Intern. J. Computer Math., Vol. 55, pp. 235–244
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Gordon and Breach Science Publishers SA
Printed in Malaysia

A NEW DIMENSION—REDUCING METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

T. N. GRAPSA and M. N. VRAHATIS*

Department of Mathematics, University of Patras, GR-261.10 Patras, Greece

(Received 8 February 1994)

A new method for the numerical solution of systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$ is presented. Firstly, this method reduces the dimensionality of the system in such a way that it can lead to an iterative approximate formula for the computation of $n-1$ components of the solution and subsequently it perturbs the corresponding Jacobian by using proper perturbation parameters. The remaining component of the solution is evaluated separately using the final approximations of the other components. This reduced iterative formula generates a sequence of points in $\mathbb{R}^{n-1}$ which converges quadratically to the $n-1$ components of the solution. Moreover, it does not require a good initial guess for one component of the solution and it does not directly perform function evaluations. Thus, it can be applied to problems with imprecise function values. A proof of convergence is given and numerical applications are presented.

KEY WORDS: Newton's method, dimension-reducing methods, nonlinear SOR, reduction to one-dimensional equations, imprecise function values, systems of nonlinear equations, numerical solution, zeros, quadratic convergence.

C.R. CATEGORY: G.1.5.

MATHEMATICS SUBJECT CLASSIFICATION (1991): 65H10.

1 INTRODUCTION

Recently, new methods have been proposed [2–4] for the numerical solution of a system of nonlinear algebraic and/or transcendental equations:

$$F(x) = \Theta^n = (0, 0, \ldots, 0), \tag{1.1}$$

where $F = (f_1, \ldots, f_n): \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping on an open neighborhood $\mathscr{D}^* \subset \mathscr{D}$ of a solution $x^* \in \mathscr{D}$ of System (1.1). These methods incorporate the advantages of Newton and nonlinear SOR algorithms [7]. Specifically, although these methods use reduction to simpler one-dimensional nonlinear equations, they converge quadratically.

* Part of the work of the second author was done at the European Organization for Nuclear Research (CERN, Geneva) and the National Institute of Nuclear Physics (INFN, Bologna).

In this paper, we derive and apply a new iterative procedure, for the numerical solution of systems (1.1), which also incorporates the advantages of Newton and SOR algorithms. The idea behind this method is the reduction of the dimensionality of the system as well as the perturbation of the Jacobian of the reduced system, by using proper perturbation parameters $A'_j, j = 1, \ldots, n-1$. Although this new procedure uses reduction to simpler one-dimensional nonlinear equations, it generates a quadratically converging sequence of points in $\mathbb{R}^{n-1}$ which converges to the $n-1$ components of the solution, while a proper choice of the parameters A'_j accelerates the convergence even further. The remaining component of the solution is evaluated separately using the final approximations of the others. For this component an initial guess is not necessary and it is also at the user's disposal to choose which will be the remaining component, according to the problem. Also this method does not directly need any function evaluation and it compares favourably with Newton's method when it is difficult to evaluate the function values accurately, as well as when the Jacobian at the solution is singular or when the Jacobian is ill-conditioned.

2 THE NEW METHOD AND ITS CONVERGENCE

NOTATION 2.1 *Throughout this paper* $\mathbb{R}^n$ *is the n-dimensional real space of column vectors x with components* $x_1, x_2, \ldots, x_n$; $(y;z)$ *represents the column vector with components* $y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_k$; $\partial_i f(x)$ *denotes the partial derivative of* $f(x)$ *with respect to the* i*-th variable* x_i; $\bar{\mathscr{A}}$ *denotes the closure of the set* $\mathscr{A}$ *and* $f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ *defines the mapping obtained by holding* $x_1, \ldots, x_{i-1}$, $x_{i+1}, \ldots, x_n$ *fixed.*

