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Topological Conjugacies of Piecewise Monotone Interval Maps

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1. Introduction

A map $f : [0,1] \to [0,1]$ defines an one-dimensional dynamical system. We are interested in the asymptotic properties of its iterates and a central notion is that of the *orbit* of a point x, which is the set $O(x) = \{x, f(x), f^2(x), \ldots\}$. The ω*-limit* set of x is the set of accumulation points of $O(x)$ and is denoted by $\omega(x)$. An interesting question is when should we consider two dynamical systems $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ as being the same? There are many ways to answer this question depending on the features of dynamics we want to be preserved. According to the following definition two dynamical systems are the same if they have the same orbits.

Definition 1.1. Two dynamical systems $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ are called *topologically conjugate* if there exists a homeomorphism $h : [0,1] \to [0,1]$ (called *conjugacy*) such that

$$h \circ f = g \circ h.$$

In that case h maps f-orbit of x to g-orbit of $h(x)$. So if x is a periodic point of f, then $h(x)$ is a periodic point of g with the same period. Futhermore, if x is an attracting periodic point, so is $h(x)$.

The study of such topics has commenced with H. Poincaré in the 1880s. He considered homeomorphisms $f : S^1 \to S^1$ of the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ and showed that if f has no periodic points then there exists a unique rotation $R : S^1 \to S^1$ such that, for every $n \in \mathbb{N}$ and $x \in S^1$,

$$\{x, f^{-1}(x), \ldots, f^{-n}(x)\} \text{ and } \{x, R^{-1}(x), \ldots, R^{-n}(x)\}$$

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are ordered in the same way on the unit circle S^1. From the above result follows that there exists a continuous, surjective and monotone map $h : S^1 \to S^1$ (called *semi-conjugacy*) such that $h \circ f = R \circ h$. A. Denjoy (1932) showed that if f is a C^2 diffeomorphism, then h is injective hence it is a conjugacy.

Similar results for piecewise monotone maps $f : [0,1] \to [0,1]$ were proved later by W. Parry [Pa] and J. Milnor and W. Thurston (1977). So, if f is continuous, piecewise monotone with positive topological entropy $h(f)$, then there exists a piecewise linear map T with slope $\pm s = \exp(h(f))$ and a semiconjugacy h such that $h \circ f = T \circ h$. A semiconjugacy becomes conjugacy, if there are no attracting periodic points and no wandering intervals for f. The non existence of wandering intervals has been proved for a large class of functions satisfying some mild smoothness conditions ([BL] [Gu] [Lyu] [MMS]).

In this paper we consider the family $\mathcal{M}$ of functions which are piecewise monotone but not necessarily continuous. Particularly, $f : [0,1] \to [0,1]$ belongs in the family $\mathcal{M}$ if there exists a partition $0 = a_0 < a_1 < \ldots < a_r = 1$, $(r \geq 2)$ of [0,1] such that $f|(a_{i-1}, a_i)$ $(i = 1, 2, \ldots, r)$ is a monotone C^1 function and $f(a_{i-1}, a_i) = (0,1)$, $i = 1, 2, \ldots, r$. Futhermore, we assume that there is $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for almost every $x \in [0,1]$, in which case, f is called *expansive.* Our aim is to show that every $f \in \mathcal{M}$ is topologically conjugate to a map T which is linear and surjective on each interval $[\frac{i-1}{r}, \frac{i}{r}]$ $(i = 1, 2, \ldots, r)$. Next we prove a more general result. We consider a larger set of functions than $\mathcal{M}$ by replacing the condition $f(a_{i-1}, a_i) = (0,1)$, $i = 1, 2, \ldots, r$, with the following Markov condition: for every $i = 1, 2, \ldots, r$, there exist $l(i), r(i) \in \{0, 1, \ldots, r\}$ with $l(i) < r(i)$ such that $f(a_{i-1}, a_i) = (a_{l(i)}, a_{r(i)})$. We denote by $\mathcal{M}'$ this class of functions. In the last section, we consider the class $\mathcal{M}_\infty$ where $[0,1]$ accepts a countable partition accumulating to 1.

Convention: if $I \subset [0,1]$ is an interval, we denote $|I|$ its length.

