6

Optimal Rational Approximation Number Sets: Application to Nonlinear Dynamics in Particle Accelerators

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In Honor of Constantin Carathéodory

Abstract We construct optimal multivariate vectors of rational approximation numbers with common denominator and whose coordinate decimal expansion string of digits coincides with the decimal expansion digital string of a given sequence of mutually irrational numbers as far as possible. We investigate several numerical examples and we present an application in Nuclear Physics related to the beam stability problem of particle beams in high-energy hadron colliders.

1 Introduction

A central problem in number theory is how to construct "optimal" rational approximants to irrational numbers [49]. In spite of its simple formulation, the fraction that is closer to an irrational number than any other rational approximant with a smaller denominator depends strongly on the denominator of the convergent of the continued fraction expansion of the irrational number [28], and thus, from a numerical point of view, one might expect that the "optimal" rational approximants lack most of their practical usefulness appeal.

There is another independent reason advocating for this expectation. It is perhaps surprising that the "innocent" generalization of this problem to the *simultaneous rational approximants to several mutually irrational numbers* is considered a

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difficult, essentially unsolved problem in number theory. The precise generalization of rational approximants to a single irrational number is to define a sequence:

$$\left\{\sigma^{(k)} = \left(\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots, \sigma_n^{(k)}\right) \in \mathbb{Q}^n, \;\; k = 1, 2, \ldots\right\}$$

of ordered sets of n rational numbers:

$$\sigma_j^{(k)} = \left(p_j^{(k)}/r^{(k)}\right), \;\; j = 1, 2, \ldots, n,$$

each set with *common* denominator $r^{(k)} \in \mathbb{Z} \setminus \{0\}$, which converges to a given n-vector of mutually irrational numbers $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n$. In this direction, S. Kim and S. Ostlund gave ordered sets of two rational approximants to pairs of mutually irrational numbers [29]. The ordered sets of two rational approximants generated by their algorithm are in fact the best ordered pairs of rational approximants relative to a criterion of weak convergence [29]. However, their algorithm does not always give "optimal" simultaneous rational approximation to mutually irrational numbers for any $n > 2$. For $n \geqslant 2$, the only well-known efficient approximation method reveals the Jacobi–Perron classical algorithm (JPA) [7]. Under general enough circumstances, this inductive algorithm generates sequences of optimal n-vectors containing mutually rational numbers with common denominator and approximating the given n-vector of irrational numbers. The approximation method is convergent, but the resultant construction depends strongly on the associated function defining the algorithm's transformation. So, numerators and (common) denominator in Jacobi–Perron rational approximation n-vectors are completely determined by this function, and no freedom is left.

The principal aim of the paper at hand is to show how rational approximation theory can be cleared of its strong dependence on "optimal" approximants and reconnected to original ideas of *numerical approximation*. To do so, we will investigate multivariate vectors of rational approximation numbers—the so-called Optimal Rational Approximation NUmber Sets or simply ORANUS—whose decimal expansion string of digits coincides with the decimal expansion digital string of mutually irrational numbers as far as possible. More precisely, we will look at a direct numerical construction of simultaneous rational approximations with *arbitrary common denominator*. *The advantage of these approximants over Jacobi–Perron approximants lies in the completely free choice of the common denominator which may lead to a better approximation.*

The paper is organized as follows. Section 2 gives a concise overview of JPA's classical applications. Section 3 recalls basic results of rational approximation to analytic functions, while Sect. 4 develops and analyzes the multivariate rational approximation method of the paper. In Sects. 5 and 6, we study the efficacy of the method, and in Sect. 7 we give an application related to the beam stability problem in circular particle accelerators. Finally, Sect. 8 summarizes and gives concluding remarks.

2 The Jacobi–Perron Algorithm

Let $\langle a^{(k)}\rangle \equiv \left(a^{(0)}, a^{(1)}, \ldots, a^{(k)}, \ldots\right)$ be a sequence of vectors in $\mathbb{R}^n$. Let also $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be any mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that

$$\Phi\left(a^{(k)}\right) = b^{(k)} = \left(b_1^{(k)}, b_2^{(k)}, \ldots, b_n^{(k)}\right) \Longrightarrow a_1^{(k)} \neq b_1^{(k)} \quad (k = 0, 1, 2, \ldots).$$

Definition 1.

(i) The sequence $\langle a^{(k)}\rangle$ is called a *Jacobi–Perron algorithm* (in short JPA) of the vector $a^{(0)} \in \mathbb{R}^n$, if there exists a T-transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that:

(a) $T\left(a^{(k)}\right) = a^{(k+1)}$ and

(b) $T\left(a^{(k)}\right) = \left(a_1^{(k)} - b_1^{(k)}\right)^{-1}\left(a_2^{(k)} - b_2^{(k)}, \ldots, a_n^{(k)} - b_n^{(k)}, 1\right)$.

In such a case, we shall call the sequence $\left\langle\Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)}\rangle$ a *T-function*.

(ii) The T-function $\left\langle\Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)}\rangle$ is said to be *P-bounded*, if there is a constant C independent of k and satisfying Perron's conditions $0 < 1/b_n^{(k)} \leqslant C$ and $0 \leqslant b_i^{(k)}/b_n^{(k)} \leqslant C$ (for any $i = 1, 2, \ldots, n$ and $k = 0, 1, \ldots$).

(iii) Define numbers $d_i^{(j)}$ as follows:

$$d_i^{(j)} = \begin{cases} d_i^{(j)} = \delta_{i,j}\,, & i, j = 0, 1, \ldots, n \\ d_i^{(n+1+k)} = \displaystyle\sum_{j=0}^{n} b_j^{(k)} d_i^{(k+j)} \ (b_0^{(k)} = 1), & i = 0, 1, \ldots, n;\ k = 0, 1, \ldots \end{cases}$$

where $\delta_{i,j}$ denotes the Krönecker's delta. Then:

(a) The JPA of the vector $a^{(0)}$ is said to be *convergent*, if

$$a_i^{(0)} = \lim_{k\to\infty} d_i^{(k)}/d_0^{(k)},$$

whenever $i = 1, 2, \ldots, n$.

(b) The JPA of the vector $a^{(0)}$ is said to be *ideally convergent*, if the sequences $\left\langle d_i^{(k)} - a_i^{(0)} d_0^{((k)}\right\rangle$ $(i = 1, 2, \ldots, n)$ are all null sequences. □

Notation 1. *It is clear that a JPA which is ideally convergent is also convergent if and only if* $\left|d_0^{(j)}\right| > 1$, *for any* $j > j_0$. *It follows from* $\left|d_i^{(k)} - a_i^{(0)} d_0^{(k)}\right| < \varepsilon$ *for* $j > j_0(\varepsilon)$ *that*

$$\left|a_i^{(0)} - \frac{d_i^{(k)}}{d_0^{(k)}}\right| < \frac{\varepsilon}{\left|d_0^{(k)}\right|} < \varepsilon \quad (i = 1, 2, \ldots, n). \qquad \square$$