Our interest lies in obtaining a sequence $\{x^p\}, p = 0, 1, \ldots$ of points in $\mathbb{R}^n$ which converges to a solution $x^* = (x_1^*, \ldots, x_n^*) \in \mathscr{D}$ of the system (1.1). To do this, we consider the sets $\mathscr{B}_i, i = 1, \ldots, n$ to be those connected components of $f_i^{-1}(0)$ containing x^* on which $\partial_n f_i \neq 0$, for $i = 1, \ldots, n$ respectively. Next, applying the Implicit Function Theorem [7, 2–4] for each one of the components $f_i, i = 1, \ldots, n$ of F we can find open neighborhoods $\mathscr{A}_i^* \subset \mathbb{R}^{n-1}$ and $\mathscr{A}_{2,i}^* \subset \mathbb{R}, i = 1, \ldots, n$ of the points $y^* = (x_1^*, \ldots, x_{n-1}^*)$ and x_n^* respectively, such that for any $y = (x_1, \ldots, x_{n-1}) \in \bar{\mathscr{A}}_1^*$ there exist unique mappings φ_i defined and continuous in $\mathscr{A}_1^*$ such that:

$$x_n = \varphi_i(y) \in \bar{\mathscr{A}}_{2,i}^*, \quad i = 1, \ldots, n, \tag{2.1}$$

and

$$f_i(y; \varphi_i(y)) = 0, \quad i = 1, \ldots, n. \tag{2.2}$$

Moreover, the partial derivatives $\partial_j \varphi_i, j = 1, \ldots, n-1$ exist in $\mathscr{A}_1^*$ for each $\varphi_i, i = 1, \ldots, n$, they are continuous in $\mathscr{A}_1^*$ and they are given by:

$$\partial_j \varphi_i(y) = -\partial_j f_i(y; \varphi_i(y)) / \partial_n f_i(y; \varphi_i(y)), \quad i = 1, \ldots, n, \; j = 1, \ldots, n-1. \tag{2.3}$$

Next, working exactly as in [3], we can obtain the following iterative scheme for the computation of the $n-1$ components of x^*:

$$y^{p+1} = y^p + A_p^{-1} V_p, \quad p = 0, 1, \ldots, \tag{2.4}$$

where:

$$\begin{aligned} y^p &= [x_i^p], \quad i = 1, \ldots, n-1, \\ \mathscr{A}_p &= [a_{ij}] = [\hat{\partial}_j f_i(y^p; x_n^{p,i}) / \hat{\partial}_n f_i(y^p; x_n^{p,i}) \\ &\quad - \hat{\partial}_j f_n(y^p; x_n^{p,n}) / \hat{\partial}_n f_n(y^p; x_n^{p,n})], \quad i,j = 1, \ldots, n-1, \\ V_p &= [v_i] = [x_n^{p,i} - x_n^{p,n}], \quad i = 1, \ldots, n-1, \end{aligned} \tag{2.5}$$

with $x_n^{p,i} = \varphi_i(y^p)$, while after a desired number of iterations of (2.4), say $p = m$, the nth component of x^* is approximated by means of the following relation:

$$x_n^{m+1} = x_n^{m,n} - \sum_{j=1}^{n-1} (x_j^{m+1} - x_j^m) \hat{\partial}_j f_n(y^m; x_n^{m,n}) / \hat{\partial}_n f_n(y^m; x_n^{m,n}). \tag{2.6}$$

Consider now the mapping:

$$\begin{aligned} &\Lambda = (\lambda_1, \ldots, \lambda_{n-1}): \bar{\mathscr{A}}_1^* \subset \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \text{ by} \\ &\lambda_i(y) = -\varphi_i(y) + \varphi_n(y), \quad i = 1, \ldots, n-1, \end{aligned} \tag{2.7}$$

and denote the corresponding Jacobian by Λ'; then it is obvious that the iterative scheme (2.4) is equivalent to the following one:

$$y^{p+1} = y^p + s^p, \quad p = 0, 1, \ldots, \tag{2.8}$$

where s^p is the solution of the linear system:

$$\Lambda'(y^p) s^p = -\Lambda(y^p). \tag{2.9}$$

To accelerate the convergence of the iterates (2.8), we perturb the corresponding Jacobian matrix using proper perturbation parameters $A_j', j = 1, \ldots, n-1$. To this

end we consider the mapping:

$$G = (g_1, \ldots, g_{n-1}): \bar{\mathscr{A}}_1^* \subset \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \text{ by}$$

$$g_i(y) = \lambda_i(y) + \sum_{j=1}^{n-1} A'_j x_j, \quad i = 1, \ldots, n-1, \tag{2.10}$$

where the vector A' of the perturbation parameters $A'_j, j = 1, \ldots, n-1$, is taken such that the inner product $\langle x, A' \rangle = 0, \forall\, x \in \bar{\mathscr{A}}_1^*$.

