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Abstracts of Invited Speakers

FRAMES AND MV-ALGEBRAS

L. P. Belluce and A. Di Nola

Department of Mathematics
British Columbia University
Vancouver, B.C. Canada
e-mail:belluce@math.ubc.ca

Department of Mathematics and Informatics
University of Salerno
e-mail: adinola@unisa.it

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Abstract

We present a preliminary study of a class of MV-algebras which is a natural generalization of the class of "algebras of continuous functions". More specifically, we're interested in the algebra of frame maps $Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})$ in the category $\mathcal{F}$ of frames, where A is a topological MV-algebra, $\Omega(A)$ the lattice of open sets of A, and $\mathbf{K}$ an arbitrary frame.

Given a topological space X and a topological MV-algebra A, we have the algebra $C(X, A)$ of continuous functions from X to A. We can look at this from a frame point of view. Among others we have the result: if $\mathbf{K}$ is spatial, then $\mathcal{C}(pt(\mathbf{K}), A)$, $pt(\mathbf{K})$ the points of $\mathbf{K}$, embeds into $Hom_{\mathcal{F}}(\Omega(A), \mathbf{K})$ analogous to the case of $\mathcal{C}(X, A)$ embedding into $Hom_{\mathcal{F}}(\Omega(A), \Omega(X))$.

BOSBACH STATES ON PSEUDO BL-ALGEBRAS

George Georgescu*

Faculty of Mathematics, University of Bucharest, Str. Academiei 14, Bucharest, Romania

Abstract

The concept of BL-algebra arises from the algebraic structure induced by a continuous t-norm on [0,1]. MV-algebras, product algebras and Gődel algebras are the main classes of BL-algebras. On the other hand, BL-algebras are the algebraic models of Hájek's Basic Logic.

Recently, the pseudo-MV algebras were introduced as a noncommutative generalization of the MV-algebras. Dvurečenskij proved that the category of pseudo-MV algebras is equivalent to the category of l-groups with strong unit. This theorem extends the classical Mundici result that the MV-algebras and the abelian l-groups are categorically equivalent.

Pseudo-BL algebras constitute a common extension of BL-algebras and pseudo-MV algebras. This structure seems to be a very general algebraic concept in order to express the noncommutative reasoning. We remark that a pseudo-BL algebra has two implications and two negations.

Probability theory for MV-algebras is a very actual subject. A natural problem is to develop a probability theory for other types of fuzzy structures. Riečan introduces a notion of state for BL-algebras and Dvurečenskij studies the states on pseudo-MV algebras.

This paper is concerned with states on pseudo-BL algebras. The first problem is to find a good concept of state. For our case, the notion of state defined by Bosbach for right complementary semigroups seems to be the most appropriate. Actually, we work with states verifying the Bosbach condition for each of the two implications.

We prove several results on Bosbach states and we establish the connection between state-morphisms, normal maximal filters and extremal states. For pseudo-MV algebras, the Bosbach states coincide with the states defined by Dvurečenskij. We extend Riečan states for the case of good pseudo-BL algebras and we prove that a Bosbach state is a Riečan state.

Conditional states on MV-algebras were studied by several authors. We present another way to associate a conditional state with a continuous state

*Corresponding author. *E-mail address*: georgescu@funinf.math.unibuc.ro (G. Georgescu)

defined on a σ-complete MV-algebra A. The main idea is to associate with any element $x \in A$ two boolean elements $\varphi_1(x)$, $\varphi_2(x)$ and to use these elements in order to define the conditional state. A notion of conditional state is also introduced for BL-algebras.

Universes of Fuzzy Sets – a Survey of t-Norm Related Approaches

Siegfried Gottwald

Institut für Logik und Wissenschaftstheorie, Universität Leipzig
Beethovenstr. 15, D-04107 LEIPZIG, Germany
E-mail: gottwald@uni-leipzig.de

Almost from the initial phase of fuzzy set theory in the 1960s there have been approaches toward the development of universes of fuzzy sets. One of their main aims always has been to make visible, and mathematically graspable, the similarities between classical and fuzzy set theory.

Another one of their aims has been the development of universes of fuzzy sets which are closed under the formation of fuzzy subsets, making precise in this way the notion of fuzzy set of higher level.