Example 1. Let us consider the special case:

$$\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right] := \left(\left[a_1^{(k)}\right], \left[a_2^{(k)}\right], \ldots, \left[a_n^{(k)}\right]\right), \quad k = 0, 1, \ldots,$$

where $[x]$ denotes the integer part of x. For $n = 2$, the JPA with the T-function $\Phi\left(a^{(k)}\right) = b^{(k)} \equiv \left[a^{(k)}\right]$ becomes the Euclidean algorithm and yields the expansion of any real number by simple continued fractions. If the JPA of a vector $a^{(0)} \in \mathbb{R}^n$ is associated with the T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$, then $\left|d_0^{(j)}\right| > 1$ for any $j \geqslant n+1$ so that ideal convergence here always implies convergence. But, as the reader can easily verify, the JPA of a $a^{(0)} \in \mathbb{R}^n$ with the associated T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$ is always convergent, since in this case $a_n^{(k)} = \left(a_1^{(k-1)} - b_1^{(k-1)}\right)^{-1}$ $(k = 1, 2, \ldots)$ so that $b_n^{(k)} \geqslant 1$, and it is also easily verified that $0 \leqslant \left(b_i^{(k)}/(b_n^{(k)}\right) < 1$ for any $i = 1, 2, \ldots, n$ and $k = 0, 1, 2.\ldots$ But we can also always achieve that $b_n^{(0)} \geqslant 1$; thus, if $\left[a_n^{(0)}\right] = -l < 0$, then, substituting $a_n^{(0)\prime} = a_n^{(0)} + l + 1$, we obtain $b_n^{(0)\prime} = 1$; the same holds for $a_i^{(0)}$ $(i = 1, 2, \ldots, n-1)$. Since the JPA of $a^{(0)} \in \mathbb{R}^n$ with the T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$ is convergent, we obtain for a rational approximation of the $a_i^{(0)}$:

$$a_i^{(0)} = \lim_{k\to\infty} \frac{d_i^{(k)}}{d_0^{(k)}}, \quad \text{whenever} \quad i = 1, 2, \ldots n. \qquad \square$$

With the notation of Definition 1, the main convergence criterion of JPA can be stated as follows.

Theorem 1 ([7]). *The JPA of the vector* $a^{(0)} \in \mathbb{R}^n$ *is convergent if its* T*-function* $\left\langle\Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)}\rangle$ *is P-bounded.* $\square$

3 Rational Approximation to Analytic Functions

Let $F(z) = \sum_{v=0}^{\infty} a_v^{(F)} z^v$ be a function analytic in the open disk $\Delta(0;\varrho) = \{z \in \mathbb{C} : |z| < \varrho\}$, and let Λ_F be the $\mathbb{C}$-linear defined on the space $\mathbb{P}(\mathbb{C})$ of all analytic polynomials by $\Lambda_F(x^v) := a_v^{(F)}$ $(v = 0, 1, 2, \ldots)$. By density, Λ_F extends on the space $\mathcal{O}(\overline{\Delta(0;[1/\varrho])})$ of the functions which are analytic in an open neighborhood of the closed disk $\overline{\Delta(0;[1/\varrho])}$ and holds that $F(z) = \Lambda_F\big((1-xz)^{-1}\big)$ for any $z \in \Delta\,(0;\varrho)$. If $p_k(x,z)$ is the unique Hermite polynomial with degree at most k, which interpolates $(1-xz)^{-1}$ at the $(k+1)$ points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$, then the expression $\Lambda_F(p_k(x,z))$ is a function with numerator and denominator degrees at most k and $k+1$, respectively. In fact, by setting $V_{k+1}(x) = \gamma \prod_{i=0}^{k}(x - \pi_i)$

($\gamma \in \mathbb{C} \setminus \{0\}$), it is easily seen that $W_k(z) = \varLambda_F([V_{k+1}(x) - V_{k+1}(z)] / [x - z])$ is a polynomial in z of degree at most k, and we obtain

$$\varLambda_F(p_k(x,z)) = \left\{\widehat{W}_k(z)/\widehat{V}_{k+1}(z)\right\} := \left\{z^k W_k(z^{-1})/z^{k+1} V_{k+1}(z^{-1})\right\}.$$

This rational function, denoted by $(k/(k+1))_F(z)$, is characterized by the property that $F(z) - (k/(k+1))_F(z) = O\left(z^{k-1}\right)$ (as $z \to 0$) and is known as a *Padé-type approximant* to the Taylor series $\sum_{v=0}^{\infty} a_v^{(F)} z^v$, whereas every polynomial $V_{k+1}(x)$ is called a *generating polynomial* of this approximation [12–15, 25].

Remark 1. It is possible to construct Padé-type approximants with various degrees in the numerator and in the denominators [12, 13]. □

A natural problem connected with the choice of the generating polynomials is the convergence of such a sequence of rational approximants. It has been completely solved by M. Eiermann.

Theorem 2 ([18, 24]). *If the generating polynomials $V_k(x)$ satisfy*

$$\lim_{k\to\infty} \left[V_k(x)/V_k\left(z^{-1}\right)\right] = 0$$

uniformly on any compact subset of an open set $\Omega \subset \mathbb{C}^2$ containing $\left\{(x,0) : x \in \mathbb{C}\right\}$, then there holds

$$\lim_{k\to\infty} (k/(k+1))_F(z) = F(z),$$

uniformly on every compact subset of

$$\left\{z \in \mathbb{C} : \lim_{k\to\infty} \left[V_k\left(\xi^{-1}\right)/V_k\left(z^{-1}\right)\right] = 0, \quad \forall\, \xi \in \mathbb{C} \setminus \varDelta(0;\varrho)\right\}. \qquad \square$$

A second reasonable question concerns the "optimal" choice of the interpolation points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$ (equivalently, of the poles of the rational approximants). Some attempts to solve this problem have been made by Magnus [31]. A general answer is given in the following results.

Theorem 3 ([19]). *Let $z \in \varDelta(0;\varrho) \setminus \{0\}$ and let k be a positive integer. If $\tilde{p}_k(x,z)$ is the unique polynomial of degree at most k, which interpolates $(1 - xz)^{-1}$ at the $(k+1)$ roots $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_k \in \mathbb{C}$ of the generating polynomial $\widetilde{V}_{k+1}(x) = \widetilde{V}_{k+1}^{(z)}(x) = x^{k+1} + 1/z(z^k - 1)x^k$, then*

(i) $|F(z) - \varLambda_F(\tilde{p}_k(x,z))| \leqslant |F(z) - \varLambda_F(p_k(x,z))|$, for any Hermite polynomial $p_k(x,z) \in \mathbb{P}(\mathbb{C})$ in x and any function F analytic in the open disk $\varDelta(0;\varrho)$.
(ii) If moreover k is even and $0 < \varepsilon < \delta < \varrho$, then

$$\left\|F(z)-\Lambda_F(\tilde{p}_k(x,z))\right\|_2^{\delta,\varepsilon} \leqslant \left\|F(z)-\Lambda_F(p_k(x,z))\right\|_2^{\delta,\varepsilon},$$

for any Hermite polynomial $p_k(x,z) \in \mathbb{P}(\mathbb{C})$ *in* x *and any function* F *analytic in the open disk* $\Delta(0;\varrho)$. *Here, we have used the notation* $\left\|g(z)\right\|_2^{\delta,\varepsilon} := \left(\int_{\delta\leqslant|z|<\varepsilon}|g(z)|^2\,dz\right)^{1/2}$. □

In spite of these results, there is no analogous possibility to determine an "optimal" uniform choice for the interpolation system $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$, since a minimum for the uniform norm:

$$\left\|F(z)-(k/(k+1))_F\,(z)\right\|_\infty^{\delta,\varepsilon} := \sup_{\delta\leqslant|z|<\varepsilon}\left|F(z)-(k/(k+1))_F\,(z)\right|,$$

of the error on a compact ring $\overline{\Delta(0;\delta,\varepsilon)} = \{z\in\mathbb{C}:\ \delta\leqslant|z|\leqslant\varepsilon\}$ is obtained at the limit points $\pi_v=\infty$ $(v=0,1,\ldots,k-i)$ and $\pi_v=0$ $(v=k-i+1,\ldots,k)$ for any $i=0,1,\ldots,k+1$. In particular, the only feasible optimal interpolation system is given by $\pi_0=\pi_1=\cdots=\pi_k=0$ [19].

