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SOLVING SYSTEMS OF NONLINEAR EQUATIONS IN $\mathbb{R}^n$ USING A ROTATING HYPERPLANE IN $\mathbb{R}^{n+1}$

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A procedure which accelerates the convergence of iterative methods for the numerical solution of systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$ is introduced. This procedure uses a rotating hyperplane in $\mathbb{R}^{n+1}$, whose rotation axis depends on the current approximation of $n-1$ components of the solution. The proposed procedure is applied here on the traditional Newton's method and on a recently proposed "dimension-reducing" method [5] which incorporates the advantages of nonlinear SOR and Newton's algorithms. In this way, two new modified schemes for solving nonlinear systems are correspondingly obtained. For both of these schemes proofs of convergence are given and numerical applications are presented.

KEY WORDS: Newton's method, dimension-reducing method, reduction to one-dimensional equations, nonlinear SOR, m-step SOR–Newton, implicit function theorem, imprecise function values, bisection method, nonlinear equations, numerical solution, zeros, quadratic convergence.

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1. INTRODUCTION

Perhaps the most familiar and often used method for solving a system of nonlinear equations

$$F(x) = \Theta^n = (0, 0, \dots, 0), \tag{1.1}$$

where $F = (f_1, \dots, f_n): \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a Fréchet differentiable mapping on an open neighborhood $\mathscr{D}^* \subset \mathscr{D}$ of a solution $x^* \in \mathscr{D}$ of the system (1.1), is Newton's iterative scheme,

$$x^{p+1} = x^p - F'(x^p)^{-1} F(x^p), \qquad p = 0, 1, \dots . \tag{1.2}$$

According to this scheme, if the Jacobian $F'(x^*)$ is nonsingular and $F'(x)$ is Lipschitz continuous in $\mathscr{D}^*$, the iterates x^p of (1.2) will converge quadratically to a solution of (1.1) x^*, provided the initial guess x^0 is sufficiently close to x^* [10, 12].

The quadratic convergence of Newton's method is attractive. However, as is well known, the method depends on a good initial approximation [3] and requires in general $n^2 + n$ function evaluations per iteration besides the solution of an $n \times n$

linear system. Moreover, the behavior of Newton's method is problematic when $F'(x^*)$ is singular since, in that case, (1.2) does not converge quadratically and, in general, is not appropriate for approximations of x^* with a high accuracy. For this reason there are procedures which under some assumptions (such as rank $F'(x^*)=n-1$) can attain a highly accurate solution x^* by enlarging the system (1.1) to one which is at least $(2n+1)$-dimensional [24, 25]. Also, Newton's method remains problematic when the values of F cannot be accurately achieved. Of course, this problem is common to all iterative procedures which directly depend on function evaluations. To overcome it, one may resort to generalized bisection methods [2, 6, 7, 18–23] since they only make use of the algebraic sign of the function involved in the equations. These methods, however, do not generally attain a quadratic convergence.

There is a class of methods for the numerical solution of system (1.1) which arise from iterative procedures used for systems of linear equation [8, 11–13, 15]. These methods use reduction to simpler one-dimensional nonlinear equations for the components $f_1, f_2, \dots, f_n$ of F. The best-known method of this type is the *nonlinear successive overrelaxation* (SOR) method which solves at the $(p+1)$st iteration the one-dimensional equation

$$f_i(x_1^{p+1}, \dots, x_{i-1}^{p+1}, x_i, x_{i+1}^p, \dots, x_n^p) = 0 \tag{1.3}$$

for x_i and then sets

$$x_i^{p+1} = x_i^p + \omega(x_i - x_i^p), \qquad i = 1, \dots, n, \quad p = 0, 1, \dots, \tag{1.4}$$

provided that $\omega \in (0, 1]$. Independent of the value of ω, the above process is called SOR process even though this nomenclature is sometimes reserved for the case $\omega > 1$. Now, a large variety of combined methods can be constructed depending on a secondary iteration and the number of steps required for solving (1.3). Thus, for example, one can obtain the *exact nonlinear SOR* or *m-step SOR–Newton process* [12, 15] and so on. Now, if the Jacobian of F at the solution x^* of (1.1) is an $\mathscr{M}$-matrix [12] the iterates of the above processes will converge linearly to x^* provided that $\omega \in (0, 1]$ [12].

New methods which incorporate the advantages of nonlinear SOR and Newton algorithms have been recently proposed [4, 5]. More specifically, although these methods use reduction to simpler one-dimensional nonlinear equations, they generate a quadratically converging sequence of points in $\mathbb{R}^{n-1}$ which converges to $n-1$ components of the solution, while the remaining component of the solution is evaluated separately using the final approximations of the other components. Moreover, these methods do not require a good initial guess for one component of the solution and do not directly need any function evaluation. Also these methods compare favourably with Newton's method when the Jacobian at the solution is singular (without making any enlargement of the system), or when the Jacobian is ill-conditioned, or when it is difficult to evaluate the function values accurately. They use tangent hyperplanes to the surfaces $x_{n+1} = f_i(x)$, $i = 1, \dots, n$ at points on the $x_{n+1} \equiv 0$ hyperplane, (see [4] for a geometric interpretation).

In this paper, we derive and apply a new procedure which can accelerate the convergence of other algorithms used for the numerical solution of systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$. The proposed procedure uses a "rotating" hyperplane in $\mathbb{R}^{n+1}$, whose rotation axis depending on the current approximation of $(n-1)$ components of the solution. This procedure is applied here on the traditional Newton's algorithm as well as on a method proposed in [5], whence two new modified schemes are obtained.

The new procedure is described in Section 2 of this paper. In Section 3 we give the new modified schemes which are derived by applying the proposed procedure on Newton's method and on the methods of [4, 5] correspondingly. In the same section we also give the corresponding proofs of convergence. Finally, in Section 4 we illustrate these modified schemes on a number of numerical applications.

2. THE METHOD OF ROTATING HYPERPLANES

In this section we give a geometric interpretation of the proposed method in $\mathbb{R}^2$ and present its generalization to n dimensions. We also illustrate the main differences between our new method and Newton's method, as well as the methods introduced in [4, 5].

NOTATION 2.1 Throughout this paper $\mathbb{R}^n$ is the n-dimensional real space of column vectors x with components $x_1, x_2, \ldots, x_n$; $(y; z)$ represents the column vector with components $y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_k$; $\partial_i f(x)$ denotes the partial derivative of $f(x)$ with respect to the ith variable x_i; $\bar{\mathscr{A}}$ denotes the closure of the set $\mathscr{A}$ and $f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ defines the mapping obtained by holding $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fixed.