2. Topological Conjugacies

When $f \in \mathcal{M}$ and $0 = a_0 < a_1 < \ldots < a_r = 1$ is the partition corresponding to f, we say that f is of *order* r. The points of the partition are called *critical points* of f. We denote by $I_1, \ldots, I_r$ the intervals of the partition, i.e. $I_j = (a_{j-1}, a_j)$. Also we denote by f_j the restriction of f to I_j. Note that every f_j is injective so there exist the inverse function f_j^{-1}. Finally we denote by $F_{j_1 j_2 \ldots j_k}$ the composition $f_{j_1}^{-1} \circ f_{j_2}^{-1} \circ \ldots \circ f_{j_k}^{-1}$. For every $x \in [0,1]$, $F_{j_1 j_2 \ldots j_k}(x)$ is the unique point which belongs to I_{j_1}.

Definition 2.1. Let $f \in \mathcal{M}$. An open interval $I \subset [0,1]$ is called a *branch* of f^n if $f^n|I$ is continuous, monotone and $f^n(I) = (0,1)$. The set of branches of f^n is denoted by $B_n(f)$.

For each $f \in \mathcal{M}$, we define the sets

$$\mathcal{C}_n(f) = \bigcup_{j=0}^{r} \bigcup_{i=0}^{n-1} f^{-i}(a_j), \quad n = 0, 1, \ldots \quad \text{and} \quad \mathcal{C}(f) = \bigcup_{j=0}^{r} \bigcup_{i=0}^{\infty} f^{-i}(a_j)$$

Frequently we write $\mathcal{C}_n$ and $\mathcal{C}$ instead of $\mathcal{C}_n(f)$ and $\mathcal{C}(f)$.

Lemma 2.1. *If $I \in B_n(f)$, then* $\text{cl}(I) \cap \mathcal{C}_n = \partial I$.

Proof. We suppose that this is not true. There are two cases:

(i) There exists a $x \in \text{int}(I) \cap \mathcal{C}_n$. Then $f^i(x) = a_j$, for some i, j. If f is not continuous at a_j, then $f^{n-j}|f^j(I) = f^n|I$ is not continuous, which is a contradiction since I is a branch of

f^n. If f is continuous at a_j, then there exists $\varepsilon > 0$ such that $f|(a_j - \varepsilon, a_j)$ and $f|(a_j, a_j + \varepsilon)$ are not of the same monotonicity type. Consequently, $f^{n-j}|f^j(I)$ is not monotone, which is a contradiction.

(ii) There exists a $x \in \partial I \setminus \mathcal{C}_n$. Then there exists an interval $T \supsetneq I$ such that $T \cap \mathcal{C}_n = \varnothing$. Then $f^n|T$ is monotone and, so, $f^{-n}(T) \supsetneq f^{-n}(I) = [0,1]$, which is a contradiction. □

From the previous lemma it follows that if $I \in B_n(f)$, then $f^k(I) \cap \mathcal{C}_1 = \varnothing$, for every $k = 0, 1, \ldots, n-1$. Thus, there exist intervals $I_{j_0}, I_{j_1}, \ldots, I_{j_{n-1}}$ in $B_1(f)$ such that $f^k(I) \subset I_{j_k}$, for $k = 0, 1, \ldots, n-1$.

Proposition 2.1. *If $f, g \in \mathcal{M}$ are of order r, s, respectively, then $f \circ g \in \mathcal{M}$ is of order rs.*

Proof. Let $0 = a_0 < a_1 < \ldots < a_r = 1$ and $0 = b_0 < b_1 < \ldots < b_s = 1$ be the partitions of f, g, respectively. We define $g_i = g|[b_{i-1}, b_i]$, $i = 1, \ldots, s$. Obviously, each g_i is invertible and $g_i(b_{i-1}, b_i) = (0,1)$. The set of points $\{g_i^{-1}(a_0), \ldots, g_i^{-1}(a_r)\}$ is a partition of $[b_{i-1}, b_i]$. Each of the intervals $[b_{i-1}, b_i]$ is devided into r subintervals and, hence, $[0,1]$ is devided into rs intervals. The endpoints of these intervals belong to $\cup_{j=0}^{r} \cup_{i=0}^{s} g_i^{-1}(a_j)$. Let z_1, z_2 be two successive points of $\cup_{j=0}^{r} \cup_{i=0}^{s} g_i^{-1}(a_j)$. Then there are unique k and m such that $\{z_1, z_2\} = \{g_k^{-1}(a_m), g_k^{-1}(a_{m+1})\}$. So, $(f \circ g)([z_1, z_2]) = [0,1]$. Futhermore, $f \circ g$ is monotone as a composition of the monotone maps $f|[a_m, a_{m+1}]$ and $g|[z_1, z_2]$. □