Now, utilizing the mapping (2.10) instead of (2.7) we obtain the following iterative scheme for the computation of the $n-1$ components of x^*:

$$y^{p+1} = y^p + U_p^{-1} V_p, \quad p = 0, 1, \ldots, \tag{2.11}$$

where:

$$\begin{aligned} y^p &= [x_i^p], \quad i = 1, \ldots, n-1, \\ U_p &= [a_{ij}] = [\partial_j f_i(y^p; x_n^{p,i}) / \partial_n f_i(y^p; x_n^{p,i}) \\ &\qquad - \partial_j f_n(y^p; x_n^{p,n}) / \partial_n f_n(y^p; x_n^{p,n}) + A'_j], \quad i,j = 1, \ldots, n-1, \\ V_p &= [v_i] = [x_n^{p,i} - x_n^{p,n}], \quad i = 1, \ldots, n-1. \end{aligned} \tag{2.12}$$

Finally, after a desired number of iterations of the above scheme, say $p = m$, we can approximate the nth component of x^* using Relation (2.6).

Remark 2.1 The perturbation parameters $A'_j, j = 1, \ldots, n-1$, can be estimated in each iteration from the equation:

$$\langle x^p, A' \rangle = 0, \quad p = 0, 1, \ldots, \tag{2.13}$$

by choosing $n-2$ arbitrary parameters and calculating at each iteration the $(n-1)$st parameter from (2.13).

Remark 2.2 The new process does not require the expressions φ_i but only the values $x_n^{p,i}$ which are given by the solution of the one-dimensional equations $f_i(x_1^p, \ldots, x_{n-1}^p, \cdot) = 0$. So, by holding $y^p = (x_1^p, \ldots, x_{n-1}^p)$ fixed, we can solve the equations:

$$f_i(y^p; r_i^p) = 0, \quad i = 1, \ldots, n, \tag{2.14}$$

for r_i^p in the interval $(\alpha, \alpha + \beta)$ with an accuracy ε. Of course, we can use any one of the well-known one-dimensional methods [7] to solve the above equations. Here we shall use the one-dimensional bisection, since frequently the steps β are long and also a few significant digits are required for the computations of the roots of Equations (2.14). A simplified version of the bisection method can be found in [2–4].

Remark 2.3 Relative procedures for obtaining x^* can be constructed by replacing x_n with any one of the components $x_1, \ldots, x_{n-1}$, for example x_i, and taking $y = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Next, we give a proof of the convergence of the new method described by the iterates (2.11) and Relation (2.6).

THEOREM 2.1 *Suppose that* $F=(f_1,\dots,f_n)\colon \mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ *is twice-continuously differentiable on an open neighborhood* $\mathscr{D}^*\subset\mathscr{D}$ *of a point* $x^*=(x_1^*,\dots,x_n^*)\in\mathscr{D}$ *for which* $F(x^*)=\Theta^n$. *Let* $\mathscr{B}_i, i=1,\dots,n$ *be those connected components of* $f_i^{-1}(0)$ *containing* x^* *on which* $\partial_n f_i\neq 0$ *for* $i=1,\dots,n$, *respectively. Suppose further that the matrix* A_*, *which is obtained from the matrix* A_p *of* (2.5) *at* $y^*=(x_1^*,\dots,x_{n-1}^*)$, *is nonsingular and that* $\Xi=[\xi_{ij}]=A_j'$ *is the rank-1* $n-1\times n-1$ *matrix where the vector* $A'=[A_j'], j=1,\dots,n-1$, *determines the perturbation parameters so that the inner product* $\langle x,A'\rangle=0, \forall\, x\in\mathscr{D}^*$ *and*

$$\|A_*^{-1}\|\,\|\Xi\|<1. \tag{2.15}$$

Then the iterates of (2.11) *and Relation* (2.6) *will converge to* x^*, *provided the initial guess* $y^0=(x_1^0,\dots,x_{n-1}^0)$ *is sufficiently close to* y^*. *Moreover, the iterates* $y^p, p=0,1,\dots,$ *of* (2.11) *have order of convergence two.*

Proof Obviously, the iterates (2.11) can be written as follows:

$$y^{p+1}=y^p-U_p^{-1}W_p,\quad p=0,1,\dots, \tag{2.16}$$

where:

$$\begin{aligned} &y^p=[x_i^p],\quad i=1,\dots,n-1,\\ &U_p=[a_{ij}]=[\partial_j f_i(y^p;x_n^{p,i})/\partial_n f_i(y^p;x_n^{p,i})\\ &\qquad\qquad -\partial_j f_n(y^p;x_n^{p,n})/\partial_n f_n(y^p;x_n^{p,n})+A_j'],i,j=1,\dots,n-1,\\ &W_p=[w_i]=[-x_n^{p,i}+x_n^{p,n}], i=1,\dots,n-1 \end{aligned} \tag{2.17}$$