Let us start by writing the iterates (1.2) of Newton's method in the $n=2$ case as follows

$$(x_1^{p+1}-x_1^p)\,\partial_1 f_i(x^p)+(x_2^{p+1}-x_2^p)\,\partial_2 f_i(x^p)+f_i(x^p)=0, \qquad i=1,2. \tag{2.1}$$

Now, the equations

$$x_3=(x_1-x_1^p)\,\partial_1 f_i(x^p)+(x_2-x_2^p)\,\partial_2 f_i(x^p)+f_i(x^p), \qquad i=1,2, \tag{2.2}$$

represent planes in the (x_1, x_2, x_3)-space which are tangent to the surfaces $x_3=f_i(x)$, $i=1,2$ at the points $(x_1^p, x_2^p, f_i(x^p))$, $i=1,2$, respectively. Thus the point x^{p+1} which is determined by the relationship (1.2) is the point of intersection of these two planes with the (x_1, x_2)-plane [4, 12]. In the method of [4], instead of the above planes, we have considered the following ones

$$x_3=(x_1-x_1^p)\,\partial_1 f_i(x_1^p; x_2^{p,i})+(x_2-x_2^{p,i})\,\partial_2 f_i(x_1^p; x_2^{p,i}), \qquad i=1,2, \tag{2.3}$$

where $x_2^{p,i}$ denotes the solutions of the equations $f_i(x_1^p, \cdot)=0$, $i=1,2$. The planes (2.3) are tangent to the surfaces $x_3=f_i(x)$, $i=1,2$ at the points $(x_1^p, x_2^{p,i}, 0)$, $i=1,2$ respectively. Then the point x^{p+1} of the next iteration of the method of [4] is the point of intersection of these two planes with the (x_1, x_2)-plane. Or, equivalently, x^{p+1} is the intersection point of the following lines in (x_1, x_2)-space

$$(x_1 - x_1^p)\,\partial_1 f_i(x_1^p; x_2^{p,i}) + (x_2 - x_2^{p,i})\,\partial_2 f_i(x_1^p; x_2^{p,i}) = 0, \qquad i = 1, 2. \tag{2.4}$$

In this paper we introduce a rotating plane which in the (x_1, x_2, x_3)-space has rotation axis $x_1 = x_1^p$ in the plane $x_3 = 0$, and is used to modify (2.2) and (2.4) correspondingly. More specifically, we use the proposed plane so that the point x^{p+1} which is determined by Newton's method is now taken to be the projection on the (x_1, x_2)-plane of the point of intersection of the planes (2.2) with the proposed plane. We also use this rotating plane to modify the method of [4] so that the lines (2.4), in the (x_1, x_2)-plane, are now taken to lie on this rotating plane. The main idea, of course, is that, with suitable changes in the direction of this plane, we may be able to bring the corresponding projections on the plane $x_3 = 0$ closer to the solution of (2.1) and thus achieve a more rapid convergence of the iterations.

It is easy to derive the equation of this plane. Starting with the general equation of a plane in $\mathbb{R}^3$

$$A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 = 0, \tag{2.5}$$

and requiring that it contain the line $x_1 = x_1^p$ in the (x_1, x_2)-plane, we are led to the condition that the normal vector of (2.5), (A_1, A_2, A_3), be perpendicular to the vector $(0, 1, 0)$, i.e.

$$(0, 1, 0) \cdot (A_1, A_2, A_3)^{\mathrm{T}} = 0, \tag{2.6}$$

or, equivalently, $A_2 = 0$. Thus, (2.5) becomes

$$A_1 x_1 + A_3 x_3 + A_4 = 0. \tag{2.7}$$

Now, since we have $x_1 = x_1^p$ for $x_3 = 0$, we must also impose the condition

$$A_4 = -A_1 x_1^p, \tag{2.8}$$

whence (2.7) is further reduced to

$$A_1 x_1 + A_3 x_3 - A_1 x_1^p = 0. \tag{2.9}$$

We assume now, without loss of generality, that $A_1 \neq 0$. (Note that if $A_1 = 0$ we only get a trivial modification of the method in [4].) Moreover, $A_3 \neq 0$ allows this plane to rotate in $\mathbb{R}^3$ around the line $x_1 = x_1^p$, $x_3 = 0$. So from (2.9) we finally have

$$x_3 = A'(x_1^p - x_1), \qquad \text{for} \quad A' = A_1 / A_3, \tag{2.10}$$

which determines the equation of the rotating plane. We, therefore, conclude that the proposed modified methods obtain correspondingly the next approximation x^{p+1} of x^p from the intersection point of the following two planes with the (x_1, x_2)-plane

$$x_3 = (x_1 - x_1^p)(\partial_1 f_i(x^p) + A') + (x_2 - x_2^p)\,\partial_2 f_i(x^p) + f_i(x^p), \qquad i = 1, 2, \tag{2.11}$$

cf. (2.2) and from the intersection point of the following lines in (x_1, x_2)-space

$$(x_1 - x_1^p)(\partial_1 f_i(x_1^p; x_2^{p,i}) + A') + (x_2 - x_2^{p,i})\,\partial_2 f_i(x_1^p; x_2^{p,i}) = 0, \qquad i = 1, 2, \tag{2.12}$$

cf. (2.4). Clearly, the parameter A' in (2.11) and (2.12), corresponds to different directions of the rotating plane and offers an additional "degree of freedom": it may be suitably varied, in each problem, so as to reach an optimal value, at which the speed of the convergence of the iterations is maximized.

Ultimately, of course, we wish to obtain an approximate solution of the system (1.1). To do this we need to extend the above ideas to n-dimensions and work with the equation of a hyperplane in $\mathbb{R}^{n+1}$

$$\sum_{n=1}^{n+1} A_i x_i + A_{n+2} = 0. \tag{2.13}$$

Assume now that the hyperplane (2.13) is parallel to some direction, for example to the nth coordinate of the basis of $\mathbb{R}^{n+1}$. In this case, we shall have $A_n = 0$ and (2.13) becomes

$$\sum_{n=1}^{n-1} A_i x_i + A_{n+1} x_{n+1} + A_{n+2} = 0, \tag{2.14}$$

cf. (2.7) for $n = 2$. Let $x^0 = (x_1^0, \ldots, x_n^0)$ be an initial estimate. Suppose further, that the rotation axis of the hyperplane (2.14), is determined from the conditions

$$x_{n+1} = 0, \quad x_i = x_i^0 \qquad \text{for} \quad i = 1, \ldots, n-1. \tag{2.15}$$

Thus we now use (2.15) to solve (2.14) for A_{n+2}

$$A_{n+2} = -\sum_{n=1}^{n-1} A_i x_i^0, \tag{2.16}$$

and substitute back in (2.14) to obtain

$$A_{n+1} x_{n+1} + \sum_{n=1}^{n-1} A_i (x_i - x_i^0) = 0, \tag{2.17}$$

cf. (2.9). Assuming now that $A_{n+1} \neq 0$, we may rewrite (2.17) in the form

$$x_{n+1}=\sum_{i=1}^{n-1} A_i'(x_i^0-x_i), \qquad \text{for} \quad A_i'=A_i/A_{n+1}, \quad i=1,\dots,n-1, \tag{2.18}$$

which determines the required hyperplane, with $n-1$ free parameters A_i', $i=1,\dots,n-1$.