From the above proposition it follows that $f^n \in \mathcal{M}$, for each $f \in \mathcal{M}$. Futhermore, if f is of order r, then f^n is of order r^n and its critical points are in $\mathcal{C}_n$. In the sequel we shall introduce some notions from symbolic dynamic. To each point x of $\mathcal{C}$, we correspond a sequence of symbols which is related with the order of the points of $O(x)$. An interesting notion in symbolic dynamics is the *shift map* σ: if $\underline{x} = \{x_n\}_{n=0}^{\infty}$, then $\sigma(\underline{x}) = \underline{y}$, where $\underline{y} = \{x_n\}_{n=1}^{\infty}$. Inductively, we have $\sigma^k(\underline{x}) = \{x_n\}_{n=k}^{\infty}$.

Definition 2.2. The *itinerary* of $x \in \mathcal{C}$ with respect to $f \in \mathcal{M}$ is a sequence $\underline{i}_f(x) = \{i_n(x)\}_{n=0}^{\infty}$, where

$$i_n(x) = \begin{cases} j, & \text{if } f^n(x) \in I_j, \\ \frac{2j+1}{2}, & \text{if } f^n(x) = a_j. \end{cases}$$

To each $f \in \mathcal{M}$ of order r, we associate a subset of $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r, \frac{2r+1}{2}\}^{\mathbb{N}}$. We describe this set in the following definition.

Definition 2.3. Let $f \in \mathcal{M}$ with partition $0 = a_0 < a_1 < \ldots < a_r = 1$. Then $\Sigma(f) = \{\underline{a} : \underline{a} = \{x_n\}_{n=0}^{\infty}\}$ is a set of sequences with entries from the set $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r, \frac{2r+1}{2}\}$ with the following restriction. Let $\underline{a} \in \Sigma(f)$, then there exists an entry x_n of the form $\frac{2k+1}{2}$, where $k = 0, 1, \ldots, r$; moreover, if x_n is the first entry of $\underline{a}$ with this property, then

$$\sigma^{n+1}(\underline{a}) = \begin{cases} \underline{i}_f(0), & \text{if } f(a_k) = 0, \\ \underline{i}_f(1), & \text{if } f(a_k) = 1. \end{cases}$$

It is possible to define an order on the set $\underline{i}_f(\mathcal{C})$ which is consistent with the natural order of real numbers. Two sequences of symbols $\underline{x} = \{x_n\}_{n=0}^{\infty}$ and $\underline{y} = \{y_n\}_{n=0}^{\infty}$ belonging to $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r, \frac{2r+1}{2}\}^{\mathbb{N}}$ are called to have *discrepancy* n if $x_i = y_i$, for $i = 0, 1, \ldots, n-1$, and $x_n \neq y_n$. If the itineraries of two points of $\mathcal{C}$ have discrepancy n, then the first n points of their orbits are visiting simultaneously the same intervals of $B_1(f)$. Moreover, we define $\frac{1}{2} \prec 1 \prec \frac{3}{2} \prec \ldots \prec r \prec \frac{2r+1}{2}$.

Definition 2.4. Let $f \in \mathcal{M}$ and $x, y \in \mathcal{C}$ with $x \neq y$. We assume that itineraries $\underline{i}_f(x)$ and $\underline{i}_f(y)$ have discrepancy n and that f is decreasing in k common intervals

(i) Let $k =$ even, then $\underline{i}_f(x) \prec \underline{i}_f(y)$ iff $i_n(x) \prec i_n(y)$.

(ii) Let $k =$ odd, then $\underline{i}_f(x) \prec \underline{i}_f(y)$ iff $i_n(y) \prec i_n(x)$.

Lemma 2.2. *Let $f \in \mathcal{M}$ be of order r and let $x, y \in \mathcal{C}$ with $x \neq y$. Then $\underline{i}_f(x) \prec \underline{i}_f(y)$ iff $x < y$.*

Proof. We assume that itineraries $\underline{i}_f(x)$ and $\underline{i}_f(y)$ have discrepancy n. That is, $i_k(x) = i_k(y) = j_k$, for $k = 0, 1, \dots, n-1$, and $i_n(x) \neq i_n(y)$. We observe that $j_0, j_1, \dots, j_{n-1}$ are not of the form $\frac{2s+1}{2}$. To prove this, we assume the contrary. Then $\underline{i}_f(x) = \underline{i}_f(y)$, which is a contrdiction since $i_n(x) \neq i_n(y)$. From Definition 2.2, x, y belong to I_{j_0} and successively visit the intervals $I_{j_1}, \dots, I_{j_{n-1}}$. So, we can write $x = F_{j_0 j_1 \dots j_{n-1}}(f^n(x))$ and $y = F_{j_0 j_1 \dots j_{n-1}}(f^n(y))$. We assume that f is decreasing in k of the intervals $I_{j_0}, I_{j_1}, \dots, I_{j_{n-1}}$. There are two cases:

(i) Let $k =$ even, then $F_{j_0 j_1 \dots j_{n-1}}$ is increasing. Assume that $\underline{i}_f(x) \prec \underline{i}_f(y)$, then from Definition 2.4 we have $i_n(x) \prec i_n(y)$. This means that $f^n(x) < f^n(y)$ and, hence, $x = F_{j_0 j_1 \dots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \dots j_{n-1}}(f^n(y))$.

(ii) Let $k =$ odd, then $F_{j_0 j_1 \dots j_{n-1}}$ is decreasing. Assume that $\underline{i}_f(x) \prec \underline{i}_f(y)$, then from Definition 2.4 we have $i_n(y) \prec i_n(x)$. This means that $f^n(x) > f^n(y)$ and, hence, $x = F_{j_0 j_1 \dots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \dots j_{n-1}}(f^n(y))$. $\square$

Lemma 2.3. *Let $f \in \mathcal{M}$ be of order r. The map $\underline{i}_f : \mathcal{C} \to \Sigma(f)$ is a bijection.*

Proof. Let $x, y \in \mathcal{C}$ with $\underline{i}_f(x) = \underline{i}_f(y)$. Let k, m be the minimal integers for which $f^k(x), f^m(y)$ are critical points of f. Assume that $k \neq m$ (let $k < m$). Since $f^k(x)$ is a critical point, then $f^{k+1}(x) = 0$ or 1, and, so, $i_{k+1}(x) = \frac{1}{2}$ or $\frac{2r+1}{2}$. From the other hand, $i_k(y) = 1, 2, \dots, r$, and, hence, $i_{k+1}(y) \neq \frac{1}{2}$ and $i_{k+1}(y) \neq \frac{2r+1}{2}$, which is a contradiction, since $i_{k+1}(x) = i_{k+1}(y)$. So, $k = m$. Futhermore, we observe that $f^k(x) = f^k(y)$, since $i_k(x) = i_k(y)$ and it is of the form $\frac{2j+1}{2}$. Consequently, $f^k(x) = f^k(y) = a_j$.

Assume that $i_n(x) = i_n(y) = j_n \in \mathbb{N}$, for $n = 0, 1, \dots, k-1$. From Definition 2.2, x, y belong to I_{j_0} and successively visit the intervals $I_{j_1}, \dots, I_{j_{k-1}}$. So, we can write $x = F_{j_0 j_1 \dots j_{k-1}}(f^k(x))$ and $y = F_{j_0 j_1 \dots j_{k-1}}(f^k(y))$. Since $f^k(x) = f^k(y)$, we have $x = y$. Thus, $\underline{i}_f$ is injective.

Let $\underline{a} = \{x_n\} \in \Sigma(f)$. We shall show that there exists a $x \in \mathcal{C}$ such that $\underline{i}_f(x) = \underline{a}$. From Definition 2.3, an entry of the sequence $\underline{a}$ is of the form $\frac{2k+1}{2}$. Let x_n be the first entry with this property. Then $x = F_{x_0 x_1 \dots x_{n-1}}(a_k)$ satisfies the desired property. $\square$

Proposition 2.2. *Let $f \in \mathcal{M}$ be of order r. Then $\mathcal{C}$ is dence in $[0, 1]$.*

Theorem 2.1. *Let $f \in \mathcal{M}$ be of order r with partition $0 = a_0 < a_1 < \dots < a_r = 1$. We consider the map $T \in \mathcal{M}$ with partition $0 =< \frac{1}{r} < \frac{2}{r} < \dots < \frac{r-1}{r} < 1$, which is linear in each interval $[\frac{i-1}{r}, \frac{i}{r}]$ and $T(\frac{i-1}{r}, \frac{i}{r}) = (0, 1)$. Futhermore, $T|[\frac{i-1}{r}, \frac{i}{r}]$ is of the same monotonicity with $f|[a_{i-1}, a_i]$ and it is continuous, continuous from the right or continuous from the left at $\frac{i}{r}$, when f is continuous, continuous from the right or continuous from the left at a_i respectively. Then f and T are topologically conjugate.*