Using the mapping (2.10) the above scheme can be written in the following form:

$$y^{p+1}=y^p-G'(y^p)^{-1}G(y^p),\quad p=0,1,\dots, \tag{2.18}$$

where G' denotes the corresponding Jacobian. For the mapping G, since Relation (2.15) holds, by Permutation Lemma (Banach Lemma) [6, 7, 4], the matrix G' at y^* is nonsingular and, consequently, by the well-known Newton's convergence theorem [6, 2–4] for an initial guess y^0 sufficiently close to y^*, the iterates $y^p, p=0,1,\dots$ of (2.11) converge to y^* and the order of convergence is two.

Suppose now that for some p, for example $p=m$, we obtain $y^m=y^*$. Then, Relation (2.6) yields:

$$x_n^{m+1}=\varphi_n(y^*), \tag{2.19}$$

or

$$x_n^{m+1} = x_n^*. \tag{2.20}$$

Thus the theorem is proved. □

3 NUMERICAL APPLICATIONS

The new method has been applied to several examples of nonlinear systems of different dimension. We found that the procedure behaved predictably and reliably and its speed of convergence was quite satisfactory. Here we present some typical computational results comparing the new scheme of Newton's method and also to the dimension-reducing methods [3,4] on three examples for Problem (1.1) (studied also in [3,4]), with $F = (f_1, f_2, \ldots, f_n)$ given by:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1^3 - x_1 x_2 x_3 = 0, \\ f_2(x_1, x_2, x_3) &= x_2^2 - x_1 x_3 = 0, \\ f_3(x_1, x_2, x_3) &= 10x_1 x_3 + x_2 - x_1 - 0.1 = 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 x_3 - x_3 e^{x_1^2} + 10^{-4} = 0, \\ f_2(x_1, x_2, x_3) &= x_1(x_1^2 + x_2^2) + x_2^2(x_3 - x_2) = 0, \\ f_3(x_1, x_2, x_3) &= x_1^3 + x_3^3 = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4, x_5) &= 2x_1 + x_2 + x_3 + x_4 + x_5 - 6 = 0, \\ f_2(x_1, x_2, x_3, x_4, x_5) &= x_1 + 2x_2 + x_3 + x_4 + x_5 - 6 = 0, \\ f_3(x_1, x_2, x_3, x_4, x_5) &= x_1 + x_2 + 2x_3 + x_4 + x_5 - 6 = 0, \\ f_4(x_1, x_2, x_3, x_4, x_5) &= x_1 + x_2 + x_3 + 2x_4 + x_5 - 6 = 0, \\ f_5(x_1, x_2, x_3, x_4, x_5) &= x_1 x_2 x_3 x_4 x_5 - 1 = 0. \end{aligned} \tag{3.3}$$

System (3.1) has two roots $r_1 = (0.1, 0.1, 0.1)$ and $r_2 = (-0.1, -0.1, -0.1)$ within the cube $[-0.1, 0.1]^3$ and its Jacobian at these roots is nonsingular. However, this system has a particular difficulty since the function values at some points (for example at points close to the origin) cannot be accurately achieved. On the other hand the Jacobian of System (3.2) at its root $r = (-0.99990001 \cdot 10^{-4}, -0.99990001 \cdot 10^{-4}, 0.99990001 \cdot 10^{-4})$ is singular. System (3.3), which is a well-known test case, (*Brown's almost linear system*) [5], has roots of the form (a, a, a, a, a^{-4}), where a satisfies the equation $a^4(5a - 6) + 1 = 0$ and its Jacobian at these roots is nonsingular. The difficulty of this system is that its Jacobian at all the above roots is ill-conditioned. For this case we shall present results for the roots $r_1 = (1, 1, 1, 1, 1)$,

$r_2 = (0.91635458253385, \ldots, 1.41822708733080)$ and $r_3 = (-0.57904308849412, \ldots, 8.89521544247060)$ reported in the tables.

In Tables 1, 2 and 3 we exhibit the number of iterations needed to obtain an approximate solution of Systems (3.1), (3.2) and (3.3) correspondingly, for required accuracy 10^{-14}, by using Newton's method and the iterative scheme (2.11) and Relation (2.6) of this paper, for several starting points $x^0 = (x_1^0, \ldots, x_n^0)$ and values $A' = [A'_j], j = 1, \ldots, n-1$. We set arbitrary $n-2$ values of the A'_j and we calculate the remaining $(n-1)$st in each iteration in such a way that $\langle x^p, A' \rangle = 0$.