Of course, similar hyperplanes can be obtained using any one of the other coordinates $i\neq n$. Moreover, as we shall see later similar results can be obtained by considering a constant rotating axis, for example the nth coordinate of the basis of $\mathbb{R}^{n+1}$.

3. THE NEW MODIFIED ITERATIVE SCHEMES AND THEIR PROOF OF CONVERGENCE

The rotating hyperplane (2.18) can be used in an iterative procedure to obtain approximate solutions of the system (1.1). This can be done, by replacing the usual hyperplane $x_{n+1}=0$, at every iteration, by this hyperplane, thus modifying Newton's method (1.2) to obtain

$$F(x^p)+F'(x^p)(x-x^p)=\Theta^n. \tag{3.1}$$

Now, by replacing the $x_{n+1}=0$ hyperplane by the hyperplane (2.18) we get

$$F(x^p)+F'(x^p)(x-x^p)=\begin{pmatrix} \sum_{i=1}^{n-1} A_i'(x_i^p-x_i) \\ \vdots \\ \sum_{i=1}^{n-1} A_i'(x_i^p-x_i) \end{pmatrix}, \tag{3.2}$$

or, after some matrix manipulations,

$$x=x^p-G'(x^p)^{-1}F(x^p), \tag{3.3}$$

where $G'(x^p)=F'(x^p)+\Xi$ and $\Xi=[\xi_{ij}]$ is the rank-1 $n\times n$ matrix with

$$\xi_{ij}=\begin{cases} A_j' & \text{if} \quad j\neq n, \\ 0 & \text{if} \quad j=n. \end{cases} \tag{3.4}$$

Finally, taking x as the new approximation of the solution we end up with the following modified Newton's scheme

$$x^{p+1}=x^p-G'(x^p)^{-1}F(x^p), \qquad p=0,1,\dots . \tag{3.5}$$

Now, the parameters A_j', $j=1,\dots,n-1$ can be chosen such that the "direction" of the hyperplane (2.18) be, for example, perpendicular to the hyperplanes which are tangent to the surfaces $x_{n+1}=f_i(x)$ for $i=1,\dots,n-1$ at the points $(x_1^0,\dots,x_n^0, f_i(x^0))$, $i=1,\dots,n-1$ respectively. One may also attempt to find relations between A'_j and the curvature of particular curves on the surfaces $x_{n+1}=f_i(x)$, $i=1,\dots,n$. In a future paper, we intend to explore these possibilities in order to find out optimal A'_j values which will accelerate the convergence of Newton's method for any class of functions. That this is indeed possible is demonstrated in Section 4 on a number of numerical applications. Of course, the scheme (3.5) can be derived (see Theorem 3.2) using a constant rotation axis of the hyperplane, for example the nth coordinate of the basis of $\mathbb{R}^{n+1}$.

In order to give a convergence result for the modified scheme (3.5) the following lemma and theorem are needed:

LEMMA 3.1 (Permutation Lemma, Banach Lemma) *Let A be an $n\times n$ nonsingular matrix. If E is an $n\times n$ matrix and $\|A^{-1}\|\,\|E\|<1$, for any arbitrary norm, then the matrix $A+E$ is nonsingular.*

Proof See [10, 12].

THEOREM 3.1 *Suppose that $F=(f_1,\dots,f_k)\colon\mathscr{E}\subset\mathbb{R}^k\to\mathbb{R}^k$ is twice-continuously differentiable on an open neighborhood $\mathscr{E}^*\subset\mathscr{E}$ of a point $x^*=(x_1^*,\dots,x_k^*)\in\mathscr{E}$ for which $F(x^*)=\Theta^k$ and $F'(x^*)$ nonsingular. Then the iterates x^p, $p=0,1,\dots$ of Newton's method*

$$x^{p+1}=x^p-F'(x^p)^{-1}F(x^p),\qquad p=0,1,\dots, \tag{3.6}$$

will converge to x^* provided the initial guess x^0 is sufficiently close to x^*. Moreover, the order of convergence will be two.

Proof See [10, 14, 17].

We note here that the condition that $F'(x)$ be Lipschitz continuous in $\mathscr{E}^*$ (which we assumed in Section 1) is ensured since the component functions f_i of F are all twice-continuously differentiable. We now proceed with the following convergence result.

THEOREM 3.2 *Suppose that $F=(f_1,\dots,f_n)\colon\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ is twice-continuously differentiable on an open neighborhood $\mathscr{D}^*\subset\mathscr{D}$ of a point $x^*=(x_1^*,\dots,x_n^*)\in\mathscr{D}$ for which $F(x^*)=\Theta^n$ and $F'(x^*)$ nonsingular. Let $\Xi=[\xi_{ij}]$ be the rank-1 $n\times n$ matrix with*

$$\xi_{ij}=\begin{cases}A'_j & \text{if}\quad j\neq n,\\ 0 & \text{if}\quad j=n,\end{cases} \tag{3.7}$$

where the vector $A'=[A'_j]$, $j=1,\dots,n$, $A'_n=0$ determine the parameters of the rotating hyperplane (2.18) such that the inner product $\langle x,A'\rangle=0$ $\forall\,x\in\mathscr{D}^*$ and that $\|F'(x^*)^{-1}\|\,\|\Xi\|<1$. Then the iterates x^p, $p=0,1,\dots$ of (3.5) will converge to x^* provided the initial guess x^0 is sufficiently close to x^*. Moreover the order of convergence will be two.

Proof Consider the mapping

$$G=(g_1,\ldots,g_n)\colon \mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n, \quad \text{by}$$
$$g_i(x_1,\ldots,x_n)=f_i(x_1,\ldots,x_n)+\sum_{j=1}^{n-1}A'_j x_j, \qquad i=1,\ldots,n. \tag{3.8}$$

By the assumptions it is obvious that

$$G(x^*)=F(x^*)=\Theta^n. \tag{3.9}$$

Moreover the Jacobian matrix G' of G is given by

$$G'(x^*)=F'(x^*)+\Xi. \tag{3.10}$$

Hence, by Lemma 3.1 we obtain that $G'(x^*)$ is nonsingular. So for the above mapping G the conditions of Theorem 3.1 are fulfilled. Consequently the iterates x^p, $p=0,1,\ldots$ of (3.5) converge quadratically to x^*. Thus the theorem is proven. $\square$

Of course, similar convergence results can be obtained by considering other mappings instead of (3.8). According to the above theorem we can estimate the free parameters A'_j, $j=1,\ldots,n-1$ of the rotating hyperplane (2.18) in each iteration from the relationships

$$\langle x^p, A'\rangle=0, \qquad p=0,1,\ldots, \tag{3.11}$$

by choosing $n-2$ arbitrary parameters and calculating at each iteration the $(n-1)$st parameter from (3.11).

