Survey on Generalizations of the Intermediate Value Theorem and Applications

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Abstract. Generalizations of the intermediate value theorem in several variables are presented. These theorems are very useful in various approaches including the existence of solutions of systems of nonlinear equations, the existence of fixed points of continuous functions as well as the existence of periodic orbits of nonlinear mappings and similarly, fixed points of the Poincaré map on a surface of section. Based on the corresponding criteria for the existence of a solution or a fixed point emanated by the intermediate value theorems, generalized bisection methods for approximating zeros or fixed points of continuous functions are given. These bisection methods require only the algebraic signs of the function values and are of major importance for studying and tackling problems with imprecise information.

Keywords: Generalizations of the intermediate value theorem · Existence theorems · Zeros · Fixed points · Systems of nonlinear algebraic and/or transcendental equations · Periodic orbits · Poincaré map

1 Introduction

Assume that $F_n = (f_1, f_2, \ldots, f_n)\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping and $\theta^n = (0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$. The problem of solving the equation:

$$F_n(x) = \theta^n, \tag{1}$$

is to find a *zero* $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \mathcal{D}$ for which $F_n(x^*) = \theta^n$. The problem (1) may be represented as follows:

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0,\\ f_2(x_1, x_2, \ldots, x_n) &= 0,\\ &\vdots\\ f_n(x_1, x_2, \ldots, x_n) &= 0. \end{aligned} \tag{2}$$

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The problem of computing the *extrema of an objective function* $f\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ can be studied and tackled by solving the following equation:

$$\nabla f(x) = \theta^n, \tag{3}$$

where $\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$, denotes the gradient of f at $x \in \mathcal{D}$.

Furthermore, the problem of finding a *fixed point* of F_n in $\mathcal{D} \subset \mathbb{R}^n$ is to find a point $x^\star \in \mathcal{D}$ which satisfies the equation:

$$F_n(x^\star) = x^\star. \tag{4}$$

Obviously, the problem of finding a fixed point is equivalent to the problem of solving Eq. (1) by using the mapping $G_n = I_n - F_n$ (where I_n indicates the identity mapping) instead of F_n and solving the equation:

$$G_n(x) = \theta^n. \tag{5}$$

The problem of computing *periodic orbits of nonlinear mappings* or *fixed points of the Poincaré map on a surface of section* can be studied and tackled by using fixed points [31]. More specifically the problem of finding periodic orbits of nonlinear mappings: $\varPhi_n = (\varphi_1, \varphi_2, \ldots, \varphi_n)\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, of *period* p amounts to finding *fixed points* $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star) \in \mathcal{D}$ *of period* p which satisfy the following equation:

$$\varPhi_n^p(x^\star) = \underbrace{\varPhi_n\left(\varPhi_n\left(\cdots \varPhi_n\big(\varPhi_n(x^\star)\big)\cdots\right)\right)}_{p \text{ times}} = x^\star. \tag{6}$$

The problem of finding periodic orbits of period p of *dynamical systems* in $\mathbb{R}^{n+1}$ amounts to fixing one of the variables, say $x_{n+1} = \text{const}$, and locating points $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$ on an n-dimensional *surface of section* $\varSigma_{t_0}$ which satisfy Eq. (6). where $\varPhi_n^p = P_{t_0} : \varSigma_{t_0} \to \varSigma_{t_0}$ is the *Poincaré map* of the system. For example, let us consider a conservative dynamical system of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{7}$$

with $\mathbf{x} = (x, \dot{x}) \in \mathbb{R}^2$ and $\mathbf{f} = (f_1, f_2)$ periodic in t with frequency ω. We obtain periodic orbits of period p of System (7) by taking as initial conditions of these orbits the points which the orbits intersect the surface of section:

$$\varSigma_{t_0} = \left\{ \big(x(t_k),\ \dot{x}(t_k)\big), \quad \text{with} \quad t_k = t_0 + k\frac{2\pi}{\omega}, \quad k \in \mathbb{N} \right\}, \tag{8}$$

at a finite number of points p. Thus the dynamics is studied in connection with a Poincaré map $\varPhi_n^p = P_{t_0} : \mathbf{\Sigma}_{t_0} \to \varSigma_{t_0}$, constructed by following the solutions of (7) in continuous time.

In the paper at hand, *generalizations of the intermediate value theorem* in several variables are presented. These theorems are very useful in various approaches including, among others, those mentioned previously. Specifically, using these theorems we can study and analyze (a) the existence of solutions of systems of nonlinear algebraic and/or transcendental equations, (b) the localization of extrema of objective functions, (c) the existence of fixed points of continuous functions, as well as (d) the existence of periodic orbits of nonlinear mappings and similarly, fixed points of the Poincaré map on a surface of section. We notice that, these theorems are of major importance for tackling problems with imprecise (not exactly known) information.

Based on the corresponding existence criteria emanated by the above theorems, methods, named *generalized bisection methods*, are given. The only computable information required by the generalized bisection methods is the algebraic sign of the function value which is the minimum possible information (one bit of information) necessary for the purpose needed, and not any additional information. Thus, these methods are of major importance for studying and tackling problems with imprecise (not exactly known) information. These problems appear in various fields of science and technology, because, in a large variety of applications, precise function values are either impossible or time consuming and computationally expensive to obtain. In other cases, it may be necessary to integrate numerically a system of differential equations in order to obtain a function value, so that the precision of the computed value is limited. Furthermore, these methods are particularly useful for studying and tackling problems where the corresponding functions obtain very large and/or very small values.

It is worthy to mention that regarding the case of *algebraic equations*, it is well known that these equations are very important in studying and solving problems on geometric, kinematic, and other constraints in various fields of science and technology including, among others, robotics, vision, modeling and graphics, molecular biology, signal processing, and computational economics. In addition, regarding the algebraic signs of algebraic expressions there are various efficient approaches in obtaining this information, see [4,8,9] and the references thereof.