Remark 2. Another way to indicate "optimal" choice of the interpolation points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$ is by exactness properties of formal orthogonal polynomials [1–4, 6, 9, 11, 16, 20–23, 27, 30, 32, 33, 38, 41, 48, 50, 51]. □

4 ORANUS

In this section, we will show how to construct simultaneous rational approximants to several mutually irrational numbers.

Let $a=(a_1,a_2,\ldots,a_n)\in\mathbb{A}^n$ be an irrational n-vector. Suppose that the irrational coordinates a_j of a are expressed in the decimal system:

$$a_j = a_j^{(0)}.a_j^{(1)}a_j^{(2)}a_j^{(3)}\ldots a_j^{(m)}\ldots = a_j^{(0)} + \left(a_j^{(1)}/10\right)+\cdots+\left(a_j^{(m)}/10^m\right)+\cdots$$

$(a_j^{(0)}\in\mathbb{N}$ and $0\leqslant a_j^{(v)}\leqslant 9$ whenever $v=1,2,\ldots$ and $j=1,2,\ldots,n)$. The associated power series with integral coefficients

$$f_j(z)=\sum_{v=0}^{\infty}a_j^{(v)}z^v = a_j^{(0)}+a_j^{(1)}z+a_j^{(2)}z^2+\cdots+a_j^{(m)}z^m+\cdots$$

converges uniformly on any compact subset of the open unit disk $\Delta(0;1)$.

We will approximate $f_j(z)$ by using rational approximants. To do so, let us consider the $\mathbb{C}$-linear functional defined on the space of all analytic polynomials by $\Lambda_{f_j}(x^v) := a_j^{(v)}$ whenever $v=0,1,\ldots$ If $V_{k+1}(x)=\gamma\prod_{v=0}^{k}(x-\pi_v)$ $(\gamma\in\mathbb{C}\setminus\{0\})$ for some suitably chosen complex numbers $\pi_0,\pi_1,\ldots,\pi_k$, the rational map:

$$\big((k/(k+1))_{f_1}(z),\ldots,(k/(k+1))_{f_n}(z)\big) = \left(\frac{\widehat{W}_k^{(1)}(z)}{\widehat{V}_{k+1}(z)},\ldots,\frac{\widehat{W}_k^{(n)}(z)}{\widehat{V}_{k+1}(z)}\right),$$

is a *vector rational* approximant to the map $\big(f_1(z), f_2(z), \ldots, f_n(z)\big)$ with generating polynomial $V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v)$. In particular, for $z = 10^{-1}$, we obtain an ordered set of n rational numbers:

$$\begin{aligned}&\big((k/(k+1))_{f_1}(10^{-1}),\ldots,(k/(k+1))_{f_n}(10^{-1})\big)\\ &= \left(pt_1 := \frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})},\ldots,pt_n := \frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right)\end{aligned}$$

with *arbitrary common denominator* $\widehat{V}_{k+1}(10^{-1})$ and approximating $a = (a_1, a_2, \ldots, a_n)$ in the sense that

$$\left(a_1 - \frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})},\ldots,a_n - \frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right) = \left(\underbrace{O\left(10^{-k-1}\right),\ldots,O\left(10^{-k-1}\right)}_{n-\text{times}}\right).$$

This means that the decimal coordinate expression:

$$\left(pt_1^{(0)}.\,pt_1^{(1)}\ldots pt_1^{(m_1)}, pt_2^{(0)}.pt_2^{(1)}\ldots pt_2^{(m_2)},\ldots,pt_n^{(0)}.pt_n^{(1)}\ldots pt_n^{(m_n)}\right)$$

of $(pt_1, pt_2, \ldots, pt_n)$ $(= (pt_{1,k}, pt_{2,k}, \ldots, pt_{n,k}))$ agrees with the decimal coordinate expression:

$$\left(a_1^{(0)}.\,a_1^{(1)}\ldots a_1^{(m)}\ldots, a_2^{(0)}.a_2^{(1)}\ldots a_2^{(m)}\ldots,\ldots,a_n^{(0)}.\,a_n^{(1)}\ldots a_n^{(m)}\ldots\right)$$

of $(a_1, a_2, \ldots, a_n)$ up to the k first decimal digits:

$$\begin{aligned}&\left(pt_1^{(0)}.\,pt_1^{(1)}\ldots pt_1^{(k)}, pt_2^{(0)}.pt_2^{(1)}\ldots pt_2^{(k)},\ldots,pt_n^{(0)}.pt_n^{(1)}\ldots pt_n^{(k)}\right)\\ &= \left(a_1^{(0)}.\,a_1^{(1)}\ldots a_1^{(k)}, a_2^{(0)}.a_2^{(1)}\ldots a_2^{(k)},\ldots,a_n^{(0)}.a_n^{(1)}\ldots a_n^{(k)}\right).\end{aligned}$$

Definition 2. The vector:

$$\left(pt_1 := \frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}},\ldots,pt_n := \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}},\right) := \left(\frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})},\ldots,\frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right)$$

is said to be an *ORANUS (Optimal Rational Approximation Number Set)* to the irrational vector a. It will be denoted by $(\mathrm{ORANUS}/k)_a$. The polynomial:

$$V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v) \ \ (\gamma \in \mathbb{C} \setminus \{0\})$$

is called the *generating polynomial* of this approximation. □

We are thus in position to formulate a first theoretical method for approximating irrational vectors:

Framework for Approximating Irrational Vectors by ORANUS

1. Given an irrational n-vector:

$$a = \left(\underbrace{a_1^{(0)}.\,a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots}_{a_1},\ \underbrace{a_2^{(0)}.\,a_2^{(1)}a_2^{(2)}a_2^{(3)}\ldots}_{a_2},\ \ldots,\ \underbrace{a_n^{(0)}.\,a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots}_{a_n} \right)$$

2. Let $k \in \mathbb{N}$.
3. Let $\pi_0, \pi_1, \ldots, \pi_k$ be arbitrarily chosen complex numbers.
4. Put $V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v)$ $(\gamma \in \mathbb{C} \setminus \{0\})$.
5. Set $\widehat{V}_{k+1} := 10^{-(k+1)} V_{k+1}(10)$.
6. For each $j = 1, 2, \ldots, n$ and each $v = 0, 1, 2, \ldots$, define

$$\Lambda_j(x^v) := a_j^{(v)}$$

and

$$\widehat{W}_k^{(j)} := 10^{-k} \Lambda_j \left(\frac{V_{k+1}(x) - V_{k+1}(10)}{x - 10} \right).$$