Now, we use the rotating hyperplane (2.18) to derive a modified scheme of the method in [5]. This scheme is derived in such a way that it can incorporate the advantages of nonlinear SOR and Newton's method. It is important to note that although we shall use reduction to simpler one-dimensional nonlinear equations we will still produce a quadratically converging sequence of points in $\mathbb{R}^{n-1}$.

The following theorem and corollary are seminal to the development of our analysis.

THEOREM 3.3 (Implicit Function Theorem) *Suppose that* $F=(f_1,\ldots,f_n)\colon\mathscr{D}\subset\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$ *is defined and continuously differentiable on an open neighborhood* $\mathscr{D}^0\subset\mathscr{D}$ *of a point* $(x^0;y^0)=(x_1^0,\ldots,x_m^0,y_1^0,\ldots,y_n^0)\in\mathscr{D}$ *such that* $F(x^0;y^0)=\Theta^n$ *and that the Jacobian* $\partial(f_1,\ldots,f_n)/\partial(y_1,\ldots,y_n)$ *is nonsingular at* $(x^0;y^0)$. *Then there exist open neighborhoods* $\mathscr{A}_1\subset\mathbb{R}^m$ *and* $\mathscr{A}_2\subset\mathbb{R}^n$ *of* x^0 *and* y^0, *respectively, such that, for any* $x\in\bar{\mathscr{A}}_1$ *there is a unique system on n mappings* ϕ_i, $i=1,\ldots,n$ *defined and continuous on* $\bar{\mathscr{A}}_1$ *such that* $y_i=\phi_i(x)\in\bar{\mathscr{A}}_2$ *for* $i=1,\ldots,n$ *and* $f_i(x,\phi_1(x),\ldots,\phi_n(x))=0$ *for* $i=1,\ldots,n$ *and any* $x\in\bar{\mathscr{A}}_1$. *Moreover the function* $\Phi=(\phi_1,\ldots,\phi_n)$ *is continuously differentiable in* $\mathscr{A}_1$ *and the Jacobian matrix* $\Phi'(x)$ *is equal to* $-B^{-1}C$, *where C (respectively B) is obtained by replacing* y_i *by* $\phi_i(x)$, $i=1,\ldots,n$ *in the Jacobian matrix* $[\partial f_i/\partial x_k]$ *(respectively* $[\partial f_i/\partial y_j]$*).*

Proof See [1,12].

A direct corollary of the above theorem is the following.

COROLLARY 3.1 *Suppose that $f:\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}$ is defined and continuously differentiable on an open neighborhood $\mathscr{D}^0\subset\mathscr{D}$ of a point $x^0=(x_1^0,\ldots,x_n^0)$ for which $f(x^0)=0$ and $\partial_n f(x^0)\neq 0$. Then there exist open neighborhoods $\mathscr{A}_1\subset\mathbb{R}^{n-1}$ and $\mathscr{A}_2\subset\mathbb{R}$ of the points $y^0=(x_1^0,\ldots,x_{n-1}^0)$ and x_n^0 respectively, such that, for any $y=(x_1,\ldots,x_{n-1})\in\bar{\mathscr{A}}_1$ there is a unique mapping ϕ defined and continuous on $\bar{\mathscr{A}}_1$ such that $x_n=\phi(y)\in\bar{\mathscr{A}}_2$ and $f(y;\phi(y))=0$ for any $y\in\bar{\mathscr{A}}_1$. Moreover the mapping $\phi:\mathscr{A}_1\to\mathbb{R}$ has continuous partial derivatives in $\mathscr{A}_1$ which are given by*

$$\partial_j\phi(y)=-\partial_j f(y;\phi(y))/\partial_n f(y;\phi(y)),\qquad j=1,\ldots,n-1. \tag{3.12}$$

Of course, similar corollaries can be obtained using any one of the components $x_1,\ldots,x_n$, for example x_i, instead of x_n and taking $y=(x_1,\ldots,x_{i+1},x_{i+1},\ldots,x_n)$.

Assume now that $F=(f_1,\ldots,f_n):\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ is twice-continuously differentiable on an open neighborhood $\mathscr{D}^*\subset\mathscr{D}$ of a solution $x^*\in\mathscr{D}$ of the system of nonlinear equations $F(x)=\Theta^n$. Our interest lies in obtaining a sequence $\{x^p\}$, $p=0,1,\ldots$ of points in $\mathbb{R}^n$ which converges to x^*. To do this we shall make use of the rotating hyperplane (2.18). For simplicity we shall assume that the rotating axis is the nth coordinate of the basis $\mathbb{R}^{n+1}$. We assume further that for the vector A' of the parameters A'_j, $j=1,\ldots,n-1$, $A'_n=0$ of the rotating hyperplane holds that the inner product $\langle x,A'\rangle=0\ \forall x\in\mathscr{D}^*$. Of course similar results can be obtained using any other similar rotating axis. Next, we define the mapping

$$G=(g_1,\ldots,g_n):\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n,\quad\text{by}\quad g_i(x_1,\ldots,x_n)=f_i(x_1,\ldots,x_n)+\sum_{j=1}^{n-1}A'_jx_j. \tag{3.13}$$

It is evident that the solutions of the equations $g_i(x_1^p,\ldots,x_{n-1}^p,\cdot)=0$, for $i=1,\ldots,n$ are identical with the corresponding solutions of $f_i(x_1^p,\ldots,x_{n-1}^p,\cdot)=0$ in $\mathscr{D}^*$. Moreover, it is obvious that $g_i(x_1^*,\ldots,x_n^*)=f_i(x_1^*,\ldots,x_n^*)=0$ for $i=1,\ldots,n$. Now working exactly as in [5], we consider the sets $\mathscr{B}_i$, $i=1,\ldots,n$ to be those connected components of $g_i^{-1}(0)$ containing x^* on which $\partial_n g_i\neq 0$, for $i=1,\ldots,n$ respectively. Next, we apply Corollary 3.1 for each one of the components g_i, $i=1,\ldots,n$ of G. So, according to the above corollary, there exist open neighborhoods $\mathscr{A}_1^*\subset\mathbb{R}^{n-1}$ and $\mathscr{A}_{2,i}^*\subset\mathbb{R}$, $i=1,\ldots,n$ of the points $y^*=(x_1^*,\ldots,x_{n-1}^*)$ and x_n^* respectively, such that for any $y=(x_1,\ldots,x_{n-1})\in\bar{\mathscr{A}}_1^*$ there exist unique mappings ϕ_i defined and continuous in $\mathscr{A}_1^*$ such that

$$x_n=\phi_i(y)\in\bar{\mathscr{A}}_{2,i}^*,\qquad i=1,\ldots,n, \tag{3.14}$$

and

$$g_i(y;\phi_i(y))=0,\qquad i=1,\ldots,n. \tag{3.15}$$

Moreover there exist the partial derivatives $\partial_j\phi_i$, $j=1,\ldots,n-1$ in $\mathscr{A}_1^*$ for each ϕ_i, $i=1,\ldots,n$, they are continuous in $\bar{\mathscr{A}}_1^*$ and they are given by

$$\partial_j \phi_i(y) = -\partial_j g_i(y; \phi_i(y))/\partial_n g_i(y; \phi_i(y)), \qquad i=1,\dots,n, \quad j=1,\dots,n-1. \quad (3.16)$$