3. The Linear Case

In this section we shall restrict to the maps of $\mathcal{M}$ which are piecewise linear. We will not assume that the intervals of the partition have the same length. More precisely we consider functions for which there exists a partition $0 = a_0 < a_1 < \ldots < a_r = 1$ such that $f_j(x) = \frac{x-a_{j-1}}{a_j-a_{j-1}}$ or $f_j(x) = \frac{x-a_j}{a_{j-1}-a_j}$. We shall show that it is possible to estimate the length of the intervals of the partition of f^n, for each $n \geq 1$.

Lemma 3.1. *Let $f \in \mathcal{M}$ be of order r and $I \subset [0,1]$. Then $|f_j^{-1}(I)| = |I||I_j|$.*

Proof. Let $I = [x_1, x_2]$, then $f_j^{-1}(I) = [(a_j - a_{j-1})x_1 + a_{j-1}, (a_j - a_{j-1})x_2 + a_{j-1}]$ or $f_j^{-1}(I) = [a_j - (a_j - a_{j-1})x_2, a_j - (a_j - a_{j-1})x_1]$, if f_j is increasing or decreasing, respectively. In both cases, $|f_j^{-1}(I)| = (a_j - a_{j-1})(x_2 - x_1) = |I_j||I|$. □

Proposition 3.1. *Let $f \in \mathcal{M}$ be of order r and $I \in B_n(f)$. Assume that $I_{j_0}, I_{j_1}, \ldots, I_{j_{n-1}}$ are intervals of $B_1(f)$ such that $f^k(I) \subset I_{j_k}$, for $k = 0, 1, \ldots, n-1$. Then*

$$|I| = \prod_{i=1}^{r} |I_i|^{n_i},$$

where n_i is the number of I_i's in the sequence of intervals $I_{j_0}, I_{j_1}, \ldots, I_{j_{n-1}}$.

A consequence of the above proposition is that we can calculate the points of $\mathcal{C}_n$. Let $I \in B_n(f)$. Then according to the observation that comes after Lemma 2.1, there exist numbers $j_0, j_1, \ldots, j_{n-1}$ such that $f^k(I) \subset I_{j_k}$, for $k = 0, 1, \ldots, n-1$. These number are the first n elements of itineraries of I. With these numbers we can order the intervals $J_1, J_2, \ldots, J_{r^n}$ of $B_n(f)$ according to Lemma 2.2. If $0 = c_0 < c_1 < c_2 < \ldots < c_{r^n} = 1$ are the points of $\mathcal{C}_n$, then

$$c_k = \sum_{i=1}^{k} |J_i|, \qquad k \geq 1.$$

4. Markov Maps

An important difference between the families of functions $\mathcal{M}$ and $\mathcal{M}'$ is that if $f \in \mathcal{M}'$ then f_i^{-1} is not defined, for all $x \in (0,1)$. Consequently, $F_{j_1 j_2 \ldots j_k}$ is not defined for a random choice of $j_1, j_2, \ldots, j_k$. So we modify Definition 2.3.

Definition 4.1. Let $f \in \mathcal{M}'$ with partition $0 = a_0 < a_1 < \ldots < a_r = 1$. We define the set of sequences $\Sigma'(f) = \{\underline{a} : \underline{a} = \{x_n\}_{n=0}^{\infty}\}$ with entries from the set $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r, \frac{2r+1}{2}\}$ which satisfy the following conditions.

(i) Let $\underline{a} = \{x_n\} \in \Sigma'(f)$. Then there exists an entry x_n of $\underline{a}$ of the form $\frac{2k+1}{2}$, where $k = 0, 1, \ldots, r$. Futhermore, if x_N is the first entry of $\underline{a}$ with this property, then $\sigma^N(\underline{a}) = \underline{i}_f(a_k)$.

(ii) If $n < N-1$ and $x_n = j$, then $l(j) + 1 \leq x_{n+1} \leq r(j)$.

Observe that if $\underline{a} = \{x_n\} \in \Sigma'(f)$, then there exists $x \in \mathcal{C}$ such that $\underline{i}_f(x) = \underline{a}$. Hence, a proposition similar to Lemma 2.3, holds, i.e., if $f \in \mathcal{M}'$, then $\underline{i}_f : \mathcal{C} \to \Sigma'(f)$ is a bijection.

