Abelian lattice-ordered group with strong unit. The fundamental significance of the MV-algebra $[0, 1]$ is underlined by Chang's Completeness Theorem: an MV-equation holds in every MV-algebra iff it holds in $[0, 1]$. GMV–algebras (= pseudo MV-algebras) have been recently introduced as non-commutative generalizations of MV-algebras. They can be viewed as models of an algebraic semantics for a non-commutative generalization of multi-valued reasoning. It is proved that every GMV-algebra is analogously obtained from a (not necessarily Abelian) lattice-ordered group with a strong unit. Because of the importance of the real interval $[0, 1]$ for MV-algebras and fuzzy logics, a natural question arises: Is it possible to represent every GMV-algebra as a GMV-algebra of real-valued functions? We provide a positive answer for normal-valued GMV-algebras (including all MV-algebras).

Adding involution to residuated structures

J. Raftery

Abstract: Two methods of embedding residuated structures into ones with an involution operator will be discussed. Various applications will be given. The constructions allow one to deduce the decidability of the equational theory of commutative distributive residuated lattices from a result of R.T. Brady concerning the logic RW. One of the constructions helps to show that the variety generated by Grishin's L0-algebras satisfies no nontrivial idempotent Mal'cev condition.

Functional representation theorems for monadic $n \times m$–valued Łukasiewicz algebras with negation

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Abstract

$n \times m$–valued Łukasiewicz algebras with negation (or $NS_{n\times m}$–algebras) were introduced by the author in [3]. These algebras constitute an extension of matrix Łukasiewicz algebras ([4]) and they coincide with n–valued Łukasiewicz algebras ([2]) in the particular case $m = 2$.
By adding to $NS_{n\times m}$–algebras a unary operator which is called existential quantifier, monadic $n \times m$–valued Łukasiewicz algebras with negation (or $MNS_{n\times m}$–algebras) are introduced. This new class of algebras represents a natural generalization of that of monadic n–valued Łukasiewicz algebras ([1]). In this work, three functional representation theorems for these algebras are given. From some of the results established in [4], the first representation is described. A second one is obtained by applying P. Halmos's functional representation for monadic Boolean algebras to the set of Boolean elements of an $MNS_{n\times m}$–algebra. Finally, rich $MNS_{n\times m}$–algebras are introduced and characterized, and a third representation for these algebras is obtained.

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Direct decompositions of $DR\ell$-monoids and GMV-algebras

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Commutative dually residuated lattice ordered monoids ($DR\ell$-monoids) were introduced as a common generalization of abelian lattice ordered groups (ℓ-groups) and Brouwerian algebras. Moreover, e.g. MV-algebras and BL-algebras, which are algebraic counterparts of the Łukasiewicz infinite valued propositional logic and Hájek basic fuzzy logic, can be considered as special cases of commutative $DR\ell$-monoids. General $DR\ell$-monoids, not necessarily commutative, involve not only arbitrary ℓ-groups but also GMV-algebras (called also pseudo MV-algebras) and pseudo BL-algebras.

We introduce inner direct decompositions of $DR\ell$-monoids, we describe properties of their ideals which are direct factors, and characterize the sets of all direct factors of $DR\ell$-monoids. The results are applicable to all above mentioned special cases of $DR\ell$-monoids. Moreover, we engage in their specifying for GMV-algebras.

Visser Algebras As Generalized Heyting Algebras

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Abstract: Visser algebras correspond with Albert Visser's propositional calculus of 1981, as Heyting algebras correspond with intuitionistic propositional calculus. We sketch some of the major properties of the category of Visser algebras. Of particular interest is a decomposition result which somehow measures how much a complete Visser algebra looks like a Heyting algebra, modulo a quotient Visser algebra with explicit fixed points.

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Decidability for Knotted Extensions of Propositional Linear Logic

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We consider propositional Classical Linear Logic (***LL***) extended by *knotted structural rules*:

$$\frac{\Gamma, x^n \Rightarrow y}{\Gamma, x^m \Rightarrow y} \ (n, m)$$

where n, m are integers, $n \geq 0$, $m \geq 1$, $m \neq n$ and x^n represents a sequence of n copies of x. The simplest cases of knotted structural rules are *weakening* $(0, 1)$, *contraction* $(2, 1)$ and *mingle* $(1, 2)$.