In these tables "A'" indicates the vector of the perturbation parameters, "j" the coordinate for which the equation $\langle x^p, A' \rangle = 0$ is solved, "IT" the number of iterations, "FE" the number of function evaluations, "AS" the total number of algebraic

Table 1 Comparison of Newton's Method with New Dimension-Reducing Method for System (3.1).

Newton's method						*New Dimension-Reducing*			
						$A' = (-0.1, 0)$, $j = 2$			
x_1^0	x_2^0	x_3^0	IT	FE	r_i	IT	FE	AS	r_i
0.4	0.5	0.5	53	636	r_2	7	63	210	r_1
0.5	−0.5	2	28	336	r_2	4	36	120	r_2
2	−2	−2	43	516	r_1	5	45	150	r_2
−2	2	−3	64	768	r_2	6	54	180	r_2
−4	−2	−2.5	53	636	r_2	5	45	150	r_1
−4	−2	−3	46	552	r_1	5	45	150	r_1
−4.5	−2	−2	71	852	r_1	5	45	150	r_1
−5	−2	−3	43	516	r_2	5	45	150	r_1
−10	−2	−2.5	63	756	r_2	6	54	180	r_1
−100	100	−2	72	864	r_1	6	54	180	r_2
50	−50	−200	62	744	r_1	6	54	180	r_2
100	−100	50	122	1464	r_1	6	54	180	r_2

Table 2 Comparison of Newton's Method with New Dimension-Reducing Medthod for System (3.2).

Newton's method					*New Dimension-Reducing*		
					$A' = (-0.00001, 0)$, $j = 2$		
x_1^0	x_2^0	x_3^0	IT	FE	IT	FE	AS
−0.5	−0.4	−0.1	119	1428	4	36	120
0.1	0.1	−3	86	1032	2	18	60
0.5	0.4	0.1	108	1296	4	36	120
−2	−2	−1	213	2556	2	18	60
−3	−3	−3	53	636	3	27	90
−5	−5	−4.8	232	2784	4	36	120
−10	−10	−2	252	3024	3	27	90
2	2	1	303	3636	2	18	60
3	3	3	122	1464	3	27	90
10	10	2	253	3036	3	27	90
15	15	14	267	3204	3	27	90
15	15	15	247	2964	3	27	90

Table 3 Comparison of Newton's Method with New Dimension Reducing-Method for System (3.3).

Newton's method								*New Dimension-Reducing*			
								$A' = (0.2, 0.2, 0.2, 0)$, $j = 4$			
x_1^0	x_2^0	x_3^0	x_4^0	x_5^0	*IT*	*FE*	r_i	*IT*	*FE*	*AS*	r_i
−0.25	−0.25	0.25	−0.25	−0.625	185	5550	r_3	6	150	300	r_3
−0.5	−0.5	−0.5	−0.5	−0.625	178	5340	r_3	5	125	250	r_3
−0.5	−0.5	0.5	−0.5	−1.25	175	5250	r_3	7	175	350	r_2
−1	2	−1.5	2	1.5	42	1260	r_3	6	150	300	r_2
−2	−2	−2	−2	−2.5	184	5520	r_1	7	175	350	r_1
−3	−3	−3	−3	−3.75	189	5670	r_3	7	175	350	r_1
−4	−4	−4	2	1.5	85	2520	r_1	6	150	300	r_2
−4	−4	4	2	1.5	80	2400	r_3	5	125	250	r_1
−8	−8	−8	−8	−10	189	5670	r_1	7	175	350	r_1
−10	3	4	2	1.5	87	2610	r_1	6	150	300	r_1
−20	−20	20	20	1.5	177	5310	r_3	6	150	300	r_1
10	10	10	10	1.5	138	4140	r_3	7	175	350	r_1

signs that are required for applying the modified one-dimensional bisection method [2–4] and "r_i" denotes the root to which the corresponding method converges.

We applied the above schemes using Crout's method with partial pivoting for the corresponding linear systems.

From the results shown in the tables we observe that the new method of this paper compares favourably with Newton's method.

Moreover, we have compared the new method with the dimension-reducing methods of [3, 4] and observed that the number of iterations of the new method is less than or equal to the corresponding iterations of the dimension-reducing methods. Some of these comparisons are exhibited in Tables 4, 5 and 6.