Applications of the presented generalizations of the intermediate value theorem for obtaining methods related to systems of nonlinear algebraic and/or transcendental equations, as well as fixed points of continuous functions are presented. Furthermore, an application is presented which concerns the computation of all the periodic orbits (stable and unstable) of any period and accuracy which occur, among others, in the study of beam dynamics in circular particle accelerators like the Large Hadron Collider (LHC) machine at the European Organization for Nuclear Research (CERN).

2 Generalizations of the Intermediate Value Theorem

2.1 Definitions and Notations

Let us give some necessary definitions and notations.

Notation 1. We denote by ϑA the boundary of a set A, by $\mathrm{cl}A$ its closure, by $\mathrm{int}A$ its interior, by $\mathrm{card}\{A\}$ its cardinality (i.e., the number of elements in the set A) and by $\mathrm{co}A$ its convex hull (i.e., the set of all finite convex combinations of elements of A).

Notation 2. We shall use the index sets $N^n = \{0, 1, \ldots, n\}$, $N^n_{\neg 0} = \{1, 2, \ldots, n\}$ and $N^n_{\neg i} = \{0, 1, \ldots, i-1, i+1, \ldots, n\}$. Also, for a given set $I = \{i, j, \ldots, \ell\} \subset N^n$ we denote by $N^n_{\neg I}$ or equivalently by $N^n_{\neg ij\cdots\ell}$ the set $\{k \in N^n \mid k \notin I\}$.

Definition 1. For any positive integer n, and for any set of points $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ in some linear space which are affinely independent (i.e., the vectors $\{\upsilon^1 - \upsilon^0, \upsilon^2 - \upsilon^0, \ldots, \upsilon^n - \upsilon^0\}$ are linearly independent) the convex hull $\mathrm{co}\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is called the *n-simplex with vertices* $\upsilon^0, \upsilon^1, \ldots, \upsilon^n$. For each subset of $(m+1)$ elements $\{\omega^0, \omega^1, \ldots, \omega^m\} \subset \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$, the m-simplex $[\omega^0, \omega^1, \ldots, \omega^m]$ is called an *m-face* of $[\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. In particular, 0-faces are vertices and 1-faces are edges. The m-faces are also called *facets* of the n-simplex. An m-face of the n-simplex is called the *carrier* of a point p if p lies on this m-face and not on any sub-face of this m-face.

Notation 3. We denote the n-simplex with set of vertices $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ by $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. Also, we denote the $(n-1)$-simplex that determines the i-th $(n-1)$-face of σ^n by $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$. Furthermore, for a given index set $I = \{i, j, \ldots, \ell\} \subset N^n$ with cardinality $\mathrm{card}\{I\} = \kappa$, we denote by $\sigma^n_{\neg I}$ or equivalently by $\sigma^n_{\neg ij\cdots\ell}$ the $(n-\kappa)$-face of σ^n with vertices $\upsilon^m, m \in N^n_{\neg I}$.

Definition 2 [26,29]. The *diameter* of an m-simplex σ^m in $\mathbb{R}^n$, $m \leqslant n$, denoted by $\mathrm{diam}(\sigma^m)$, is defined to be the length of the longest edge (1-face) of σ^m while the *microdiameter*, $\mu\mathrm{diam}(\sigma^m)$, of σ^m is defined to be the length of the shortest edge of σ^m.

Definition 3. Let $\sigma^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^m]$ be an m-simplex in $\mathbb{R}^n$, $m \leqslant n$. Then the *barycenter* of σ^m denoted by K is the point $K = (m+1)^{-1} \sum_{i=0}^{m} \upsilon^i$ in $\mathbb{R}^n$.

Remark 1. By convexity it is obvious that the barycenter of any m-simplex σ^m in $\mathbb{R}^n$ is a point in the relative interior of σ^m.

Definition 4. An n-simplex is *oriented* if an order has been assigned to its vertices. If $\langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is an orientation of $\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ this is regarded as being the same as any orientation obtained from it by an even permutation of the vertices and as the opposite of any orientation obtained by an odd permutation of the vertices. We shall denote oriented n-simplices by $\sigma^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$, and we shall write, for example, $\langle \upsilon^0, \upsilon^1, \upsilon^2, \ldots, \upsilon^n \rangle = -\langle \upsilon^1, \upsilon^0, \upsilon^2, \ldots, \upsilon^n \rangle = \langle \upsilon^2, \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$. The *boundary* $\vartheta\sigma^n$ of an oriented n-simplex $\sigma^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is given by $\vartheta\sigma^n = \sum_{i=0}^{n} (-1)^i \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n \rangle$. The oriented $(n-1)$-simplex $\langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n \rangle$ will be called the ith *face* of σ^n.

Definition 5. An n-dimensional *polyhedron* Π^n is a union of a finite number of oriented n-simplices σ^n_i, $i = 1, 2, \ldots, k$ such that the σ^n_i have pairwise-disjoint interiors. We write $\Pi^n = \sum_{i=1}^{k} \sigma^n_i$ and $\vartheta\Pi^n = \sum_{i=1}^{k} \vartheta\sigma^n_i$.

Definition 6. Let $\psi \in \mathbb{R}$, then the *sign (or signum) function*, denoted by sgn, maps ψ to the set $\{-1, 0, 1\}$ as follows:

$$\operatorname{sgn}\psi = \begin{cases} -1, & \text{if} \quad \psi < 0, \\ 0, & \text{if} \quad \psi = 0, \\ 1, & \text{if} \quad \psi > 0. \end{cases} \tag{9}$$

Furthermore, for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ the *sign* of a, denoted $\operatorname{sgn} a$, is defined as $\operatorname{sgn} a = (\operatorname{sgn} a_1, \operatorname{sgn} a_2, \ldots, \operatorname{sgn} a_n)$.