Algebraic models for ***LL*** are *classical linear algebras with storage*, which are commutative residuated lattices with bounds $\top$, $\bot$, a constant $\mathbf{0}$ and identity $(x \to \mathbf{0}) \to \mathbf{0} = x$ (equivalently, $\sim\sim x = x$, where $\sim x = x \to \mathbf{0}$) and a unary 'storage operator' !. If we extend ***LL*** by the rule (n, m) the corresponding algebraic models must also satisfy $x^m \leq x^n$. We show that in each such case, the class of algebras has the *finite embeddability property*, meaning that every finite partial subalgebra of an algebra in the class may be embedded into a finite full member of the class. The finite model property, and hence also decidability, for each knotted extension of ***LL*** follows. This contrasts with the fact that ***LL*** is undecidable.

The method does not rely on the presence of $\mathbf{0}$ and so may also be used to show that each knotted extension of Intuitionistic Linear Logic (***ILL***) has the finite model property with respect to its algebraic semantics, and is decidable. The storage operator ! is also not essential so results for the exponent-free versions of ***LL*** and ***ILL*** also follow.

On two fragments with negation and without implication of the logic of residuated lattices

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Abstract

The logic of (commutative integral bounded) residuated lattices is known under different names in the literature: monoidal logic [7], intuitionistic logic without contraction [1], H_{BCK} [10], etc. It is usually given, up to definitional equivalence, in the language $\langle \vee, \wedge, *, \neg, \rightarrow, 0, 1\rangle$. In this talk we explain several results obtained in [4] about the $\langle \vee, *, \neg, 0, 1\rangle$-fragment and the $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-fragment of this logic.

As regards the algebraic aspects of this study, we introduce the notion of pseudocomplementation with respect to the monoidal operation $*$ (see [2] in the case of $\wedge$). We then define two new classes of algebras: the class of commutative integral bounded semilatticed pseudocomplemented monoids, denoted by $\mathbb{CIBPM}^{s\ell}$, and the class of commutative integral bounded latticed pseudocomplemented monoids, denoted by $\mathbb{CIBPM}^{\ell}$. We show that these classes of algebras are varieties whose quasiequational theories are decidable. Their members are exactly the subreducts of the variety of residuated lattices, i.e., every $\mathbb{CIBPM}^{s\ell}$-algebra and every $\mathbb{CIBPM}^{\ell}$-algebra is embeddable into a (complete) residuated lattice. It can be seen that it is impossible to build this embedding in such a way that all existing (infinite) joins are preserved. It is also true that if the reduct of a residuated lattice is subdirectly irreducible in $\mathbb{CIBPM}^{\ell}$ then it is subdirectly irreducible as a residuated lattice, while the reverse implication is false.

As regards the logical aspects of this study, we introduce two sequent calculi: $\mathbf{FL_{ew}}[\vee, *, \neg, 0, 1]$ and $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg, 0, 1]$. They are obtained from the well known contraction-free calculus $\mathbf{FL_{ew}}$ (in the language $\langle \vee, \wedge, *, \neg, \rightarrow, 0, 1\rangle$) [9]. The former, $\mathbf{FL_{ew}}[\vee, *, \neg, 0, 1]$, is obtained by deleting the rules of additive conjunction $\wedge$ and implication $\rightarrow$ from $\mathbf{FL_{ew}}$. And $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg, 0, 1]$ results by deleting the rules of implication $\rightarrow$ from $\mathbf{FL_{ew}}$. It can be shown that $\mathbb{CIBPM}^{s\ell}$ ($\mathbb{CIBPM}^{\ell}$) is the equivalent variety semantics [11, 6] of the intuitionistic Gentzen system associated to the sequent calculi $\mathbf{FL_{ew}}[\vee, *, \neg, 0, 1]$ ($\mathbf{FL_{ew}}[\vee, \wedge, *, \neg, 0, 1]$). As a consequence, we have that the variety $\mathbb{CIBPM}^{s\ell}$ ($\mathbb{CIBPM}^{\ell}$) is an algebraic semantics, with defining equation $p \approx 1$, for the external deductive system $\mathcal{S}_e[\vee, *, \neg, 0, 1]$ ($\mathcal{S}_e[\vee, \wedge, *, \neg, 0, 1]$) associated to $\mathbf{FL_{ew}}[\vee, *, \neg, 0, 1]$ ($\mathbf{FL_{ew}}[\vee, \wedge, *, \neg, 0, 1]$). Moreover, we show a generaliza-