Suppose now that $x^0=(x_1^0,\dots,x_n^0)$ is an initial approximation of the solution x^* where $y^0=(x_1^0,\dots,x_{n-1}^0)\in\bar{\mathscr{A}}_1^*$, then using Taylor's formula we can expand the $\phi_i(y)$, $i=1,\dots,n$ about y^0 where $y=(x_1,\dots,x_{n-1})$. So, we can obtain that

$$\phi_i(y) \simeq \phi_i(y^0) + \sum_{j=1}^{n-1} (x_j - x_j^0)\, \partial_j \phi_i(y^0), \qquad i=1,\dots,n. \quad (3.17)$$

Now, using the relationships (3.14) and (3.15) we form the following system of equations,

$$x_n = x_n^{0,i} - \sum_{j=1}^{n-1} (x_j - x_j^0)\, \partial_j g_i(y^0; x_n^{0,i})/\partial_n g_i(y^0; x_n^{0,i}), \qquad i=1,\dots,n, \quad (3.18)$$

where $x_n^{0,i}=\phi_i(y^0)$, $i=1,\dots,n$ are the corresponding solutions of the one-dimensional equations of one unknown $g_i(x_1^0,\dots,x_{n-1}^0,\,\cdot\,)=0$, $i=1,\dots,n$ which, as we have mentioned before, are identical with the corresponding solutions of the equations $f_i(x_1^0,\dots,x_{n-1}^0,\,\cdot\,)=0$.

Now, using (3.13) we form the following system

$$x_n = x_n^{0,i} - \sum_{j=1}^{n-1} \{(x_j - x_j^0)(\partial_j f_i(y^0; x_n^{0,i}) + A_j')/\partial_n f_i(y^0; x_n^{0,i})\}, \qquad i=1,\dots,n. \quad (3.19)$$

Next, from the nth equation of the above set of equations we can obtain that

$$x_n = x_n^{0,n} - \sum_{j=1}^{n-1} \{(x_j - x_j^0)(\partial_j f_n(y^0; x_n^{0,n}) + A_j')/\partial_n f_n(y^0; x_n^{0,n})\}. \quad (3.20)$$

By substituting (3.20) in the remaining equations (3.19) we obtain the following system of $n-1$ linear equations

$$\sum_{j=1}^{n-1} (x_j - x_j^0)\{(\partial_j f_i(y^0; x_n^{0,i}) + A_j')/\partial_n f_i(y^0; x_n^{0,i}) - (\partial_j f_n(y^0; x_n^{0,n}) + A_j')/\partial_n f_n(y^0; x_n^{0,n})\}$$
$$= x_n^{0,i} - x_n^{0,n}, \quad i=1,\dots,n-1, \quad (3.21)$$

which, in matrix form, becomes

$$U_0(y-y^0) = V_0, \quad (3.22)$$

where

$$U_0 = [u_{ij}] = [(\partial_j f_i(y^0; x_n^{0,i}) + A_j')/\partial_n f_i(y^0; x_n^{0,i}) - (\partial_j f_n(y^0; x_n^{0,n}) + A_j')/\partial_n f_n(y^0; x_n^{0,n})],$$
$$i,j=1,\dots,n-1,$$

$$y=[x_i], \quad y^0=[x_i^0], \quad i=1,\dots,n-1 \tag{3.23}$$

$$V_0=[v_i]=[x_n^{0,i}-x_n^{0,n}], \quad i=1,\dots,n-1.$$

Assuming that U_0 is nonsingular, the solution y of the linear system (3.22) gives a new approximation of the first $n-1$ components of the solution x^* of (1.1) and finally, by replacing y in (3.20) we can approximate the nth component of x^*. Thus in general we can obtain the following iterative scheme for the computation of the $n-1$ components of x^*

$$y^{p+1}=y^p+U_p^{-1}V_p, \qquad p=0,1,\dots, \tag{3.24}$$

where

$$y^p=[x_i^p], \qquad i=1,\dots,n-1,$$

$$\begin{aligned} U_p=[a_{ij}]=[(\partial_j f_i(y^p;x_n^{p,i})+A_j')/\partial_n f_i(y^p;x_n^{p,i}) \\ -(\partial_j f_n(y^p;x_n^{p,n})+A_j')/\partial_n f_n(y^p;x_n^{p,n})], \quad i,j=1,\dots,n-1, \end{aligned} \tag{3.25}$$

$$V_p=[v_i]=[x_n^{p,i}-x_n^{p,n}], \qquad i=1,\dots,n-1.$$

Finally, after a desired number of iterations of the above scheme, say $p=m$, using (3.20) we can approximate the nth component of x^* using the following relationship

$$x_n^{m+1}=x_n^{m,n}-\sum_{j=1}^{n-1}\{(x_j^{m+1}-x_j^m)(\partial_j f_n(y^m;x_n^{m,n})+A_j')/\partial_n f_n(y^m;x_n^{m,n})\}. \tag{3.26}$$

Of course, relative procedures for obtaining x^* can be constructed by replacing x_n in Corollary 3.1 with any one of the components $x_1,\dots,x_{n-1}$, for example x_i, and taking $y=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

We would like to mention here that the above process does not require the expressions ϕ_i but only the values $x_n^{p,i}$ which are given by the solution of the one-dimensional equations $f_i(x_1^p,\dots,x_{n-1}^p,\cdot)=0$. So, by holding $y^p=(x_1^p,\dots,x_{n-1}^p)$ fixed we can solve the equations

$$f_i(y^p;r_i^p)=0, \qquad i=1,\dots,n, \tag{3.27}$$

for r_i^p in the interval $(\alpha,\alpha+\beta)$ with an accuracy δ. Of course, we can use any one of the well-known one-dimensional methods [12, 14, 15, 17] to solve the above equations. Here we shall use the one-dimensional bisection, (see [2, 16] for a discussion of its advantages), since frequently the steps β are long and also a few significant digits are required for the computations of the roots of the equations

(3.27). A simplified version of the bisection method can be found in [18, 20–23] and in the Appendix.