7. The ordered set:

$$(\text{ORANUS}/k)_a := \left(\frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}},\ \frac{\widehat{W}_k^{(2)}}{\widehat{V}_{k+1}},\ \ldots,\ \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}} \right)$$

is a rational n-vector of n rational numbers:

$$pt_1 := \frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}},\ \ pt_2 := \frac{\widehat{W}_k^{(2)}}{\widehat{V}_{k+1}},\ \ \ldots,\ \ pt_n := \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}}$$

with common denominator $\widehat{V}_{k+1}$. The decimal coordinate expansion:

$$\left(pt_1^{(0)}.pt_1^{(1)}\ldots pt_1^{(m_1)},\ \ pt_2^{(0)}.pt_2^{(1)}\ldots pt_2^{(m_2)},\ \ \ldots,\ \ pt_n^{(0)}.pt_n^{(1)}\ldots pt_n^{(m_n)} \right)$$

of $(\text{ORANUS}/k)_a$ matches the decimal coordinate expansion:

$$\left(a_1^{(0)}.a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots,\ \ a_2^{(0)}.a_2^{(1)}a_2^{(2)}a_2^{(3)}\ldots,\ \ \ldots,\ \ a_n^{(0)}.a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots \right)$$

of a up to the k first decimal digits:

$$pt_1^{(0)}.pt_1^{(1)}\dots pt_1^{(k)} = a_1^{(0)}.a_1^{(1)}\dots a_1^{(k)},$$
$$pt_2^{(0)}.pt_2^{(1)}\dots pt_2^{(k)} = a_2^{(0)}.a_2^{(1)}\dots a_2^{(k)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$pt_n^{(0)}.pt_n^{(1)}\dots pt_n^{(k)} = a_n^{(0)}.a_n^{(1)}\dots a_n^{(k)}.$$

8. End.

5 Convergence of ORANUS' Sequences

A main question on ORANUS' asymptotic behavior is the "optimal" choice of the interpolation points $\pi_0, \pi_1, \dots, \pi_k$. We will study this question in the next section. Another problem connected with the choice of the π_v's is the problem of convergence of a given vector $a = (a_1, a_2, \dots, a_n)$ of n irrational numbers. According to Theorem 2, we have:

Theorem 4. *If the generating polynomials* $V_k(x) = \gamma \prod_{v=0}^{k-1}(x-\pi_v)$ $(\gamma \in \mathbb{C}\setminus\{0\})$ *satisfy*

$$\lim_{k\to\infty}\left[V_k(x)/V_k\left(z^{-1}\right)\right] = 0,$$

uniformly on any compact subset of an open set $\Omega \subset \mathbb{C}^2$ *containing*

$$\mathbb{C}\times\{0\}\cup\Delta(0;1)\times\{10^{-1}\},$$

then it holds

$$\lim_{k\to\infty}(\text{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1},\dots,\widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1,\dots,a_n). \qquad \square$$

We shall now give some examples of ORANUS' sequences and apply Theorem 4 to these special cases.

Example 2. Let the generating polynomials $V_k(x) = \gamma \prod_{v=0}^{k-1}(x-\pi_v)$ $(\gamma \in \mathbb{C}\setminus\{0\})$ have the form:

$$V_k(x) = (x-\beta)^k \;\; (\beta \in \mathbb{C},\; k = 0, 1, 2, \dots).$$

If the complex number β is chosen so that

$$|10-\beta| \geqslant \sup_{|\xi|\leqslant 1} |\xi-\beta|,$$

then

$$\lim_{k\to\infty} (\text{ORANUS}/k)_a = \lim_{k\to\infty} \left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}, \dots, \widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1, \dots, a_n). \qquad \square$$

Example 3. Assume that the generating polynomials are given by:

$$V_k(x) = \prod_{i=0}^{k-1} (x-\beta_i) \ \ \big(\beta_i \in \mathbb{C},\ i = 0, 1, 2, \dots, k-1\big)$$

i.e., the zeroes of V_k do not depend on k. Further, suppose the limit $b = \lim_{k\to\infty} \beta_i$ exists, thus:

(i) If $b = 0$, then $\lim_{k\to\infty}(\text{ORANUS}/k)_a = (a_1, \dots, a_n)$.
(ii) If $b \neq 0$ and $\Re(b) < 5$, then $\overline{\lim}_{k\to\infty}(\text{ORANUS}/k)_a = (a_1, \dots, a_n)$. $\square$

Example 4. Assume that the generating polynomials have the form:

$$V_k(x) = \prod_{i=0}^{k-1} (x-b_i)^{k-1}, \quad k = 1, 2, \dots.$$

Suppose further that the sequence $(b_i,\ i = 0, 1, 2, \dots)$ has k limit points $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ approached cyclically, i.e.:

$$\lim_{s\to\infty} \beta_{sk+j} = \gamma_j, \quad j = 0, 1, \dots, k-1.$$

We do not require that the limit points $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ are distinct. Define

$$q_k(x) = \prod_{i=0}^{k-1} (x-\gamma_i).$$

For each positive number ρ, let $\mathcal{L}_\rho$ denote the interior of the lemniscates with foci $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ and radius ρ, i.e., the set of all points z satisfying the inequality:

$$|q_k(z)| < \rho.$$

If

$$10 \notin \overline{\mathcal{L}}_{\rho_0} \ \left(\text{with } \rho_0 = \sup_{|\xi|\leqslant 1} |q_k(\xi)|\right),$$

then

$$\lim_{k\to\infty}(\text{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\,\widehat{V}_{k+1},\ldots,\widehat{W}_k^{(n)}/\,\widehat{V}_{k+1}\right) = (a_1,\ldots,a_n). \qquad \square$$

Example 5. We choose the zeros of the Chebyshev polynomials as $\pi_0, \pi_1, \ldots, \pi_k$, i.e.:

$$V_k(x) = \prod_{i=0}^{k-1}\left(x - \cos\left[\frac{2i+1}{2(k+1)}\pi\right]\right), \quad k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty}(\text{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\,\widehat{V}_{k+1},\ldots,\widehat{W}_k^{(n)}/\,\widehat{V}_{k+1}\right) = (a_1,\ldots,a_n). \qquad \square$$

6 Best Choice of ORANUS

A natural question which now arises is the "optimal" choice of an ORANUS. In the present section, we discuss this problem. Let

$$a = (a_1,\ldots,a_n) = \left(\underbrace{a_1^{(0)}.\,a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots}_{a_1},\ \ldots,\ \underbrace{a_n^{(0)}.\,a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots}_{a_n}\right) \in \mathbb{A}^n.$$

As usually, for any $j = 1, 2, \ldots, n$ and $v = 0, 1, \ldots$, we set $\Lambda_j(x^v) := a_j^{(v)}$ and $\widehat{W}_k^{(j)} := 10^{-k}\Lambda_j([V_{k+1}(x) - V_{k+1}(10)]/[x-10])$, where $V_{k+1}(x) = \gamma\prod_{i=0}^{k}(x - \pi_i)$ ($\gamma \in \mathbb{C}\setminus\{0\}$), the $\pi_0, \pi_1, \ldots, \pi_k$ being arbitrarily chosen complex numbers. Obviously, the coordinate absolute differences $\left|a_j - \left[\widehat{W}_k^{(n)}/\,\widehat{V}_{k+1}\right]\right|$, with $\widehat{V}_{k+1} = 10^{-(k+1)}V_{k+1}(10)$, are the *coordinate absolute errors* of the considered approximation. The above asked question can be rephrased in terms of the coordinate absolute errors as follows. Given a $k \in \mathbb{N}$, find an

$$(\widetilde{\text{ORANUS}/k})_a = \left(\widetilde{\widehat{W}_k^{(1)}}/\,\widetilde{\widehat{V}_{k+1}},\quad \widetilde{\widehat{W}_k^{(2)}}/\,\widetilde{\widehat{V}_{k+1}},\quad \ldots\quad, \widetilde{\widehat{W}_k^{(n)}}/\,\widetilde{\widehat{V}_{k+1}}\right),$$

minimizing all coordinate absolute errors, in the sense that

$$\left|a_j - \widetilde{\widehat{W}_k^{(j)}}/\,\widetilde{\widehat{V}_{k+1}}\right| \left|a_j - \widehat{W}_k^{(j)}/\,\widehat{V}_{k+1}\right|, \quad j = 1, 2, \ldots, n,$$

whenever

$$\left(\widehat{W}_k^{(1)} / \widehat{V}_{k+1}, \quad \widehat{W}_k^{(2)} / \widehat{V}_{k+1}, \quad \ldots \quad , \widehat{W}_k^{(n)} / \widehat{V}_{k+1}\right),$$

is an ORANUS to a.