2.2 Bolzano's Intermediate Value Theorem

The fundamental and pioneering well-known and widely applied Bolzano's theorem states the following [3,12]:

Theorem 1 (Bolzano's theorem). *If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if it holds that $f(a)f(b) < 0$, then there is at least one $x \in (a,b)$ such that $f(x) = 0$.*

The above theorem is also called *intermediate value theorem* since it can be easily given as follows:

Theorem 2 (Bolzano's intermediate value theorem). *If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if y_0 is a real number such that:*

$$\min\{f(a), f(b)\} < y_0 < \max\{f(a), f(b)\},$$

then there is at least one $x_0 \in (a,b)$ such that $f(x_0) = y_0$.

Remark 2. Obviously, Theorem 2 can be deduced from Theorem 1 by considering the function $g(x) = f(x) - y_0$.

Remark 3. The above theorem has been independently proved by Bolzano in 1817 [3] and Cauchy in 1821 [6]. These proofs were crucial in the procedure of *arithmetization of analysis*, which was a research program in the foundations of mathematics during the second half of the 19th century.

2.3 Bolzano-Poincaré-Miranda Intermediate Value Theorem

A straightforward generalization of Bolzano's intermediate value theorem to continuous mappings in several variables was proposed (without proof) by Poincaré in 1883 and 1884 in his work on the *three body problem* [20,21]. This generalization, known as Bolzano-Poincaré-Miranda theorem, states that [17,25,30]:

Theorem 3 (Bolzano - Poincaré - Miranda theorem). *Suppose that $P = \bigl\{x \in \mathbb{R}^n \mid |x_i| < L, \text{ for } 1 \leqslant i \leqslant n\bigr\}$ and let the mapping $F_n = (f_1, f_2, \ldots, f_n)\colon P \to \mathbb{R}^n$ be continuous on $\mathrm{cl}P$ such that $\theta^n \notin F_n(\vartheta P)$, and*

(a) $f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geqslant 0, \quad \textit{for} \quad 1 \leqslant i \leqslant n,$
(b) $f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \leqslant 0, \quad \textit{for} \quad 1 \leqslant i \leqslant n.$

Then, there is at least one $x \in P$ *such that* $F_n(x) = \theta^n$.

Remark 4. The Bolzano-Poincaré-Miranda theorem is closely related to important theorems in analysis and topology and constitutes an invaluable tool for verified solutions of numerical problems by means of interval arithmetic. For various interesting relations between the theorems of Bolzano-Poincaré-Miranda, Borsuk, Kantorovich and Smale with respect to the existence of a solution of a system of nonlinear equations, we refer the interested reader to [1].

Remark 5. Theorem 3 it has come to be known as Miranda's theorem since in 1940 Miranda [17] proved that it is equivalent to the traditional Brouwer fixed point theorem [5]. Also, this theorem has been named Miranda-Vrahatis theorem [2]. For a short proof and a generalization of the Bolzano-Poincaré-Miranda theorem using topological degree theory we refer the interested reader to [30]. Following the proof of [30] it is easy to see that Theorem 3 is also true, if L is dependent of i. That is, P can also be an n-dimensional rectangle and need not to be necessarily an n-dimensional cube. In addition, for generalizations with respect to an arbitrary basis of $\mathbb{R}^n$ that eliminate the dependence of the Bolzano-Poincaré-Miranda theorem on the standard basis of $\mathbb{R}^n$ see [11,30].

2.4 Intermediate Value Theorem for Simplices

The intermediate value theorem for simplices (cf. Theorem 4 below) is proposed in [33]. The obtained proof is based on the following *Knaster-Kuratowski-Mazurkiewicz covering principle* [15]:

Lemma 1 (Knaster-Kuratowski-Mazurkiewicz). *Let* $C_i,\ i \in N^n$ *be a family of* $(n+1)$ *closed subsets of an* n*-simplex* $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ *in* $\mathbb{R}^n$ *satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) For each $\emptyset \neq I \subset N^n$ *it holds that* $\bigcap_{i \in I} \sigma^n_{\neg i} \subset \bigcup_{j \in N^n_{\neg I}} C_j\,.$

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Remark 6. Lemma 1 is often referred in the literature as *KKM Lemma*.

Remark 7. The three well known and widely applied fundamental and pioneering classical results, namely, the Brouwer fixed point theorem [5], the Sperner lemma [24], and the KKM lemma [15] are mutually equivalent in the sense that each one can be deduced from another.

Similar to KKM covering principle, the following covering principles have been proposed by Sperner [24]:

Lemma 2 (Sperner covering principle). *Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) $\sigma^n_{\neg i} \cap C_i = \emptyset, \quad \forall\, i \in N^n\,.$

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

A similar covering principle is the following:

Lemma 3 (Sperner covering principle). *Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) $\sigma^n_{\neg i} \subset C_i\,, \quad \forall\, i \in N^n\,.$

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Remark 8. Based on the above *Sperner covering principles* two short proofs of the intermediate value theorem for simplices (cf. Theorem 4 below) are given in [34].

Next, we give the intermediate value theorem for simplices [33,34]:

Theorem 4 (Intermediate value theorem for simplices). *Assume that $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is an n-simplex in $\mathbb{R}^n$. Let $F_n = (f_1, f_2, \ldots, f_n)\colon \sigma^n \to \mathbb{R}^n$ be a continuous function such that $f_j(\upsilon^i) \neq 0$, $\forall\, j \in N^n_{\neg 0} = \{1, 2, \ldots, n\}$, $i \in N^n = \{0, 1, \ldots, n\}$ and $\theta^n \notin F_n(\vartheta \sigma^n)$. Assume that the vertices υ^i, $i \in N^n$ are reordered such that the following hypotheses are fulfilled:*

$$(a) \quad \mathrm{sgn} f_j(\upsilon^j)\, \mathrm{sgn} f_j(x) = -1, \qquad \forall\, x \in \sigma^n_{\neg j}\,, \;\; j \in N^n_{\neg 0}\,, \tag{10}$$

$$(b) \quad \mathrm{sgn} F_n(\upsilon^0) \neq \mathrm{sgn} F_n(x), \qquad \forall\, x \in \sigma^n_{\neg 0}\,, \tag{11}$$

where $\mathrm{sgn} F_n(x) = \big(\mathrm{sgn} f_1(x), \mathrm{sgn} f_2(x), \ldots, \mathrm{sgn} f_n(x)\big)$ and $\sigma^n_{\neg i}$ denotes the face opposite to vertex υ^i. Then, there is at least one point $x \in \mathrm{int}\, \sigma^n$ such that $F_n(x) = \theta^n$.