And at the same time the approaches started to base set algebraic operations for fuzzy sets not only on de Morgan algebras but on structures with more general, usually also non-idempotent operations.

Furthermore the approaches diverge concerning the fact whether they allow for graded identity relations or whether they are restricted to crisp identity relations only.

The methods to attack the related problem of the construction of a fuzzy analog to the cumulative universe of – standard, i.e. crisp – sets fall essentially into three classes:

- approaches which try to form cumulative universes of fuzzy sets rather similar to the construction of the cumulative universe of sets via an transfinite iteration of the power set operation;
- approaches which try to form cumulative universes of fuzzy sets rather similar to Boolean valued models for classical set theory;
- approaches which intend to suitably generalize the categorical characterization of the category SET of all sets and mappings to a similar characterization of some category FSET of all fuzzy sets and of suitable mappings between them.

These approaches shall be discussed, some recent results explained, and some open problems mentioned.

These approaches toward universes of fuzzy sets there are paralleled by a large amount of different purely axiomatic approaches toward a fuzzy set theory. These approaches shall, however, mentioned only rather sketchily.

Sheaves and Many Valued Topologies

Ulrich Höhle
FB C Mathematik und Naturwissenschaften
Bergische Universität, D-42097 Wuppertal, Germany

Abstract

Let Ω be a complete Heyting algebra and $sh(\Omega)$ be the category of sheaves on Ω. Since $sh(\Omega)$ is a topos, $sh(\Omega)$ has (internal) topological space objects (cf. [9]). On the other hand, a pair (X, τ) is an Ω-valued topological space iff τ is a subframe of Ω^X (cf. [5]). The aim of this talk is to explore various relationships between topological space objects in $sh(\Omega)$ and Ω-valued topological spaces. In particular, I will focus on the following facts:

1. There exists an adjunction between the category of topological space objects in $sh(\Omega)$ and the category of separated presheaves with values in **TOP** (see also [10]).
2. The category of stratified, Ω-valued topological spaces forms a full subcategory of the category of topological space objects in $sh(\Omega)$ (cf. [7]).
3. There exists an adjunction between the category of limit spaces and stratified, Ω-valued topological spaces (cf. [4]).
4. Fibrewise topological spaces with base space $(X, \mathcal{O})$ give rise to topological space objects in $sh(\Omega)$ (cf. [7]).
5. Ω-valued normed spaces generate stratified, Ω-valued topological spaces (cf. [6]).

Typical examples of Ω-valued normed spaces are Ω-probilistic normed spaces (in the case of $\Omega = [0,1]$ see [8]) and separated presheaves of normed spaces (cf. [1]). In this context it is interesting to see that the *Browerian motion* creates a probabilitsic normed space which plays a prominent role in the construction of the stochastic integral (cf. [3]).

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The Space of Penrose Tilings

Christopher J. Mulvey,
University of Cambridge/University of Sussex.

The concept of quantale is an instance of that of residuated lattice that is adapted to the non-commutative logics that arise in both mathematics and physics. In this talk, we shall examine the way it may be applied to describe the non-commutative space of Penrose tilings, providing a particularly straightforward motivation of the approach taken by Connes in introducing a C*-algebra to represent this non-commutative geometric construct. In doing so, we shall more generally consider the way that quantales may be obtained by introducing propositional geometric theories within non-commutative logic, generalising the way that such theories may be applied to obtain spaces within constructive logic. In the present context, we shall see how the C*-algebra introduced by Connes is obtained by considering the theory of Penrose tilings within non-commutative logic.

Characterizing free MV-algebras and projective l-groups

Daniele Mundici

Abstract: All existing characterizations of free MV-algebras rely on McNaughton representation theorem. Also, all known characterizations of projective l-groups are representation dependent: they rely on the Baker Beyon theory. In both cases one represents free objects as algebras of piecewise linear functions, and free generators as identity functions. Things are different for, say, boolean algebras: as is well known, the free boolean algebra over countably many generators is the countable atomless boolean algebra. In such characterization neither free variables nor boolean functions are mentioned. Using singular homology theory, we provide a similar representation-free characterization of free MV-algebras and projective l-groups.

Dept. of Math. "U.Dini"
University of Florence
Viale Morgagni 67/A
50134 Florence, Italy

RESIDUATED LATTICES: AN ALGEBRAIC PERSPECTIVE

CONSTANTINE TSINAKIS
VANDERBILT UNIVERSITY
U.S.A.