Application of Theorem 3 shows that the generating polynomial:

$$\widetilde{\widehat{V}_{k+1}}(x) = x^{k+1} + 10\left(10^{-k} - 1\right)x^k, \;\; k \in \mathbb{N}$$

leads to the ORANUS:

$$\left(\widetilde{\widehat{W}_k^{(1)}} / \widetilde{\widehat{V}_{k+1}}, \quad \widetilde{\widehat{W}_k^{(2)}} / \widetilde{\widehat{V}_{k+1}}, \quad \ldots \quad , \widetilde{\widehat{W}_k^{(n)}} / \widetilde{\widehat{V}_{k+1}}\right),$$

minimizing all coordinate absolute errors. Especially, since

$$\widetilde{\widehat{V}_{k+1}} = 10^{-(k+1)}\, \widetilde{\widehat{V}_{k+1}}(10) = 10^{-k+2}$$

and

$$\begin{aligned}\widetilde{\widehat{W}_k^{(j)}} &= 10^{-k} \Lambda_j \left(\frac{x^{k+1} + 10\left(10^{-k} - 1\right)x^k - 10^{k+1} - 10\left(10^{-k} - 1\right)10^k}{x - 10} \right) \\ &= 10^{-k} \Lambda_j \left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{x - 10} \right),\end{aligned}$$

we infer that

$$\frac{\widetilde{\widehat{W}_k^{(j)}}}{\widetilde{\widehat{V}_{k+1}}} = \frac{1}{100} \Lambda_j \left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{x - 10} \right).$$

So, we have obtained the following:

Theorem 5. *An "optimal" ORANUS to an n-vector of mutually irrational numbers $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n$ is given by:*

$$\left(\widetilde{\mathrm{ORANUS}/k}\right)_a = (\Lambda_1, \ldots, \Lambda_n) \left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{10^2\,[x - 10]} \right). \qquad \square$$

7 Application

A very important application of the abovementioned work can be in the field of the beam stability problem in circular accelerators like the large hadron collider (LHC) machine at the European Organization for Nuclear Research (CERN). The LHC is considered "one of the great engineering achievements of mankind" and the largest and highest-energy particle accelerator in the world. It remains one of the largest and most complicate experimental machine ever constructed and is expected to address some of the still unsolved questions of science. Particularly, the application at hand concerns the stability of particle beams in high-energy hadron colliders, where symplectic mappings naturally arise due to the periodically repeated (and of very brief duration) effects of beam-beam collisions or beam passage through magnetic focusing elements [5, 10, 17, 39, 43, 44, 46, 47]. The main open problems in such mappings (particularly in the n-dimensional case, $n \geqslant 2$) concern the long-term stability of orbits, which can slowly diffuse away from the origin through thin chaotic layers, leading, e.g., to particle loss in the storage rings of an accelerator or, in similar cases, stars escaping from a galaxy [47]. A well-studied and widely applied such mapping is the following *symplectic mapping T*:

$$T: \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}, \tag{1}$$

which describes the (instantaneous) effect experienced by a hadronic particle as it passes through a magnetic focusing element of the FODO cell type [8, 34–36, 42, 44, 46, 47]. The coordinates x_1 and x_3 represent the particle's deflections from the ideal (circular) orbit, in the horizontal and vertical directions, respectively, and x_2, x_4 are the associated "momenta," while ω_1, ω_2 are related to the accelerator's betatron frequencies (or "tunes") q_x, q_y by:

$$\omega_1 = 2\pi q_x, \qquad \omega_2 = 2\pi q_y$$

and constitute the main parameters that can be varied by an experimentalist [47].

In general, the accurate computation of periodic orbits and the knowledge of their stability properties play a central role for studying the behavior of various such mappings. We say that $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$ is a *fixed point of a mapping T of order p* or a *periodic orbit of period p*, if:

$$x^\star = T^p(x^\star) \equiv \underbrace{T\Big(T\big(\cdots T\big(T(x^\star)\big)\cdots\big)\Big)}_{p \text{ times}}, \quad p = 1, 2, 3, \ldots .$$

For efficient methods of computing periodic orbits, we refer the interested reader to [35, 37, 44–47].

In previous papers of ours [46, 47], we have studied the structure and breakdown of invariant tori of the 4-D symplectic mapping (1) which, as we have already mentioned, arises in a realistic application related to the beam stability problem in circular particle accelerators. Our original goal was to examine the structure of tori by approximating them with sequences of periodic orbits whose *rational* rotation numbers converge to the pair of *irrational* rotation numbers of an invariant torus. Particularly, the sequence of rational rotation numbers:

$$\left(\sigma_1^{(k)},\, \sigma_2^{(k)}\right) = \left(\frac{p_k}{r_k},\, \frac{q_k}{r_k}\right), \quad k = 0, 1, 2, \ldots, \tag{2}$$

have been taken to converge, as $k \to \infty$, to a pair of incommensurate irrationals. In the problem studied in [46, 47], we have chosen the example:

$$\left(\sigma_1^{(k)},\, \sigma_2^{(k)}\right) \underset{n\to\infty}{\longrightarrow} (\sigma_1, \sigma_2) = \left(\frac{\sqrt{5}-1}{2},\, \sqrt{2}-1\right) = (0.61803\ldots, 0.41421\ldots)\,. \tag{3}$$

The choice of σ_1, σ_2 is arbitrary, but it may be useful for comparison purposes with the 2-D case [26]. Next, by selecting linear frequencies $q_x = 0.61903$, $q_y = 0.4152$, we approximated the (σ_1, σ_2)-invariant torus by periodic orbits characterized by the rotation numbers of the Jacobi–Perron sequence [7, 40] which are recursively obtained from the relation:

$$s_{k+1} = l_{k+1} s_k + m_{k+1} s_{k-1} + s_{k-2}, \quad k = 0, 1, \ldots\ ,$$

$(s_k = p_k, q_k, r_k)$, with the integers l_k, m_k determined as follows:

$$\left(s_1^{(k+1)},\, s_2^{(k+1)}\right) = \left(\left\{\frac{1}{s_2^{(k)}}\right\},\, \left\{\frac{s_1^{(k)}}{s_2^{(k)}}\right\}\right), \left(l_{k+1},\, m_{k+1}\right) = \left(\left[\frac{1}{s_2^{(k)}}\right],\, \left[\frac{s_1^{(k)}}{s_2^{(k)}}\right]\right),$$

where $[x]$ and $\{x\}$ denote to the integer and fractional part of the number x, respectively, and $\left(s_1^0,\, s_2^0\right) = \left(\sigma_1,\, \sigma_2\right), (p_0, q_0, r_0) = (0,\, 0,\, 1), (p_{-1}, q_{-1}, r_{-1}) = (1,\, 0,\, 0), (p_{-2}, q_{-2}, r_{-2}) = (0,\, 1,\, 0)$.