Theorem 4.1. *Let $f \in \mathcal{M}'$ be of order r with partition $0 = a_0 < a_1 < \ldots < a_r = 1$. We consider the map $T \in \mathcal{M}'$ with partition $0 =< \frac{1}{r} < \frac{2}{r} < \ldots < \frac{r-1}{r} < 1$ which is linear in each interval $[\frac{i-1}{r}, \frac{i}{r}]$ and $T(\frac{i-1}{r}, \frac{i}{r}) = (\frac{l(i)}{r}, \frac{r(i)}{r})$. Futhermore, $T|[\frac{i-1}{r}, \frac{i}{r}]$ is of the same monotonicity with $f|[a_{i-1}, a_i]$ and it is continuous, continuous from the right or continuous from the left at $\frac{i}{r}$, when f is continuous, continuous from the right or continuous from the left at a_i respectively. Then f and T are topologically conjugate.*

5. Maps of Countable Order

In previous sections we studied functions with a finite partition. Here we study a special class of functions with countable partition. Some modifications are necessary.

Definition 5.1. A map $f : [0,1] \to [0,1]$ belongs to the class of functions $\mathcal{M}_\infty$ if there exists a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$ with $0 = a_0 < a_1 < a_2 < \dots$ and $\lim_{n\to\infty} a_n = 1$ such that:

(i) f is C^1 and monotone on each interval (a_{i-1}, a_i) of the partition.
(ii) For every $i \in \mathbb{N}^*$, there exist unique $l(i), r(i) \in \mathbb{N}$ such that $f(a_{i-1}, a_i) = (a_{l(i)}, a_{r(i)})$.
(iii) There exists $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for all $x \in [0,1]$.

In this case $\mathcal{C}(f) = \cup_{j=0}^{\infty} \cup_{i=0}^{\infty} f^{-i}(a_j)$.

Definition 5.2. Let $f \in \mathcal{M}_\infty$ with partition $0 = a_0 < a_1 < a_2 < \dots < 1$. We define the set of sequences $\Sigma_\infty(f) = \{\underline{a} : \underline{a} = \{x_n\}_{n=0}^{\infty}\}$ with entries from $\{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ which satisfy the following conditions.

(i) Let $\underline{a} = \{x_n\} \in \Sigma_\infty(f)$. Then there exists an entry x_n of $\underline{a}$, of the form $\frac{2k+1}{2}$, where $k = 0, 1, \dots$. Futhermore if x_N is the first entry of $\underline{a}$ with this property, then $\sigma^N(\underline{a}) = \underline{i}_f(a_k)$.

(ii) If $n < N - 1$ and $x_n = j$, then $l(j) + 1 \leq x_{n+1} \leq r(j)$.

Theorem 5.1. *Let $f \in \mathcal{M}_\infty$ with partition $0 = a_0 < a_1 < a_2 < \dots < 1$. We consider the map $T \in \mathcal{M}_\infty$ with partition $0 < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \dots < 1$ which is linear in each interval $[\frac{i-1}{i}, \frac{i}{i+1}]$ and $T(\frac{i-1}{i}, \frac{i}{i+1}) = (\frac{l(i)}{r}, \frac{r(i)}{r})$. Futhermore, $T|[\frac{i-1}{i}, \frac{i}{i+1}]$ is of the same monotonicity with $f|[a_{i-1}, a_i]$ and it is continuous, continuous from the right or continuous from the left at $\frac{i}{i+1}$, when f is continuous, continuous from the right or continuous from the left at a_i respectively. Then f and T are topologically conjugate.*

References

[BL] A. Blokh, M. Lyubich, *Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems II. The smooth case*, Ergod. Th. & Dynam. Sys. **9** (1989), 751-758.

[Gu] J. Guckenheimer, *Sensitive dependence to initial conditions for one dimensional maps*, Commun. Math. Phys. **70** (1979), 133-160.

[Lyu] M. Lyubich, *Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems I. The case of negative Schwarzian derivative*, Ergod. Th. & Dynam. Sys. **9** (1989), 737-750.

[MMS] M. Martens, W. de Melo, S. van Strien, *Julia - Fatou - Sullivan theory for real one dimensional dynamics*, Acta Math. **168** (1992), 273-318.

[MS] W. de Melo, S. van Strien, *One Dimensional Dynamics*, A Series of Mondern Surveys in Mathematics **25**, Springer Verlag, Berlin (1993).

[Pa] W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. A.M.S. **122** (1966), 368-378.