Abstract. A *residuated lattice* (RL) is an algebraic system $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that:
(1) $\langle L, \wedge, \vee\rangle$ is a lattice;
(2) $\langle L, \cdot, e\rangle$ is a monoid; and
(3) for all $x, y, z \in L$, $xy \leq z \;\Leftrightarrow\; x \leq z/y \;\Leftrightarrow\; y \leq x\backslash z$.

The class, $\mathcal{RL}$, of all RLs is a congruence permutable and congruence distributive variety. Its members have been studied in several branches of mathematics and are the bound-free reducts of the algebraic counterparts of the *full Lambek propositional logic*. This logic is obtained from the Gentzen-type sequent calculus of the intuitionistic propositional logic by deleting all structural rules: exchange, contraction and weakening. The elimination of the requirement that an RL have a smallest element or a greatest element has led to the development of a surprisingly rich theory that includes the study of various important varieties of cancellative RLs, such as the variety of lattice-ordered groups.

We will highlight recent results of the Vanderbilt group, by paying particular attention to the algebraic aspects of the theory. To this end, we present a generalization of the notion of an MV-algebra in the context of residuated lattices that includes non-commutative and unbounded structures. We prove that each RL in the resulting subvariety – $\mathcal{GMV}$, of *generalized MV-algebras* – can be obtained from lattice-ordered groups via a truncation construction that generalizes the Chang-Mundici-Dvurečenskij construction. This correspondence extends to a categorical equivalence. [This portion of the talk is based on joint research with Nikolaos Galatos.]

Motivated by the preceding considerations, we provide a simple equational basis for the join, $\mathcal{IRL} \vee \mathcal{LG}$, of the variety $\mathcal{LG}$ of lattice-ordered groups and the variety $\mathcal{IRL}$ of integral residuated lattices. In the process of deriving this result, we will obtain a simple axiomatic basis for the variety $\mathcal{IRL}\times_s\mathcal{LG}$, consisting of all semi-direct products of members of $\mathcal{IRL}$ by members of $\mathcal{LG}$. We conclude the talk by presenting a general method for constructing such semi-direct products, including wreath products. [The last part of the talk is based on joint research with Bjarni Jónsson.]

Glivenko properties of substructural logics

Hiroakira Ono
Japan Advanced Institute of Science and Technology, Japan
e-mail ono@jaist.ac.jp

This is a joint work with Nick Galatos. In 1929, V. Glivenko showed that for any formula α, α is provable in classical logic if and only if its double negation $\neg\neg\alpha$ is provable in intuitionistic logic. In their recent paper, R. Cignoli and A. Torrens proved that α is provable in classical logic if and only if $\neg\neg\alpha$ is provable in the logic **SBL**, an extension of Hájek's basic logic **BL**, and also that the same relation holds also between Łukasiewicz infinite valued logic and **BL**.

Let us consider Glivenko-type theorems in a general setting. For substructural logics **L** and **K** over **FL**, we say that *Glivenko property* holds for **L** relative to **K** when for any formula α, α is provable in **K** if and only if its double negation is provable in **L**. Then, our questions are the following:

- When does Glivenko property hold for **L** relative to **K**?
- What conditions should **K** satisfy, when for some logics Glivenko property holds relative to **K**?
- When Glivenko property holds for some logics relative to **K**, is there a smallest logic among logics for which Glivenko property holds relative to **K**? If so, what is the smallest one in case of classical logic?

We will answer all of these questions. An important notion here is an equivalence relation on substructural logics, called *Glivenko equivalence.* For each logic **L**, the equivalence block to which **L** belongs is convex, and moreover has both the greatest logic **M**(**L**) and the smallest **G**(**L**). Another important notion is *involutiveness.* Actually, there are three types of involutiveness. Corresponding to each of these three types of involutiveness, we can introduce three types of Glivenko properties, i.e. ordinary one, deductive one and equational one. Our theorem says that if **M**(**L**) is involutive then Glivenko property holds relative to **M**(**L**) for any logic in the equivalence block. We give also an explicit axiomatization of **G**(**L**) for classical logic **L**. This logic **G**(**L**) is shown to satisfy neither exchange rule nor weakening rule.