In Table 1 we exhibit Jacobi–Perron approximates to the irrationals (3), up to $k = 16$, cf. (2). Notice that the convergence is rather slow, as one might expect of quadratic irrationals, like σ_1, σ_2.

Using the approach of the paper at hand, we can construct simultaneous rational approximants to the given pair of irrational numbers:

$$\sigma_1 = \frac{\sqrt{5}-1}{2} \cong 0.6180339887499\ldots \text{ and } \sigma_2 = \sqrt{2}-1 \cong 0.4142135623731\ldots$$

In fact, let us define the corresponding two $\mathbb{C}$-linear functionals Λ_1 and Λ_2 as they are exhibited in Table 2.

Table 1 Rational approximants of the Jacobi–Perron algorithm of the quadratic irrationals $\sigma_1 = (\sqrt{5}-1)/2$ and $\sigma_2 = \sqrt{2}-1$ and the period p of the corresponding periodic orbit [47]

k	p_k	q_k	$p_k/r_k - \sigma_1$	$q_k/r_k - \sigma_2$	$p = r_k$
4	1	1	-0.11803399	$0.85786438 \times 10^{-1}$	2
5	3	2	0.13196601	$0.85786438 \times 10^{-1}$	4
6	3	2	$-0.18033989 \times 10^{-1}$	$-0.14213562 \times 10^{-1}$	5
7	91	61	$-0.31691239 \times 10^{-2}$	$-0.20514002 \times 10^{-2}$	148
8	94	63	$0.38706388 \times 10^{-3}$	$0.26012184 \times 10^{-3}$	152
9	755	506	$0.31162960 \times 10^{-3}$	$0.20085204 \times 10^{-3}$	1221
10	846	567	$-0.64668078 \times 10^{-4}$	$-0.42634689 \times 10^{-4}$	1369
11	940	630	$-0.19524582 \times 10^{-4}$	$-0.12378941 \times 10^{-4}$	1521
12	8181	5483	$0.63527172 \times 10^{-5}$	$0.41606759 \times 10^{-5}$	13237
13	9027	6050	$-0.30396282 \times 10^{-6}$	$-0.22538829 \times 10^{-6}$	14606
14	9967	6680	$-0.21167340 \times 10^{-5}$	$-0.13716371 \times 10^{-5}$	16127
15	37142	24893	$0.18932888 \times 10^{-6}$	$0.12549818 \times 10^{-6}$	60097
16	83311	55836	$0.13587919 \times 10^{-6}$	$0.87478537 \times 10^{-7}$	134800

Table 2 The corresponding two $\mathbb{C}$-linear functionals Λ_1 and Λ_2

Λ_1	Λ_2
$\Lambda_1(1) = 0$	$\Lambda_2(1) = 0$
$\Lambda_1(x) = 6$	$\Lambda_2(x) = 4$
$\Lambda_1(x^2) = 1$	$\Lambda_2(x^2) = 1$
$\Lambda_1(x^3) = 8$	$\Lambda_2(x^3) = 4$
$\Lambda_1(x^4) = 0$	$\Lambda_2(x^4) = 2$
$\Lambda_1(x^5) = 3$	$\Lambda_2(x^5) = 1$
$\Lambda_1(x^6) = 3$	$\Lambda_2(x^6) = 3$
$\Lambda_1(x^7) = 9$	$\Lambda_2(x^7) = 5$
$\Lambda_1(x^8) = 8$	$\Lambda_2(x^8) = 6$
$\Lambda_1(x^9) = 8$	$\Lambda_2(x^9) = 2$
$\Lambda_1(x^{10}) = 7$	$\Lambda_2(x^{10}) = 3$
$\Lambda_1(x^{11}) = 4$	$\Lambda_2(x^{11}) = 7$
$\Lambda_1(x^{12}) = 9$	$\Lambda_2(x^{12}) = 3$
$\Lambda_1(x^{13}) = 9$	$\Lambda_2(x^{13}) = 1$
$\vdots$	$\vdots$

(a) Let us now choose

$$k = 3 \text{ and } \pi_0 = \pi_1 = \pi_2 = 0, \ \pi_3 = -\tfrac{i}{2}.$$

The corresponding generating polynomial is

$$V_4(x) = x^4 + i\frac{x^3}{2},$$

and therefore

$$\widehat{W}_3^{(1)} = 10^{-3}\Lambda_1\left(\frac{V_4(x)-V_4(10)}{x-10}\right) = 10^{-3}\left(680+i\frac{121}{2}\right),$$

$$\widehat{W}_3^{(2)} = 10^{-3}\Lambda_2\left(\frac{V_4(x)-V_4(10)}{x-10}\right) = 10^{-3}\left(440+i\frac{81}{2}\right),$$

$$\widehat{V}_4 = 10^{-4}\,V_4(10) = 10^{-4}\left(10^4+i\frac{10^3}{2}\right).$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$\begin{aligned}(\text{ORANUS}/3)_\sigma &:= \left(\frac{\widehat{W}_3^{(1)}}{\widehat{V}_4},\ \frac{\widehat{W}_3^{(2)}}{\widehat{V}_4}\right)\\ &= (0.6813216957606 + i0.313216957606,\\ &\quad 0.4409226932688 + i0.0184538653367).\end{aligned}$$

(b) Similarly, for $k = 3$, we can choose the zeros of the Tchebycheff polynomial:

$$T_4(x) = \cos(4\arccos x)$$

divided by $\sqrt{\pi}$ as interpolation nodes, i.e.:
$\pi_0 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{\pi}{7}\right)$, $\pi_1 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{3\pi}{7}\right)$, $\pi_2 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{5\pi}{7}\right)$ and $\pi_3 = \frac{1}{\sqrt{\pi}}\cos(\pi)$.
The generating polynomial is

$$V_4(x) = x^4 + 0.282x^3 - 0.318x^2 - 0.067x - 0.012$$

and we have

$$\widehat{W}_3^{(1)} = 10^{-3}\Lambda_1\left(\frac{V_4(x)-V_4(10)}{x-10}\right) = 633.293\times 10^{-3},$$

$$\widehat{W}_3^{(2)} = 10^{-3}\Lambda_2\left(\frac{V_4(x)-V_4(10)}{x-10}\right) = 424.29\times 10^{-3},$$

$$\widehat{V}_4 = 10^{-4}V_4(10) = 10314.458\times 10^{-4}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$\begin{aligned}(\text{ORANUS}/3)_\sigma &:= \left(\frac{\widehat{W}_3^{(1)}}{\widehat{V}_4},\ \frac{\widehat{W}_3^{(2)}}{\widehat{V}_4}\right)\\ &= (0.6139866971197,\ \ 0.4113546247413).\end{aligned}$$