Remark 9. The only computable information required by the hypotheses (10) and (11) of Theorem 4 is the algebraic sign of the function values on the boundary of the n-simplex σ^n. Thus, Theorem 4 is applicable whenever the signs of the function values are computed correctly. Theorem 4 has been applied for the localization and approximation of fixed points and zeros of continuous mappings using a simplicial subdivision of a simplex [34]. For an interesting application of this theorem see [16].

3 Applications of the Intermediate Value Theorems

Applications of the corresponding existence criteria emanated by the above intermediate value theorems are given below.

3.1 Bisection Method

Based on the hypotheses of Bolzano's theorem (Theorem 1), a very useful criterion for the existence of a zero of a continuous mapping $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ within an interval (a,b) is the following *Bolzano's existence criterion*:

$$f(a)\, f(b) < 0, \tag{12}$$

or equivalently:

$$\operatorname{sgn} f(a) \operatorname{sgn} f(b) = -1, \tag{13}$$

where sgn denotes the sign function (9).

Remark 10. The Bolzano existence criterion is well-known and widely used and it can be generalized to higher dimensions, see [30,33] (cf. Sect. 2.3 and Sect. 2.4). Note that when the condition (12) (or the condition (13)) is not fulfilled, then in the interval (a,b) either no zero exists or there are zeros for which the sum of their multiplicities is an even number (e.g., two simple zeros, one double and two simple zeros, one triple and one simple zeros etc.).

The well-know and widely applied *bisection method* is based on the Bolzano existence criterion in order to approximate a zero of a continuous function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ in a given interval (a,b). A simplified version described in [27] is the following:

$$x^{p+1} = x^p + c \operatorname{sgn} f(x^p) \,/\, 2^{p+1}, \quad p = 0, 1, \ldots\,, \tag{14}$$

where $x^0 = a$ and $c = \operatorname{sgn} f(a)\,(b-a)$. Instead of the iterative formula (14) we can also use the following [27]:

$$x^{p+1} = x^p - \hat{c} \operatorname{sgn} f(x^p) \,/\, 2^{p+1}, \quad p = 0, 1, \ldots, \tag{15}$$

where $x^0 = b$ and $\hat{c} = \operatorname{sgn} f(b)\,(b-a)$.

The sequences (14) and (15) converge with certainty to a zero $r \in (a,b)$ if for some x^p it holds that:

$$\operatorname{sgn} f(x^0) \operatorname{sgn} f(x^p) = -1, \quad \text{for} \quad p = 1, 2, \ldots.$$

Furthermore, the number of iterations ν required to obtain an approximate zero r^* such that $|r - r^*| \leqslant \varepsilon$ for some $\varepsilon \in (0,1)$ is given by:

$$\nu = \left\lceil \log_2 (b-a)\, \varepsilon^{-1} \right\rceil, \tag{16}$$

where $\lceil x \rceil = \operatorname{ceil}(x)$ denotes the ceiling function that maps a real number x to the least integer greater than or equal to x.

Remark 11. The main characteristics of the iterative schemes (14) and (15) are the following:

(a) They *converge with certainty* within the given interval (a, b).
(b) They are *globally convergent methods* in the sense that they converge to a zero from remote initial guesses.
(c) Using relation (16), the *number of iterations* that are required for the attainment of an approximate zero to a given accuracy is known a priori.
(d) They are *worst-case optimal*. That is, they possess asymptotically the best possible rate of convergence in the worst case [23]. This means that they are guaranteed to converge within the predefined number of iterations, and, moreover, no other method has this important property.
(e) They require only the *algebraic signs* of the function values to be computed, as is evident from (14) and (15); thus they can be applied to problems with imprecise function values.

For applications of the iterative schemes (14) and (15) we refer the interested reader, among others, to [27,28].

3.2 Generalized Bisection Methods

The conditions of the Bolzano-Poincaré-Miranda theorem give an invaluable existence criterion for a solution of Eq. (1). Similarly to Bolzano's criterion, the Bolzano - Poincaré - Miranda criterion requires only the algebraic sings of the function values to be computed on the boundary of the n-cube P. On the other hand, for general continuous functions, in contrary to Bolzano's criterion, the hypotheses (a) and (b) of Theorem 3 are not always fulfilled or it is impossible to be verified for a given n-cube P.

Next, the characteristic polyhedron criterion and the characteristic bisection method are briefly presented. These approaches, in contrary to Bolzano - Poincaré - Miranda criterion require only the algebraic sings of the function values to be computed on the vertices of the considered polyhedron.

There are various generalized bisection methods that require the computation of the topological degree [19] in order to localize a solution of Eq. (1) (see, e.g., [14,26]). The important Kronecker's theorem [19] states that if the value of topological degree is not zero Eq. (1) has at least one zero within $\mathcal{D}$. To this end, several methods for the computation of the topological degree have been proposed in the past few years (see, e.g., [14,25]). One such method is the fundamental and pioneering Stenger's method [25] that in some classes of functions is an almost optimal complexity algorithm (see, e.g., [18,23,25]).

Once we have obtained a domain for which the value of the topological degree relative to this domain is nonzero, we are able to obtain upper and lower bounds for solution values. To this end, by computing a sequence of bounded domains with nonzero values of topological degree and decreasing diameters, we are able to obtain a region with arbitrarily small diameter that contains at least one solution of Eq. (1). However, although the nonzero value of topological degree plays an important role in the existence of a solution of Eq. (1), the computation of this value is a time-consuming procedure.