We now end this section with a proof of the convergence of the new method described by the iterates (3.24) and the relationship (3.26).

THEOREM 3.4 *Suppose that* $F=(f_1,\ldots,f_n)\colon \mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ *is twice-continuously differentiable on an open neighborhood* $\mathscr{D}^*\subset\mathscr{D}$ *of a point* $x^*=(x_1^*,\ldots,x_n^*)\in\mathscr{D}$ *for which* $F(x^*)=\Theta^n$. *Let* $\mathscr{B}_i$, $i=1,\ldots,n$ *be those connected components of* $g_i^{-1}(0)$, *containing* x^* *on which* $\partial_n g_i\neq 0$ *for* $i=1,\ldots,n$ *respectively where the functions* g_i *are defined in (3.13). Then the iterates of (3.24) and the relationship (3.26) will converge to* x^* *provided the matrix* U_* *which is obtained from the matrix* U_p *of (3.25) at* x^* *is nonsingular and also provided the initial guess* $y^0=(x_1^0,\ldots,x_{n-1}^0)$ *is sufficiently close to* $y^*=(x_1^*,\ldots,x_{n-1}^*)$. *Moreover the iterates* y^p, $p=0,1,\ldots$ *of (3.24) have order of convergence two.*

Proof Obviously, the iterates (3.24) can be written as follows

$$y^{p+1}=y^p-W_p^{-1}V_p,\qquad p=0,1,\ldots,\tag{3.28}$$

where

$$y^p=[x_i^p],\qquad i=1,\ldots,n-1,$$

$$W_p=[w_{ij}]=[-(\partial_j f_i(y^p;x_n^{p,i})+A_j')/\partial_n f_i(y^p;x_n^{p,i})+(\partial_j f_n(y^p;x_n^{p,n})+A_n')/\partial_n f_n(y^p;x_n^{p,n})],$$

$$i,j=1,\ldots,n-1,\tag{3.29}$$

$$V_p=[v_i]=[x_n^{p,i}-x_n^{p,n}],\qquad i=1,\ldots,n-1,$$

or using (3.14) and (3.16) we can form W_p and V_p as follows

$$\begin{aligned} W_p&=[w_{ij}]=[\partial_j\phi_i(y^p)-\partial_j\phi_n(y^p)],\qquad i,j=1,\ldots,n-1,\\ V_p&=[v_i]=[\phi_i(y^p)-\phi_n(y^p)],\qquad i=1,\ldots,n-1.\end{aligned}\tag{3.30}$$

Consider now the mapping,

$$\begin{aligned}&\Lambda=(\lambda_1,\ldots,\lambda_{n-1})\colon \bar{\mathscr{A}}_1^*\subset\mathbb{R}^{n-1}\to\mathbb{R}^{n-1},\quad\text{by}\\ &\lambda_i(y)=\phi_i(y)-\phi_n(y),\qquad i=1,\ldots,n-1.\end{aligned}\tag{3.31}$$

Then for the above mapping Λ and for $k=n-1$ the conditions of Theorem 3.1 are fulfilled. Consequently, the iterates y^p, $p=0,1,\ldots$ of (3.24) converge to y^* and the order of convergence is two.

Suppose now that for some p, for example $p=m$, we obtain $y^m=y^*$. Then from the relationship (3.26) we can obtain that

$$x_n^{m+1}=\phi_n(y^*), \tag{3.32}$$

or

$$x_n^{m+1}=x_n^* \tag{3.33}$$

Thus the theorem is proven. □

4. NUMERICAL APPLICATIONS

The new methods described in Section 3 have been applied to several examples of nonlinear systems of different dimension. We found that the procedures behaved predictably and reliably and their speed of convergence was quite satisfactory. Here, we present some typical computational results comparing the new schemes to the dimension-reducing method [5] and also to the more familiar Newton's method on three examples (1.1) (studied also in [5]), with $F=(f_1, f_2, \ldots, f_n)$ given by:

$$\begin{aligned} f_1(x_1,x_2,x_3)&=x_1^3-x_1x_2x_3=0\\ f_2(x_1,x_2,x_3)&=x_2^2-x_1x_3=0\\ f_3(x_1,x_2,x_3)&=10x_1x_3+x_2-x_1-0.1=0 \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_1(x_1,x_2,x_3)&=x_1x_3-x_3e^{x_1^2}+10^{-4}=0\\ f_2(x_1,x_2,x_3)&=x_1(x_1^2+x_2^2)+x_2^2(x_3-x_2)=0\\ f_3(x_1,x_2,x_3)&=x_1^3+x_3^3=0 \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
f_1(x_1,x_2,x_3,x_4,x_5) &= 2x_1+x_2+x_3+x_4+x_5-6=0\\
f_2(x_1,x_2,x_3,x_4,x_5) &= x_1+2x_2+x_3+x_4+x_5-6=0\\
f_3(x_1,x_2,x_3,x_4,x_5) &= x_1+x_2+2x_3+x_4+x_5-6=0\\
f_4(x_1,x_2,x_3,x_4,x_5) &= x_1+x_2+x_3+2x_4+x_5-6=0\\
f_5(x_1,x_2,x_3,x_4,x_5) &= x_1x_2x_3x_4x_5-1=0
\end{aligned}
\tag{4.3}
$$

System (4.1) has two roots $r_1=(0.1,0.1,0.1)$ and $r_2=(-0.1,-0.1,-0.1)$ within the cube $[-0.1,0.1]^3$ and its Jacobian at these roots is nonsingular. However, this system has a particular difficulty since the function values at some points (for example at points close to origin) cannot be accurately achieved. On the other hand, the Jacobian of system (4.2) at its root $r=(-0.99990001\cdot 10^{-4}$, $-0.99990001\cdot 10^{-4}$, $0.99990001\cdot 10^{-4})$ is singular. While the system (4.3) is a well-known test case (*Brown's almost linear system*) [7,9]. It has roots of the form (a,a,a,a,a^{-4}), where a satisfies the equation $a^4(5a-6)+1=0$, and its Jacobian at these roots is nonsingular. The difficulty of this system is that its Jacobian at all the above roots is ill-conditioned. For this case we shall present results for the roots $r_1=(1,1,1,1,1)$, $r_2=(0.91635458253385, \ldots, 1.41822708733080)$ and $r_3=(-0.57904308849412, \ldots, 8.89521544247060)$ reported in the tables.