(c) Continuing, we can choose $k = 12$ and

$$\pi_0 = \pi_1 = \cdots = \pi_{11} = 0, \quad \pi_{12} = -1.$$

Then the generating polynomial is

$$V_{13}(x) = x^{13} + x^{12}$$

and therefore

$$\widehat{W}_{12}^{(1)} = 10^{-12} \Lambda_1 \left(\frac{V_{13}(x) - V_{13}(10)}{x - 10} \right) = 679837387579 \times 10^{-12},$$

$$\widehat{W}_{12}^{(2)} = 10^{-12} \Lambda_2 \left(\frac{V_{13}(x) - V_{13}(10)}{x - 10} \right) = 455634918610 \times 10^{-12},$$

$$\widehat{V}_{13} = 10^{-13} V_{13}(10) = 11000000000000 \times 10^{-13}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\mathrm{ORANUS}/12)_\sigma := \left(\frac{\widehat{W}_{12}^{(1)}}{\widehat{V}_{13}}, \quad \frac{\widehat{W}_{12}^{(2)}}{\widehat{V}_{13}} \right)$$

$$= (0.6180339887082, \quad 0.4142130169182).$$

(d) Next, we may choose $k = 6$ and

$$\pi_0 = \pi_1 = \cdots = \pi_6 = 0.$$

This choice implies

$$V_7(x) = x^7$$

and therefore

$$\widehat{W}_6^{(1)} = 10^{-6} \Lambda_1 \left(\frac{V_7(x) - V_7(10)}{x - 10} \right) = 618053 \times 10^{-6},$$

$$\widehat{W}_6^{(2)} = 10^{-6} \Lambda_2 \left(\frac{V_7(x) - V_7(10)}{x - 10} \right) = 414213 \times 10^{-6},$$

$$\widehat{V}_7 = 10^{-7} V_7(10) = 1.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/6)_\sigma := \left(\frac{\widehat{W}_6^{(1)}}{\widehat{V}_7}, \ \frac{\widehat{W}_6^{(2)}}{\widehat{V}_7} \right) = (0.618033, \ \ 0.414213) \,.$$

e) Finally, let $k = 4$ and let $V_5\,(x)$ be the Legendre polynomial:

$$V_5(x) = \frac{1}{8}\, \left(63x^5 - 70x^3 + 15x\right).$$

Then

$$\widehat{W}_4^{(1)} = 10^{-4}\, \Lambda_1 \left(\frac{V_5\,(x) - V_5\,(10)}{x - 10} \right) = 48133 \times 10^{-4},$$

$$\widehat{W}_4^{(2)} = 10^{-4}\, \Lambda_2 \left(\frac{V_5\,(x) - V_5\,(10)}{x - 10} \right) = 32259.5 \times 10^{-4},$$

$$\widehat{V}_5 = 10^{-5}\, V_4\,(10) = 778768.75 \times 10^{-5}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/4)_\sigma := \left(\frac{\widehat{W}_4^{(1)}}{\widehat{V}_5}, \ \frac{\widehat{W}_4^{(2)}}{\widehat{V}_5} \right) = (0.6180750062198, \ \ 0.4142372174025) \,.$$

Summarizing, it should also be noted that, for small values of k (this means a few interpolation points $\pi_0, \pi_1, \ldots, \pi_k$ and a low-degree generating polynomial $V_{k+1}\,(x)$), we can achieve good rational approximations.

We can group the above numerical results in Table 3, which are directly comparable with those exhibited in Table 1.

8 Epilogue and Synopsis

In the paper at hand, we investigated multivariate rational approximation numbers whose decimal expansion string of digits coincides with the decimal expansion digital string of mutually irrational numbers as far as possible. The main advantage of these approximants over Jacobi–Perron approximants lies in the completely free choice of their common denominator which may lead to a better and increasingly rapid approximation.

Table 3 Optimal rational approximation number sets of the quadratic irrationals $\sigma_1 = \left(\sqrt{5}-1\right)/2$ and $\sigma_2 = \sqrt{2}-1$

k	$V_{k+1}(x)$	$\left\|\sigma_1 - \left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}\right)\right\|$	$\left\|\sigma_2 - \left(\widehat{W}_k^{(2)}/\widehat{V}_{k+1}\right)\right\|$	$\left\|\sigma - (\text{ORANUS}/k)_\sigma\right\|$
3	$x^4 + i\frac{x^3}{2}$ (interpolation points $\pi_0 = \pi_1 = \pi_2 = 0$, $\pi_3 = -i/2$)	0.3195468610246	0.0010539228191	0.3195485990331
3	$x^4 + 0.282x^3 - 0.318x^2 - 0.067x - 0.012$ (interpolation points $\pi_0 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{\pi}{7}\right)$, $\pi_1 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{3\pi}{7}\right)$, $\pi_2 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{5\pi}{7}\right)$ and $\pi_3 = \frac{1}{\sqrt{\pi}}\cos(\pi)$, the zeros of the Tchebycheff polynomial $T_4(x) = \cos(4\arccos x)$)	$0.0040472916302 \approx 0.40472916 \times 10^{-2}$	$0.0028589376318 \approx 0.28589376 \times 10^{-2}$	$0.0049552087645 \approx 0.49552088 \times 10^{-2}$
12	$x^{13} + x^{12}$ (interpolation points $\pi_0 = \pi_1 = \cdots = \pi_{11} = 0$, $\pi_{12} = -1$)	$0.0000000000417 = 0.417 \times 10^{-10}$	$0.0000005454549 = 0.5454549 \times 10^{-6}$	$0.0000005477226 = 0.5477226 \times 10^{-6}$
6	x^7 (interpolation points $\pi_0 = \pi_1 = \cdots = \pi_6 = 0$)	$0.000000987499 = 0.987499 \times 10^{-6}$	$0.0000005623731 = 0.5623731 \times 10^{-6}$	$0.0000011401754 = 0.11401754 \times 10^{-5}$
4	$\frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$ (Legendre polynomial)	$0.00004101747 = 0.4101747 \times 10^{-4}$	$0.000023655029 = 0.23655029 \times 10^{-4}$	$0.0002400801949 = 0.2400801949 \times 10^{-3}$