The bisection method which is briefly described below, avoids all calculations concerning the topological degree by implementing the concept of the *characteristic n-polyhedron criterion* for the existence of a solution of Eq. (1) within a given bounded domain. This criterion is based on the construction of a *characteristic n-polyhedron (CP)* [27,28,35,37]. This can be done as follows. Let $\mathcal{M}_n$ be the $2^n \times n$ matrix whose rows are formed by all possible combinations of -1 and 1. Consider now an oriented n-polyhedron Π^n, with vertices V_k, $k = 1, 2, \ldots, 2^n$. If the $2^n \times n$ *matrix of signs associated with F_n and Π^n*, $\mathcal{S}(F_n; \Pi^n)$, whose entries are the vectors $\operatorname{sgn} F_n(V_k) = \big(\operatorname{sgn} f_1(V_k), \operatorname{sgn} f_2(V_k), \ldots, \operatorname{sgn} f_n(V_k)\big)$, is identical to $\mathcal{M}_n$, possibly after some permutations of these rows, then Π_n is called *characteristic polyhedron relative to F_n*. Furthermore, if F_n is continuous, then, under some suitable assumptions on the boundary of Π^n, the topological degree of F_n relative to Π^n is not zero (see [37] for a proof), which implies the existence of a solution within Π^n. For more details on how to construct a CP and locate a desired solution see [27,31].

Next, we describe a generalized bisection method. This method combined with the above mentioned CP criterion, produces a sequence of characteristic polyhedra of decreasing size always containing the desired solution. We call it *characteristic bisection method.* This version of bisection does not require the computation of the topological degree at each step, as others do [14,26]. It can be applied to problems with imprecise function values, since it depends only on their signs.

The method simply amounts to constructing another refined characteristic polyhedron, by bisecting a known one, say Π^n. To do this, we compute the midpoint M of the longest edge $\langle V_i, V_j\rangle$, of Π^n (where the distances are measured in Euclidean norms). Then we obtain another characteristic polyhedron, Π^n_*, by comparing the sign, $\operatorname{sgn} F_n(M)$, of $F_n(M)$ with that of $F_n(V_i)$ and $F_n(V_j)$ and substituting M for that vertex for which the signs are identical [27,28,31]. Then we select the longest edge of Π^n_* and continue the above process. If one of the $\operatorname{sgn} F_n(V_i)$, $\operatorname{sgn} F_n(V_j)$ does not coincide with $\operatorname{sgn} F_n(M)$, we either continue with another edge or perform a relaxation process (for details see [27,28,31]).

The minimum number ζ of bisections of the edges of Π^n required to obtain a characteristic polyhedron Π^n_* whose longest edge length satisfies $\Delta(\Pi^n_*) \leqslant \varepsilon$, for some accuracy $\varepsilon \in (0,1)$, is given by [37]:

$$\zeta = \left\lceil \log_2 \left(\Delta(\Pi^n)\, \varepsilon^{-1} \right) \right\rceil . \tag{17}$$

Remark 12. Notice that ζ is independent of n and that the bisection algorithm has the same number of iterations as the bisection in one-dimension which is optimal and possesses asymptotically the best rate of convergence [22].

3.3 Generalized Method of Bisection for Simplices

Definition 7 [13]. Let $\sigma_0^m = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^m\rangle$ be an oriented m-simplex in $\mathbb{R}^n$, $m \leqslant n$, suppose that $\langle \upsilon^i, \upsilon^j\rangle$ is the longest edge of σ_0^m and let $\Upsilon = (\upsilon^i + \upsilon^j)/2$ be the midpoint of $\langle \upsilon^i, \upsilon^j\rangle$. Then the *bisection* of σ_0^m is the order pair of m-simplices $\langle \sigma_{10}^m, \sigma_{11}^m\rangle$ where:

$$\sigma_{10}^m = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \Upsilon, \upsilon^{i+1}, \ldots, \upsilon^j, \ldots, \upsilon^m \rangle,$$
$$\sigma_{11}^m = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^i, \ldots, \upsilon^{j-1}, \Upsilon, \upsilon^{j+1}, \ldots, \upsilon^m \rangle.$$

The m-simplices σ_{10}^m and σ_{11}^m will be called *lower simplex* and *upper simplex* respectively corresponding to σ_0^m while both σ_{10}^m and σ_{11}^m will be called *elements of the bisection* of σ_0^m. Suppose that $\sigma_0^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is an oriented n-simplex in $\mathbb{R}^n$ which includes at least one solution of Eq. (1). Suppose further that $\langle \sigma_{10}^n, \sigma_{11}^n \rangle$ is the bisection of σ_0^n and that there is at least one solution of the system (1) in some of its elements. Then this element will be called *selected n-simplex produced after one bisection of* σ_0^n and it will be denoted by σ_1^n. Moreover if there is at least one solution of the system (1) in both elements, then the selected n-simplex will be the lower simplex corresponding to σ_0^n. Suppose now that the bisection is applied with σ_1^n replacing σ_0^n giving thus the σ_2^n. Suppose further that this process continues for p iterations. Then we call σ_p^n *the selected n-simplex produced after p iterations of the bisection of* σ_0^n .

Definition 8 [29]. The *barycentric radius* $\beta(\sigma^m)$ of an m-simplex σ^m in $\mathbb{R}^n$ is the radius of the smallest ball centered at the barycenter of σ^m and containing the simplex. The barycentric radius $\beta(A)$ of a subset A of $\mathbb{R}^n$ is the supremum of the barycentric radii of simplices with vertices in A.