tion of [8, Corollary 9]: $\mathcal{S}_e[\vee, *, \neg, 0, 1]$ ($\mathcal{S}_e[\vee, \wedge, *, \neg, 0, 1]$) is the $\langle \vee, *, \neg, 0, 1 \rangle$-fragment ($\langle \vee, \wedge, *, \neg, 0, 1 \rangle$-fragment) of the logic of residuated lattices. Then, $\mathcal{S}_e[\vee, *, \neg, 0, 1]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg, 0, 1]$ are decidable. We also show that $\mathcal{S}_e[\vee, *, \neg, 0, 1]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg, 0, 1]$ are not protoalgebraic [3, 5], so there is no definable implication connective for them.

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A Gentzen System for Involutive Residuated Lattices

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An involutive residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \cdot, e, ' \rangle$ is an involutive lattice with a residuated monoid operation, such that

$$x \cdot y \leq z \quad \text{iff} \quad x \leq (y \cdot z')' \quad \text{iff} \quad y \leq (z' \cdot x)',$$

for all $x, y, z \in L$. An involutive residuated lattice, whose underlying lattice is complete, also corresponds to a Girard quantale. We will describe a cut-free Gentzen system for involutive residuated lattices from an algebraic point of view.

C. C. Chang's MV*-algebras

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Abstract

The pioneering works [1], [2] of C. C. Chang are well–known, developed and utilized in the scientific community. Much less attention, however, has got C. C. Chang's further study [3] in which he introduced MV*-algebras to prove the completeness of what he called *Logic with Positive and Negative Truth Values*, a natural extension of Lukasiewicz infinite valued logic. The aim of this presentation is to recall this study.

An *MV^*–algebra* is a system $< B, +, C, 0, 1 >$ where B is a set containing two disjoint elements 0 and 1. Moreover, B is closed under a binary operation $+$ and a unary operation C, such that the following axioms hold, for each $x, y, z \in B$,

$$x + y = y + x. \tag{1}$$

$$(1 + x) + (y + (1 + z)) = ((1 + x) + y) + (1 + z). \tag{2}$$

$$x + Cx = 0. \tag{3}$$

$$(x + 1) + 1 = 1. \tag{4}$$

$$x + 0 = x. \tag{5}$$

$$C(x + y) = Cx + Cy. \tag{6}$$

$$CCx = x. \tag{7}$$

After stipulating

$$\begin{aligned} -1 &= Cx, \\ x^+ &= 1 + (-1 + x), \\ x^- &= -1 + (1 + x), \\ x \vee y &= [x^+ + (C(x^+) + y^+)^+] + [x^- + (C(x^-) + y^-)^+], \end{aligned}$$

the remaining axioms are:

$$x + y = (x^+ + y^+) + (x^- + y^-). \tag{8}$$

$$(Cx + (x + y))^+ = C(x^+) + (x^+ + y^+). \tag{9}$$

$$x \vee y = y \vee x. \tag{10}$$

$$x \vee (y \vee z) = (x \vee y) \vee z. \tag{11}$$

$$x + (y \vee z) = (x + y) \vee (x + z). \tag{12}$$

In general, MV*–algebras are not MV–algebras. After giving examples of MV*-algebras, Chang proved some of their basic properties, e.g. that the positive part $B^+ = \{x^+ : x \in B\}$ of B can be endowed with a unary operation ^ and a binary operation $\cdot$ such that $< B^+, +, \cdot, ^\wedge, 0, 1 >$ is an MV-algebra. The same holds true for the negative part $B^- = \{x^-; x \in B\}$ of B, too, i.e. after certain stipulations $< B^+, +, \cdot, ^\sim, 0, -1 >$ is another MV-algebra (and isomorpic to B^+). Moreover, any MV*-algebra is a subalgebra of a direct product of MV*-algebras $G(p) = \{x : x \in G, -p \leq x \leq p\}$ where G is an ordered abelian group and p is a positive element of G.