In Tables 1, 2 and 3 we exhibit the number of iterations required to obtain a solution of the systems (4.1), (4.2) and (4.3) to accuracy 10^{-7} and 10^{-14} by using Newton's method, the dimension-reducing method and the iterative schemes (3.5) and (3.24)–(3.26) of this paper, for several starting points $x^0=(x_1^0,\ldots,x_n^0)$ and values A'_j, $j=1,\ldots,n-1$ of the rotating hyperplane. We set arbitrary $n-2$ values of the A'_j and we calculate the remaining $(n-1)$st in each iteration such that $\langle x^p, A'\rangle=0$. Note that the nth component of the vector A' is zero.

In these tables "A" indicates the vector of the parameters of the rotating hyperplane, "j" indicates the coordinate for which the equation $\langle x^p, A'\rangle=0$ is solved, "ε" indicates the required accuracy, "N" indicates the number of iterations, "FE" indicates the number of function evaluations, "AS" indicates the total number of algebraic signs that are required for applying the iterative scheme described in the Appendix and "r_i" denotes the root to which the corresponding method converges.

We applied the above schemes using Crout's method with partial pivoting for the corresponding linear systems. Note that this is the reason for the slight difference between the results exhibited in [5] with the results of this paper regarding Newton's method and the dimension-reducing method.

From the results shown in the tables we observe that the new modified schemes of this paper compare favourably with Newton's method and the dimension-reducing method of [5].

We also applied the new modified schemes to problems with precise function values for which the corresponding Jacobian was nonsingular and well-conditioned and we observed that the number of iterations of the new methods was less than or equal to the corresponding number of iterations of Newton's method and the dimension-reducing method.

Table 1 Results for system (4.1)

Newton's method									*Modified Newton's method*					
									$A'=(\pm 0.00001, 0, 0)$, $j=2$					
x_1^0	x_2^0	x_3^0	$\varepsilon=10^{-7}$			$\varepsilon=10^{-14}$			$\varepsilon=10^{-7}$			$\varepsilon=10^{-14}$		
			N	FE	r_i	N	FE	r_i	N	FE	r_i	N	FE	r_i
-4	-2	1	36	432	r_1	33	396	r_2	24	288	r_1	25	300	r_1
-2	-0.5	0.2	31	372	r_1	32	384	r_1	21	252	r_1	23	276	r_2
-2	2	2	30	360	r_2	32	384	r_2	24	288	r_1	25	300	r_1
-1	-2	0.6	41	492	r_2	51	612	r_1	21	252	r_2	23	276	r_2
-1	-2	1	42	504	r_1	29	348	r_2	28	336	r_1	30	360	r_1
-0.5	0.5	-0.5	25	300	r_1	26	312	r_1	21	252	r_1	23	276	r_1
0.4	0.5	0.5	28	336	r_2	53	636	r_2	17	204	r_1	19	228	r_1
0.5	-0.5	2	27	324	r_2	28	336	r_2	27	324	r_2	28	336	r_2
0.5	2	1	50	600	r_1	54	648	r_1	24	288	r_2	25	300	r_2
2	-2	-2	38	456	r_2	43	516	r_1	16	192	r_2	21	252	r_2
5	-2	-2	34	408	r_1	38	456	r_1	28	336	r_2	30	360	r_2
10	-2	-2	38	456	r_1	39	468	r_1	31	372	r_1	33	396	r_1

Dimension-reducing method										*Modified dimension-reducing method*							
										$A'=(0, -0.00001)$, $j=1$							
x_1^0	x_2^0	$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$				$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$			
		N	FE	AS	r_i	N	FE	AS	r_i	N	FE	AS	r_i	N	FE	AS	r_i
-4	-2	5	45	120	r_1	6	54	180	r_1	4	36	120	r_1	5	45	150	r_1
-2	-0.5	5	45	150	r_1	6	54	180	r_1	5	45	150	r_1	6	54	180	r_1
-2	2	5	45	150	r_2	6	54	180	r_2	5	45	150	r_2	6	54	180	r_2
-1	-2	4	36	120	r_2	5	45	150	r_2	4	36	120	r_2	5	45	150	r_2
-1	-2	4	36	120	r_2	5	45	150	r_2	4	36	120	r_2	5	45	150	r_2
-0.5	0.5	5	45	150	r_2	6	54	180	r_2	5	45	150	r_2	6	54	180	r_2
0.4	0.5	6	54	180	r_1	7	63	210	r_1	6	54	180	r_1	7	63	210	r_1
0.5	-0.5	4	36	120	r_2	5	45	150	r_2	4	36	120	r_2	5	45	150	r_2
0.5	2	5	45	150	r_2	6	54	180	r_2	5	45	150	r_2	6	54	180	r_2
2	-2	5	45	150	r_2	6	54	180	r_2	4	36	120	r_2	5	45	150	r_2
5	-2	7	63	210	r_2	8	72	240	r_2	6	54	180	r_2	7	63	210	r_2
10	-2	7	63	210	r_2	8	72	240	r_2	7	63	210	r_2	8	72	240	r_2

5. CONCLUDING REMARKS

The methods we have analysed in this paper compare favourably with Newton's method and the dimension-reducing method of [5], since they have order of convergence two for any values of A'_j of the parameters of the rotating hyperplane, while, as we show for proper values of A'_j the convergence can be significantly accelerated. Of course, having some previous knowledge about the optimal values of A'_j, e.g. by relating them to curvature of particular curves on the surfaces $x_{n+1}=f_i(x)$, $i=1,\ldots,n$, for any class of functions $F=(f_1, f_2, \ldots, f_n)$ of (1.1) would significantly improve our methods. We hope to address this question in a future publication.

Table 2 Results for system (4.2)

Newton's method							*Modified Newton's method*			
							$A' = (0, 1000, 0)$, $j = 1$			
x_1^0	x_2^0	x_3^0	$\varepsilon = 10^{-7}$		$\varepsilon = 10^{-14}$		$\varepsilon = 10^{-7}$		$\varepsilon = 10^{-14}$	
			N	FE	N	FE	N	FE	N	FE
-2	-2	-2	26	312	27	324	11	132	13	156
-1	-1	-1	28	336	29	348	7	84	9	108
-1	1	1	26	312	27	324	22	264	24	288
-0.5	-0.5	-0.5	39	468	40	480	6	72	8	96
-0.5	-0.5	0.1	22	264	23	276	18	216	20	240
0.5	0.5	0.1	41	492	42	504	41	492	42	504
0.5	0.5	0.5	45	540	46	552	5	60	7	84
1	-2	1	26	312	27	324	24	288	26	312
1	-1	1	26	312	27	324	22	264	24	288
1	1	1	26	312	27	324	6	72	8	96
2	-2	2	34	408	35	420	32	384	34	408
2	2	2	41	492	42	504	11	132	13	156

