SUMMARY

In this work, we consider first the problem of finding necessary and sufficient conditions for the singular differential equation (S.D.E.)

$$z^2 \frac{dy(z)}{dz} + a(z)y(z) = b(z), \tag{1}$$

where $a(z) = \sum_{n=0}^{\infty} a_n z^n$, $b(z) = \sum_{n=0}^{\infty} b_n z^n$ are functions analytic in some neighborhood of zero, to have solutions in the space $H_2(\Delta)$ i.e. the Hilbert space of functions

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $\sum_{n=1}^{\infty} |f_n|^2 < +\infty$.

The method we follow reduces the above problem to the problem of finding the kernel of a bounded non self-adjoint Fredholm operator A, defined on an abstract separable Hilbert space H. The method gives easily results concerning the dimension of the kernel of the operator A and solves exactly the problem when the coefficients b_n, $n = 1, 2, ...$ of the function:

$$h(z) = exp(a_2 z + a_3 \frac{z^2}{2} + a_3 \frac{z^3}{3} + ...)b(z) = \sum_{n=1}^{\infty} h_n z^{n-1}$$

can be exactly determined.

The method gives also easily results concerning the dimension of the null space of the operator which corresponds to the more general case:

$$z^m \frac{dy(z)}{dz} + a(z)y(z) = b(z), \quad m \geq 2 \tag{2}$$

and solves exactly the problem in some special cases with respect to the form of the function $a(z)$. In general we localize the computational difficulties.

Also in this work we examine the existence of solutions in $H_2(\Delta)$ of the functional differential equation:

$$z^2 y''(z) + zp(z)y'(z) + q(z)y(z) + z\sum_{i=1}^{n} a_i(z)y(q^i z) = 0, \quad |q| \leq 1 \tag{3}$$

and as a particular case we obtain the well known theory of G. Frobenius for the Fuchs differential equation of second order.

An important application of the conditions for the existence of solutions in $H_2(\Delta)$ is the study of the zeros of the ordinary Bessel functions $J_\mu(z)$.

We show that if $\varrho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ of order μ then $\frac{2}{\varrho}$ is an eigenvalue of a compact operator A_μ defined on an abstract separable Hilbert space H. In the case that μ is real and $\mu > -1$ this operator is similar to a self-adjoint compact operator S_μ. This leads to a simple proof of the well known Lommel-Hurwitz theorem concerning the zeros of the Bessel functions.

Some properties of the above operators are discussed and some relations concerning the zeros of Bessel functions are found.

Also we find a new lower bound for the zeros of the ordinary Bessel functions $J_\mu(z)$.

This bound is proved to be much better from the bounds found previously by several authors for the first zero of $J_\mu(z)$.

Finally we apply the Ritz approximation method for the derivation of the greatest positive eigenvalue of the operator S_μ and therefore, the first positive zero of $J_\mu(z)$.