However, many open problems and questions concerning MV*-algebras remain; among the most fundamental are 'Do MV*-algebras have an importance of their own?' and 'Does Logic with positive and negative truth values have any meaningful applications?'.

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Self-reference and interpolation in many-valued logic: work in progress

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Famous paradoxes like the liar paradox and Löb's paradox show that self-reference is inconsistent with classical logic. On the other hand, in classical logic one has interpolation, both in the form:
(A) $\vdash A \to B$ implies $\vdash A \to C$ and $\vdash C \to B$ for some C in the language common to A and B
and in the form
(B) $A \vdash B$ implies $A \vdash C$ and $C \vdash B$ for some C in the language common to A and to B.
Note that the forms (A) and (B) are equivalent in classical logic.

Interestingly, both the proof of the liar paradox or of Löb's paradox, as well as the equivalence between (A) and (B) are related to structural rules, and to contraction in particular. Hence both the paradoxes and the equivalence of (A) and (B) are also derivable e.g. in Intuitionistic Logic. In Łukasiewicz Logic, paradoxes are not derivable; moreover, self-reference does not give any contradiction: indeed, any Mc-Naughton function in one variable (possibly with parameters) is continuous from $[0, 1]$ into $[0, 1]$, therefore it has a fixed-point. Adding the existence of a greatest fixed point and of a least fixed-point for any McNaughton function we obtain a logic which is bi-interpretable in Łukasiewicz Logic plus Baaz projection Δ (where $\Delta(1) = 1$ and $\Delta(x) = 0$ for $x \neq 1$) plus divisibility (this means: for all x and for every positive natural number n, there is a least y such that $\underbrace{y \oplus ... \oplus y}_{n \text{ times}} = x$). Indeed, in this logic we can define all piecewise linear functions on $[0, 1]$ with rational coefficients, and the required fixed points are of this form. Viceversa, both divisibility operators and Δ can be defined as greatest or least fixed points of suitable operators. (Warning: one cannot iterate diagonalization, e.g., $\neg\Delta(p)$ has no fixed-point).

Interestingly, Łukasiewicz Logic plus Δ plus divisibility has interpolation of both kinds (A) and (B). So, interpolation and self-reference are not incompatible. Now one may wonder what happens with Hájek's logic BL.

Even though I didn't conclude my investigation yet, I could prove the following results:

- While BL has no interpolation of the form (A), it has interpolation of the form (B).
- Adding the existence of fixed-points for BL-formulas in one variable and with *one* parameter only we obtain a conservative extension of BL, i.e., we do not prove new BL-formulas.
- However, the linearly ordered models of BL plus least and greatest fixed points change: these models are ordinal sums of divisible MV-algebras (thus they cannot have any product or Gödel component).
- BL plus fixed-points plus Δ has interpolation of the form (A).

Open problems:

- Is Δ definable in BL plus existence of least and greatest fixed points?
- Is BL plus existence of least and greatest fixed points for formulas with more than one parameter conservative over BL?
- Is BL plus existence of least and greatest fixed points sufficient to get interpolation in the form (A)?

Composition on MV-algebras

Extended abstract

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1 Introduction

In this talk we shall describe an extension of MV-algebras obtained by adding an operator playing the role of composition of functions. It is well known that elements of the Lindenbaum algebra of Łukasiewicz infinite-valued logic are McNaughton functions $f : [0,1]^n \to [0,1]$ corresponding to propositions p. We shall be concerned here, for simplicity, with the case $n = 1$. The question is if the composition of functions has any "logical" meaning and if the chaotic behavior of the McNaughton functions has any interesting interpretation. The composition of McNaughton functions (for n=1) can be interpreted as substitution (in propositional many valued logic) so it does have logical interpretation. This was the starting point in considering the algebraic structures studied in this paper. But these structures proved to be interesting for themselves as algebras of "operators". The constructions presented here only for the case of MV-algebras can be extended to many other interesting algebras of logic. Finally we notice that similar problems were studied by Panti ([6]) in a different context.

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