Dimension-reducing method								*Modified dimension-reducing method*					
								$A' = (0, -3)$, $j = 1$					
x_1^0	x_2^0	$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-14}$			$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-14}$		
		N	FE	AS	N	FE	AS	N	FE	AS	N	FE	AS
-2	-2	2	18	60	3	27	90	2	18	60	3	27	90
-1	-1	2	18	60	3	27	90	2	18	60	3	27	90
-1	1	6	54	180	7	63	210	2	18	60	3	27	90
-0.5	-0.5	2	18	60	3	27	90	2	18	60	3	27	90
-0.5	-0.5	2	18	60	3	27	90	2	18	60	3	27	90
0.5	0.5	2	18	60	3	27	90	2	18	60	3	27	90
0.5	0.5	2	18	60	3	27	90	2	18	60	3	27	90
1	-2	2	18	60	3	27	90	2	18	60	3	27	90
1	-1	6	54	180	7	63	210	2	18	60	3	27	90
1	1	2	18	60	3	27	90	2	18	60	3	27	90
2	-2	6	54	180	7	63	210	2	18	60	3	27	90
2	2	2	18	60	3	27	90	2	18	60	3	27	90

Also, although the second method of this paper uses reduction to simpler one-dimensional equations, it converges quadratically to $n-1$ components of the solution, while the remaining component of the solution is evaluated separately using the final approximations of the other components. Thus it does not require a good initial estimate for one component of the solution. Moreover, this method does not directly perform function evaluations, while, using the modified bisection method described in the Appendix, it requires only their algebraic signs to be correct in finding the various $\phi_i(y)$.

Table 3 Results for system (4.3)

Newton's method											*Modified Newton's method*					
											$A'=(\pm 10000, \pm 10000, \pm 10000, 0, 0)$, $j=4$					
x_1^0	x_2^0	x_3^0	x_4^0	x_5^0	$\varepsilon=10^{-7}$			$\varepsilon=10^{-4}$			$\varepsilon=10^{-7}$			$\varepsilon=10^{-14}$		
					N	FE	r_i	N	FE	r_i	N	FE	r_i	N	FE	r_i
−8	−3	4	2	1.5	84	2520	r_3	85	2550	r_3	12	360	r_1	13	390	r_1
−4	−4	4	2	1.5	79	2370	r_3	80	2400	r_3	10	300	r_1	11	330	r_1
−2	2	4	4	1.5	72	2160	r_3	73	2190	r_3	28	840	r_1	29	870	r_1
−1	2	−1	2	1.5	37	1110	r_3	38	1140	r_3	36	1080	r_3	37	1110	r_3
−0.5	−0.6	4	2	1.5	35	1050	r_3	36	1080	r_3	39	1170	r_3	42	1260	r_3
−0.2	−0.2	−0.2	−0.2	−0.2	35	1050	r_3	36	1080	r_3	35	1050	r_3	36	1080	r_3
−0.1	−0.1	−0.1	−0.1	−0.1	48	1440	r_3	49	1470	r_3	48	1440	r_3	49	1470	r_3
−0.1	−0.1	0.1	−0.1	4	40	1200	r_3	41	1230	r_3	40	1200	r_3	41	1230	r_3
0.1	0.1	0.1	0.1	0.1	48	1440	r_3	49	1470	r_3	48	1440	r_3	49	1470	r_3
0.1	0.1	0.1	0.1	0.2	45	1350	r_1	46	1380	r_1	45	1350	r_1	46	1380	r_1
0.1	0.1	0.1	0.1	1	34	1020	r_1	35	1050	r_1	34	1020	r_1	35	1050	r_1
3	3	3	4	1.5	71	2130	r_3	72	2160	r_3	21	630	r_1	22	660	r_1
10	−3	1.5	−3	1.5	80	2400	r_3	81	2430	r_3	12	360	r_3	13	390	r_3
10	3	4	2	1.5	82	2460	r_3	83	2490	r_3	8	240	r_3	9	270	r_3

Dimension-reducing method												*Modified dimension-reducing method*							
												$A'=(\pm 0.2, \pm 0.2, \pm 0.2, 0)$, $j=4$							
x_1^0	x_2^0	x_3^0	x_4^0	$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$				$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$			
				N	FE	AS	r_i	N	FE	AS	r_i	N	FE	AS	r_i	N	FE	AS	r_i
−8	−3	4	2	6	150	300	r_1	7	175	350	r_1	5	125	250	r_1	6	150	300	r_1
−4	−4	4	2	6	150	300	r_1	7	175	350	r_1	4	100	200	r_1	5	125	250	r_1
−2	−2	4	4	6	150	300	r_1	7	175	350	r_1	5	125	250	r_1	6	150	300	r_1
−1	2	−1	2	4	100	200	r_2	5	125	250	r_2	4	100	200	r_2	5	125	250	r_2
−0.5	−0.6	4	2	9	225	450	r_2	10	250	500	r_2	8	200	400	r_2	9	225	450	r_2
−0.2	−0.2	−0.2	−0.2	9	225	450	r_3	10	250	500	r_3	9	225	450	r_3	10	250	500	r_3
−0.1	−0.1	−0.1	−0.1	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
0.1	−0.1	0.1	−0.1	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
3	3	3	4	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
10	−3	1.5	−3	6	150	300	r_1	7	175	350	r_1	6	150	300	r_1	7	175	350	r_1
10	3	4	2	6	150	300	r_1	7	175	350	r_1	5	125	250	r_1	6	150	300	r_1

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APPENDIX

For completeness, we give here a brief description of the simplified version of the bisection method mentioned in Section 3. Hence, to solve an equation of the form

$$\psi(t)=0, \tag{A.1}$$

where $\psi:[\gamma_1,\gamma_2]\subset\mathbb{R}\to\mathbb{R}$ is continuous, a simplified version of the bisection method leads to the following iterative formula

$$t_{k+1}=t_k+\operatorname{sgn}\psi(t_0)\cdot\operatorname{sgn}\psi(t_k)\cdot h/2^{k+1}, \qquad k=0,1,\dots, \tag{A.2}$$

with $t_0=\gamma_1$ and $h=\gamma_2-\gamma_1$ and where for any real number a,

$$\operatorname{sgn} a=\begin{cases}-1 & \text{if } a<0,\\ 0 & \text{if } a=0,\\ 1 & \text{if } a>0.\end{cases} \tag{A.3}$$

Of course, (A.2) converges to a root $t^*\in(\gamma_1,\gamma_2)$ if for some t_k, $k=0,1,\ldots$ holds that

$$\operatorname{sgn}\psi(t_0)\cdot\operatorname{sgn}\psi(t_k)=-1. \tag{A.4}$$

Also, the minimum number of iterations μ, that are required in obtaining an approximate root $\hat{t}$ such that $|\hat{t}-t^*|\leqq\varepsilon$, for some $\varepsilon\in(0,1)$ is given by

$$\mu=\lceil\log_2(h\cdot\varepsilon^{-1})\rceil, \tag{A.5}$$

where the notation $\lceil v\rceil$ refers to the least integer that is not less than the real number v.