Theorem 5 [29]. *Any m-simplex $\sigma^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^m]$ in $\mathbb{R}^n$, $m \leqslant n$ is enclosable by the spherical surface S_β^{m-1} with radius $\beta(\sigma^m)$ given by:*

$$\beta(\sigma^m) = \frac{1}{m+1} \max_i \left(m \sum_{\substack{j=0 \\ j\neq i}}^{m} \|\upsilon^i - \upsilon^j\|_2^2 - \sum_{\substack{p=0 \\ p\neq i}}^{m-1} \sum_{\substack{q=p+1 \\ q\neq i}}^{m} \|\upsilon^p - \upsilon^q\|_2^2 \right)^{1/2} .$$

Remark 13. The barycentric radius $\beta(\sigma^n)$ of a n-simplex σ^n in $\mathbb{R}^n$ can be used to estimate error bounds for approximate fixed points or approximate roots of mappings in $\mathbb{R}^n$, by approximating a fixed point or a root by the barycenter of σ^n. Note that the computation of $\beta(\sigma^n)$ requires only the lengths of the edges of σ^n, which are also required in order to compute the diameter $\mathrm{diam}(\sigma^n)$ of σ^n. Furthermore, since the distance of the barycenter K of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ from the barycenter K_i of the i-th face $\sigma_{\neg i}^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$ of σ^n is equal to $\|K - \upsilon^i\|_2/n$ [26,29], then using Theorem 5 we can easily compute the value of $\gamma(\sigma^n) = \min_i \|K - K_i\|_2/\mathrm{diam}(\sigma^n)$. The value $\gamma(\sigma^n)$ can be used to estimate the thickness $\theta(\sigma^n)$ of σ^n, that is:

$$\theta(\sigma^n) = \min_i \left\{ \min_{x \in \sigma_{\neg i}^n} \left\| K - x \right\|_2 \right\} / \mathrm{diam}(\sigma^n).$$

In general, the thickness $\theta(\sigma^n)$ is important to piecewise linear approximations of smooth mappings and, in general, to simplicial and continuation methods for approximating fixed points or roots of systems of nonlinear equations.

Theorem 6 [13]. *Suppose that σ_0^m is an m-simplex in $\mathbb{R}^n$ and let σ_p^m be any m-simplex produced after p bisections of σ_0^m. Then*

$$\mathrm{diam}(\sigma_p^m) \leqslant \left(\sqrt{3}/2\right)^{\lfloor p/m \rfloor} \mathrm{diam}(\sigma_0^m), \tag{18}$$

where $\mathrm{diam}(\sigma_p^m)$ *and* $\mathrm{diam}(\sigma_0^m)$ *are the diameters of* σ_p^m *and* σ_0^m *respectively and* $\lfloor p/m \rfloor$ *is the largest integer less than or equal to* p/m.

Theorem 7 [26,32]. *Suppose that* σ_0^m, σ_p^m, $\mathrm{diam}(\sigma_0^m)$ *and* $\mathrm{diam}(\sigma_p^m)$ *are as in Theorem 6 and let* K_p^m *be the barycenter of* σ_p^m. *Then for any point* T *in* σ_p^m *the following relationship is valid*

$$\|T - K_p^m\|_2 \leqslant \frac{m}{m+1} \left(\sqrt{3}/2\right)^{\lfloor p/m \rfloor} \mathrm{diam}(\sigma_0^m). \tag{19}$$

Definition 9. Let σ^n be an n-simplex in $\mathbb{R}^n$ and let $\mathrm{diam}(\sigma^n)$ and $\mu\mathrm{diam}(\sigma^n)$ be the diameter and the microdiameter of σ^n respectively. Suppose that r is a solution of Eq. (1) in σ^n. Then we define the barycenter K^n of σ^n to be an *approximation* of r and the quantity

$$\varepsilon(\sigma^n) = \frac{n}{n+1} \left(\left(\mathrm{diam}(\sigma^n)\right)^2 - \frac{n-1}{2n} \left(\mu\mathrm{diam}(\sigma^n)\right)^2 \right)^{1/2}, \tag{20}$$

to be an *error estimate* for K^n.

Theorem 8 [26,32]. *Suppose that* σ_p^n *is the selected* n*-simplex produced after* p *bisections of an* n*-simplex* σ_0^n *in* $\mathbb{R}^n$. *Let* r *be a solution of Eq. (1) which is included in* σ_p^n *and that* K_p^n *and* $\varepsilon(\sigma_p^n)$ *are the approximation of* r *and the error estimate for* K_p^n *respectively. Then the following hold:*

(a) $\varepsilon(\sigma_p^n) \leqslant \dfrac{n}{n+1} \left(\sqrt{3}/2\right)^{\lfloor p/n \rfloor} \mathrm{diam}(\sigma_0^n),$

(b) $\varepsilon(\sigma_p^n) \leqslant \left(\sqrt{3}/2\right)^{\lfloor p/n \rfloor} \varepsilon(\sigma_0^n),$

(c) $\lim_{p \to \infty} \varepsilon_p = 0,$

(d) $\lim_{p \to \infty} K_p^n = r.$

3.4 Locating and Computing Periodic Orbits

Our approaches are illustrated here for methods for *locating and computing periodic orbits of nonlinear mappings* as well as *fixed points of the Poincaré map on a surface of section.* In general, analytic expressions for locating and computing these periodic orbits on fixed points are not available.

Many problems in a variety of areas of science and technology can be studied and tackled using periodic orbits of nonlinear mappings or dynamical systems. For example, such problems appear in Quantum Mechanics where a weighted

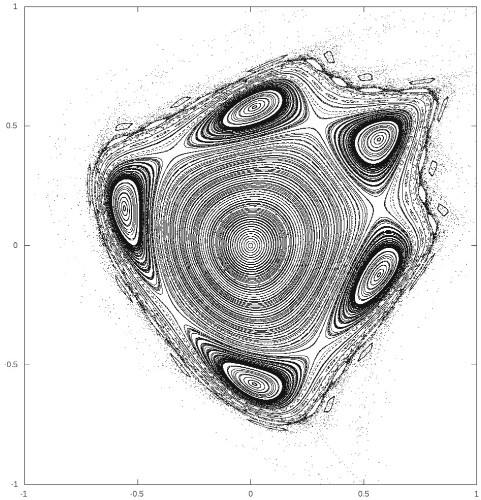

Fig. 1. Hénon mapping for $\cos\omega = 0.24$ and $g(x_1) = -x_1^2$

sum over unstable periodic orbits yields quantum mechanical energy level spacings as well as in Statistical Mechanics where a weighted, according to the values of their Liapunov exponents, sum over unstable periodic orbits can be used to calculate thermodynamic averages (see, e.g., [10]). Furthermore, periodic orbits play a major role in assigning the vibrational levels of highly excited polyatomic molecules. as well as in Celestial Mechanics and Galactic Dynamics.