We have also applied the new scheme to problems with precise function values for which the corresponding Jacobian is nonsingular and well-conditioned and observed that the number of iterations of the new method is less than or equal to the corresponding number of iterations of Newton's method and the dimensional reducing methods.

Table 4 Comparison of Scheme (2.12)–(2.14) of [3] (DR) and Scheme (3.24)–(3.26) of [4] (MDR) using $A' = (0, -0.00001)$, $j = 1$ with New Dimension-Reducing Method (PDR) using $A' = (-0.1, 0)$, $j = 2$ for System (3.1).

		DR				*MDR*				*PDR*			
x_1^0	x_2^0	*IT*	*FE*	*AS*	r_i	*IT*	*FE*	*AS*	r_i	*IT*	*FE*	*AS*	r_i
−4	−2	6	54	180	r_1	5	45	150	r_1	5	45	150	r_1
−4.5	2	8	72	640	r_2	8	72	242	r_2	7	63	210	r_2
0.5	−0.5	5	45	150	r_2	5	45	150	r_2	4	36	120	r_2
2	−2	6	54	180	r_2	6	54	180	r_2	5	45	150	r_2
−5	−2	6	54	180	r_1	6	54	180	r_1	5	45	150	r_1

Table 5 Comparison of Scheme (2.12)–(2.14) of [3] (DR) and Scheme (3.24)–(3.26) of [4] (MDR) using $A' = (0, -3)$, $j = 1$ with New Dimension-Reducing Method (PDR) using $A' = (-0.00001, 0)$, $j = 2$ for System (3.2).

		DR			*MDR*			*PDR*		
x_1^0	x_2^0	*IT*	*FE*	*AS*	*IT*	*FE*	*AS*	*IT*	*FE*	*AS*
0.5	0.5	2	18	60	3	27	90	2	18	60
−3	−3	3	27	90	3	27	90	3	27	90
15	15	4	36	120	6	54	180	3	27	90
0.1	0.1	2	18	60	2	18	60	2	18	60
−2	−2	3	27	90	3	27	90	2	18	60

Table 6 Comparison of Scheme (2.12)–(2.14) of [3] (DR) and Scheme (3.24)–(3.26) of [4] (MDR) using $A' = (\pm 0.2, \pm 0.2, \pm 0.2, 0)$, $j = 4$ with New Dimension-Reducing Method (PDR) using $A' = (\pm 0.2, \pm 0.2, 0, 0)$, $j = 4$ for System (3.3).

				DR				*MDR*				*PDR*			
x_1^0	x_2^0	x_3^0	x_4^0	*IT*	*FE*	*AS*	r_i	*IT*	*FE*	*AS*	r_i	*IT*	*FE*	*AS*	r_i
−8	−3	−4	2	7	175	350	r_1	7	175	350	r_2	6	150	300	r_2
10	3	4	2	7	175	350	r_1	6	150	300	r_1	5	125	250	r_1
−2	−2	4	4	7	175	350	r_1	6	150	300	r_1	6	150	300	r_1
10	−3	1.5	−3	7	175	350	r_1	7	175	350	r_1	5	125	250	r_1
0.1	−4	4	2	5	125	250	r_2	7	175	350	r_2	5	125	250	r_1

Moreover, we have observed that using different perturbation parameters we are able to compute different solutions starting from the same initial guess. For instance, consider System (3.1) with the initial guess $x^0 = (-2, 2)$. Taking $A' = (-2, 0)$ and $j = 2$ we obtain, after six iterations, the solution r_1, while for $A' = (-0.1, 0)\, j = 2$ we get, after six iterations, the solution r_2.

4 CONCLUDING REMARKS

The method we have analyzed in this paper compares favourably with Newton's method, since it has order of convergence two for any values of the perturbation parameters A'_j, while, for proper values of A'_j the convergence can be significantly accelerated.

Also, although the method of this paper uses reduction to simpler one-dimensional equations, it converges quadratically to $n-1$ components of the solution, while the remaining component of the solution is evaluated separately using the final approximations of the other components. Thus, it does not require a good initial estimate for one component of the solution. Besides, this method does not directly perform function evaluations, since it uses the modified one-dimensional bisection method [2–4]. It only requires correct algebraic signs in finding the various $\varphi_i(y)$.

Moreover, it is well-known that Newton's method often converges to a solution almost independently of the initial guess, while there exist several solutions close to

each other, all desirable for the applications [1, 8, 9]. Applying the new method and using various perturbation parameters we are able to compute different solutions for the same initial guess.

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