References

1. Baker, G.A. Jr.: The existence and convergence of subsequences of Padé approximants. J. Math. Anal. Appl. **43**(2), 498–528 (1973)
2. Baker, G.A. Jr.: Essentials of Padé Approximants. Academic, New York (1975)
3. Baker, G.A. Jr., Graves-Morris, P.R.: Convergence of the rows of the Padé table. J. Math. Anal. Appl. **57**, 323–339 (1977)
4. Baker, G.A. Jr., Graves-Morris, P.R.: The convergence of sequences of Padé approximants. J. Math. Anal. Appl. **87**(2), 382–394 (1982)
5. Bazzani, A., Todesco, E., Turchetti, G., Servizi, G.: A normal form approach to the theory of nonlinear Betatronic motion. CERN, Yellow Reports 94-02, European Organization for Nuclear Research, Geneva (1994)
6. Beardon, A.G.: On the location of poles of Padé approximants. J. Math. Anal. Appl. **21**, 469–474 (1968)
7. Bernstein, L.: The Jacobi-Perron Algorithm: Its Theory and Application. Lectures Notes in Mathematics, vol. 207. Springer, Berlin/Heidelberg/New York (1971)
8. Boreux, J., Carletti, T., Skokos, C., Vittot, M.: Hamiltonian control used to improve the beam stability in particle accelerator models. Commun. Nonlinear Sci. Numer. Simul. **17**, 1725–1738 (2012)
9. Borwein, P.B.: The usual behaviour of rational approximants. Can. Math. Bull. **26**(3), 317–323 (1983)
10. Bountis, T.C., Tompaidis, S.: Strong and weak instabilities in a 4D mapping model of FODO cell dynamics. In: Turchetti, G., Scandale, W. (eds.) Future Problems in Nonlinear Particle Accelerators, pp. 112–127. World Scientific, Singapore (1991)
11. Brezinski, C.: Accélération de la convergence en analyse numérique. Cours de D.E.A.. Publications du Laboratoire de Calcul de l'Université des Sciences et Techniques de Lille, Lille (1973)
12. Brezinski, C.: Rational approximation to formal power series. J. Approx. Theory **25**, 279–317 (1979)
13. Brezinski, C.: Padé-type Approximation and General Orthogonal Polynomials. Birkhäuser, Basel (1980)
14. Brezinski, C.: Outlines of Padé approximation. Computational Aspects of Complex Analysis, pp. 1–50. Reidel Publishing Company, Dordrecht (1983)
15. Brezinski, C.: Padé approximants: old and new. Jahrbruck Uberblicke Mathematik 37–63 (1983)
16. Buslaev, V.I., Gončar, A.A., Suetin, S.P.: On convergence of subsequences of the nth row of the Padé table. Math. Sbornik **120**, 162 (1983)
17. Corrigan, R., Huson, F., Month, M.: Physics of High Energy Accelerators. A.I.P. Conference Proceedings, vol. 87. American Institute of Physics, New York (1982)
18. Daras, N.J.: Padé and Padé-type approximation for 2π-periodic L_p functions. Acta Appl. Math. **62**, 245–343 (2000)
19. Daras, N.J.: On the best choice for the generating polynomial of a Padé-type approximation. BIT Numer. Math. **44**, 245–257 (2004)
20. De Bruin, M.G.: Some classes of Padé tables whose upper halves are normal. Nieuw Archief voor Wiskunde **25**, 148–160 (1977)
21. De Montessus de Ballore, R.: Sur les fractions continues algébriques. Bull. Soc. Math. Fr. **30**, 28–36 (1902)
22. Edrei, A.: The Padé table of functions having a finite number of essential singularities. Pac. J. Math. **56**, 429–453 (1975)
23. Edrei, A.: The Padé tables of entire functions. J. Approx. Theory **28**, 54–82 (1980)
24. Eiermann, M.: On the convergence of Padé-type approximants to analytic functions. J. Comput. Appl. Math. **10**, 219–227 (1984)
25. Gilewitz, J.: Approximants de Padé. Lectures Notes in Mathematics. Springer, Heidelberg (1983)

26. Greene, J.M.: A method for determining a stochastic transition. J. Math. Phys. **20**, 1183–1201 (1979)
27. Jones, W.B., Thron, W.J.: Continued Fractions: Analytic Theory and Applications, vol. 11. Addison Wesley, Reading, MA (1980)
28. Khinchin, A.Y.: Continued Fractions. University of Chicago Press, Chicago, IL (1964)
29. Kim, S., Ostlund, S.: Simultaneous rational approximations in the study of dynamical systems. Phys. Rev. A **34**(4), 3426–3434 (1981)
30. Lubinsky, D.S.: Rogers-Ramanujan and the Baker-Gammel-Wills conjecture. Ann. Math. **157**, 847–889 (2003)
31. Magnus, A.: Rate of convergence of sequences of Padé-type approximants and pole detection in the complex plane. In: de Bruin, M.G., Van Rossum, H. (eds) Padé Approximation. Lecture Notes in Mathematics, vol. 888, pp. 300–308. Springer, Heidelberg (1981)
32. Nuttall, J.: Convergence of Padé approximants of meromorphic functions. J. Math. Anal. Appl. **31**, 147–153 (1970)
33. Perron, O.: Die Lehre von den Kettenbrüchen. Chelsea, New York (1957)
34. Petalas, Y.G., Antonopoulos, C.G., Bountis, T.C., Vrahatis, M.N.: Evolutionary methods for the approximation of the stability domain and frequency optimization of conservative maps. Int. J. Bifurcation Chaos **18**(8), 2249–2264 (2008)
35. Petalas, Y.G., Parsopoulos, K.E., Vrahatis, M.N.: Stochastic optimization for detecting periodic orbits of nonlinear mappings. Nonlinear Phenom. Complex Syst. **11**(2), 285–291 (2008)
36. Petalas, Y.G., Antonopoulos, C.G., Bountis, T.C., Vrahatis, M.N.: Detecting resonances in conservative maps using evolutionary algorithms. Phys. Lett. A **373**(3), 334–341 (2009)
37. Polymilis, C., Servizi, G., Skokos, C., Turchetti, G., Vrahatis, M.N.: Topological degree theory and local analysis of area preserving maps. Chaos **13**(1), 94–104 (2003)
38. Pommerenke, C.: Padé approximants and convergence in capacity. J. Math. Anal. Appl. **41**, 775–780 (1973)
39. Rosenzweig, J.B.: Fundamentals of Beam Physics. Oxford University Press, New York (2003)
40. Schweiger, F.: The Metrical Theory of Jacobi-Perron Algorithm. Lectures Notes in Mathematics, vol. 334. Springer, Berlin/Heidelberg/New York (1973)
41. Shanks, D.: Non linear transformations of divergent and slowly convergent series. J. Math. Phys. **34**, 1–42 (1955)
42. Todesco, E.: The use of mappings for stability problems in beam dynamics. In: Benest, D., Froeschle, C. (eds.) Analysis and Modeling of Discrete Dynamical Systems with Applications to Dynamical Astronomy, pp. 301–314. Gordon and Breach, London (1997)
43. Turchetti, G., Scandale, W.: Future Problems in Nonlinear Particle Accelerators. World Scientific, Singapore (1991)
44. Vrahatis, M.N.: An efficient method for locating and computing periodic orbits of nonlinear mappings. J. Comput. Phys. **119**, 105–119 (1995)
45. Vrahatis, M.N., Servizi, G., Turchetti, G., Bountis, T.C.: A procedure to compute the fixed points and visualize the orbits of a 2D map. CERN SL/93-06 (AP), European Organization for Nuclear Research, Geneva, Switzerland (1993)
46. Vrahatis, M.N., Bountis, T.C., Kollmann, M.: Periodic orbits and invariant surfaces of 4-D nonlinear mappings. Int. J. Bifurcation Chaos **6**(8), 1425–1437 (1996)
47. Vrahatis, M.N., Isliker, H., Bountis, T.C.: Structure and breakdown of invariant tori in a 4-D mapping model of accelerator dynamics. Int. J. Bifurcation Chaos **7**(12), 2707–2722 (1997)
48. Wallin, H.: On the convergence theory of Padé approximants. In: Linear Operators and Approximation. Proceedings of the Conference Oberwolfach (1971). International Series Numerical Mathematics, vol. 20, pp. 461–469. Birkhaüser, Basel (1972)
49. Weisstein, E.W.: Rational approximation. MathWorld-A Wolfram Web Resource. Available on-line at: http://mathworld.wolfram.com/RationalApproximation.html
50. Wynn, P.: L' ε-algorithmo e la tavola di Padé. Rendiconti di Matematica di Roma **20**, 403 (1961)
51. Zinn-Justin, J.: Convergence of Padé approximants in the general case. In: Visconti, A. (ed.) Colloquium on Advanced Computing Methods in Theoretical Physics, pp. 88–102. C.N.R.S., Marseille (1971)