Let us illustrated our approaches for the following *quadratic area-preserving two-dimensional Hénon's mapping* [31]:

$$\Phi_2 : \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + g(x_1) \end{pmatrix}, \tag{21}$$

where $(x_1, x_2) \in \mathbb{R}^2$ and $\omega \in [0, \pi]$ is the *rotation angle*. By choosing $\cos\omega = 0.24$ and $g(x_1) = -x_1^2$, we observe in the corresponding Hénon's mapping phase plot, illustrated in Fig. 1, that there is a chain of five "islands" around the center of the rectangle. The center points of each island contain a *stable elliptic periodic orbit* of *period five* ($p = 5$). Additionally, the five points where the islands connect consist an *unstable hyperbolic periodic orbit* of period five [31]. These points can be computed by applying the aforementioned methods. When one of these points is computed we can either subsequently apply the same method with different starting conditions and find another point of the periodic orbit or we can iterate the mapping using one of the computed points as starting point. For example, to produce the stable periodic orbit we can iterate the mapping using the following starting point: $(x_1, x_2) = (0.5672405470221847, -0.1223202134278941)$. The rotation number of this orbit is $\sigma = m_1/m_2 = 1/5$. It produces $m_2 = 5$ points by rotating around the origin $m_1 = 1$ times. Additionally, to compute the

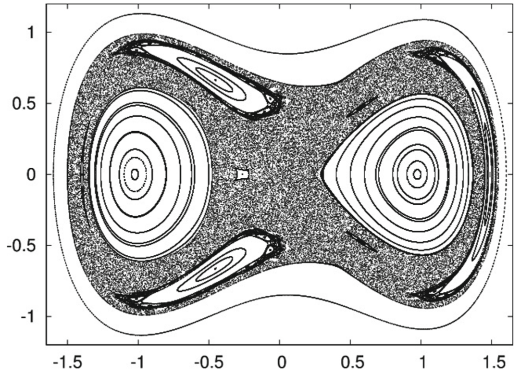

Fig. 2. A Poincaré surface of section of Duffing's oscillator for $\alpha = 0.05$ and $\beta = 2$

unstable periodic orbit, one can iterate the mapping using at starting point the $(x_1, x_2) = (0.2942106885737921, -0.4274862418615337)$ (for details see [31]).

Also, periodic orbits can be used in the study of the structure and breakdown properties of invariant tori in the case of symplectic mappings of direct relevance of the beam stability problem in circular accelerators like the Large Hadron Collider (LHC) machine at the European Organization for Nuclear Research (CERN). Such a *4-D symplectic mapping* can be defined as follows [31,36,38]:

$$\Phi_4 : \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}. \tag{22}$$

This mapping describes the (instantaneous) effect experienced by a hadronic particle as it passes through a magnetic focusing element of the FODO cell type, where x_1 and x_3 are the particle's deflections from the ideal (circular) orbit, in the horizontal and vertical directions respectively, and x_2, x_4 are the associated "momenta", while ω_1, ω_2 are related to the accelerator's betatron frequencies (or "tunes") q_x, q_y by $\omega_1 = 2\pi q_x$ and $\omega_2 = 2\pi q_y$ and constitute the main parameters that can be varied by an experimentalist see, e.g., [31,36,38] and the references thereof.

Next we consider a Poincaré surface of section for the conservative Duffing's oscillator. More specifically, the conservative Duffing's oscillator [7] can be described by the following equation:

$$\ddot{x} = x - x^3 + \alpha \cos \beta t, \tag{23}$$

which can be written as:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_1 - x_1^3 + \alpha \cos \beta t. \end{cases} \tag{24}$$

For the aforementioned dynamical system, we consider the Poincaré surface of section for the parameter values of $\alpha = 0.05$ and $\beta = 2$. Figure 2 illustrates the phase plot of this surface, in the $[-1.6, 1.6] \times [-1.2, 1.2]$ rectangle. For this example, we can observe two distinct islands along the $x_2 = 0$ axis. The center points of each island correspond to fixed points of period one ($p = 1$). Once again we can easily compute these two points by applying the aforementioned methods. The two center points correspond to $(x_1, x_2) = (-1.024572461190486, 0.0)$, and $(x_1, x_2) = (0.9746253482044169, 0.0)$.

In conclusion, our experience is that the generalized methods of bisection are very efficient and effective applied on the problems (21), (22) and (23). These is so, because, we have succeeded to compute rapidly and accurately periodic orbits (stable and unstable) for periods which reach up to the thousands. For detailed results we refer the interested reader to [7,31,36,38].

4 Synopsis

Generalizations the intermediate value theorems in several variables are presented. These theorems are very useful for the existence of solutions of systems of nonlinear equations, the existence of fixed points of continuous functions as well as the existence of periodic orbits of nonlinear mappings and similarly, fixed points of the Poincaré map on a surface of section. Based on the corresponding criteria for the existence of a solution or a fixed point emanated by the intermediate value theorems, generalized bisection methods for approximating zeros or fixed points of continuous functions are given. These bisection methods require only the algebraic signs of the function values and are of major importance for studying and tackling problems with imprecise information.

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References

1. Alefeld, G., Frommer, A., Heindl, G., Mayer, J.: On the existence theorems of Kantorovich, Miranda and Borsuk. Electron. Trans. Numer. Anal. **17**, 102–111 (2004)
2. Bánhelyi, B., Csendes, T., Hatvan, L.: On the existence and stabilization of an upper unstable limit cycle of the damped forced pendulum. J. Comput. Appl. Math. **371**, 112702 (2020)
3. Bolzano, B.: Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werten, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege. Prague (1817)
4. Brönnimann, H., Emiris, I.Z., Pan, V., Pion, S.: Sign determination in residue number systems. Theor. Comput. Sci. **210**, 173–197 (1999)
5. Brouwer, L.E.J.: Über Abbildungen von Mannigfaltigkeiten. Math. Ann. **71**, 97–115 (1912)

6. Cauchy, A.-L.: Cours d'Analyse de l'École Royale Polytechnique, Paris (1821). (Reprinted in Oeuvres Completes, Series 2, vol. 3)
7. Drossos, L., Ragos, O., Vrahatis, M.N., Bountis, T.C.: Method for computing long periodic orbits of dynamical systems. Phys. Rev. E **53**(1), 1206–1211 (1996)
8. Emiris I.Z., Mourrain B., Vrahatis M.N.: Sign methods for counting and computing real roots of algebraic systems. RR-3669, Inria (1999). inria-00073003
9. Emiris I.Z., Mourrain B., Vrahatis M.N.: Sign methods for enumerating solutions of nonlinear algebraic systems. In: Proceedings of the Fifth Hellenic European Conference on Computer Mathematics and Its Applications, vol. 2, pp. 469–473, Athens, Greece (2002)
10. Gutzwiller, M.C.: Chaos in Classical and Quantum Mechanics. Springer, New York (1990). https://doi.org/10.1007/978-1-4612-0983-6
11. Heindl, G.: Generalizations of theorems of Rohn and Vrahatis. Reliable Comput. **21**, 109–116 (2016)
12. Jarník, V.: Bernard Bolzano and the foundations of mathematical analysis. In: Bolzano and the Foundations of Mathematical Analysis, pp. 33–42. Society of Czechoslovak Mathematicians and Physicists, Prague (1981)
13. Kearfott, R.B.: A proof of convergence and an error bound for the method of bisection in $\mathbb{R}^n$. Math. Comp. **32**(144), 1147–1153 (1978)
14. Kearfott, R.B.: An efficient degree-computation method for a generalized method of bisection. Numer. Math. **32**, 109–127 (1979). https://doi.org/10.1007/BF01404868
15. Knaster, B., Kuratowski, K., Mazurkiewicz, S.: Ein Beweis des Fixpunkt-satzes für n-dimensionale Simplexe. Fund. Math. **14**, 132–137 (1929)
16. Milgrom, P., Mollner, J.: Equilibrium selection in auctions and high stakes games. Econometrica **86**(1), 219–261 (2018)
17. Miranda, C.: Un' osservatione su un theorema di Brouwer. Bollettino dell'U.M.I. **3**, 5–7 (1940)
18. Mourrain, B., Vrahatis, M.N., Yakoubsohn, J.C.: On the complexity of isolating real roots and computing with certainty the topological degree. J. Complex. **18**(2), 612–640 (2002)
19. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Classics in Applied Mathematics vol. 30. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA (2000)
20. Poincaré, H.: Sur certaines solutions particulières du problème des trois corps. Comptes rendus de l'Académie des Sciences Paris **91**, 251–252 (1883)
21. Poincaré, H.: Sur certaines solutions particulières du problème des trois corps. Bull. Astronomique **1**, 63–74 (1884)
22. Sikorski, K.: Bisection is optimal. Numer. Math. **40**, 111–117 (1982)
23. Sikorski, K.: Optimal Solution of Nonlinear Equations. Oxford University Press, New York (2001)
24. Sperner, E.: Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. Abh. Math. Sem. Hamburg **6**, 265–272 (1928)
25. Stenger, F.: Computing the topological degree of a mapping in $\mathbb{R}^n$. Numer. Math. **25**, 23–38 (1975)
26. Vrahatis, M.N.: An error estimation for the method of bisection in $\mathbb{R}^n$. Bull. Greek Math. Soc. **27**, 161–174 (1986)
27. Vrahatis, M.N.: Solving systems of nonlinear equations using the nonzero value of the topological degree. ACM Trans. Math. Softw. **14**, 312–329 (1988)
28. Vrahatis, M.N.: CHABIS: a mathematical software package for locating and evaluating roots of systems of nonlinear equations. ACM Trans. Math. Softw. **14**, 330–336 (1988)

29. Vrahatis, M.N.: A variant of Jung's theorem. Bull. Greek Math. Soc. **29**, 1–6 (1988)
30. Vrahatis, M.N.: A short proof and a generalization of Miranda's existence theorem. Proc. Amer. Math. Soc. **107**, 701–703 (1989)
31. Vrahatis, M.N.: An efficient method for locating and computing periodic orbits of nonlinear mappings. J. Comput. Phys. **119**, 105–119 (1995)
32. Vrahatis, M.N.: Simplex bisection and Sperner simplices. Bull. Greek Math. Soc. **44**, 171–180 (2000)
33. Vrahatis, M.N.: Generalization of the Bolzano theorem for simplices. Topol. Appl. **202**, 40–46 (2016)
34. Vrahatis, M.N.: Intermediate value theorem for simplices for simplicial approximation of fixed points and zeros. Topol. Appl. **275**, 107036 (2020)
35. Vrahatis, M.N.: Generalizations of the intermediate value theorem for approximating fixed points and zeros of continuous functions. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) NUMTA 2019, Part II. LNCS, vol. 11974, pp. 223–238. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-40616-5_17
36. Vrahatis, M.N., Bountis, T.C., Kollmann, M.: Periodic orbits and invariant surfaces of 4D nonlinear mappings. Int. J. Bifurcat. Chaos **6**, 1425–1437 (1996)
37. Vrahatis, M.N., Iordanidis, K.I.: A rapid generalized method of bisection for solving systems of non-linear equations. Numer. Math. **49**, 123–138 (1986). https://doi.org/10.1007/BF01389620
38. Vrahatis, M.N., Isliker, H., Bountis, T.C.: Structure and breakdown of invariant tori in a 4D mapping model of accelerator dynamics. Int. J. Bifurcat. Chaos **7**, 2707–2722 (1997)