ON THE APPLICATION OF OPTIMIZATION METHODS TO THE DETERMINATION OF MEMBERS OF FAMILIES OF PERIODIC SOLUTIONS

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Abstract. The techniques used for the numerical computation of families of periodic orbits of dynamical systems rely on predictor-corrector algorithms. These algorithms usually depend on the solution of systems of approximate equations constructed from the periodicity conditions of these orbits. In this contribution we transform the rootfinding procedure to an optimization one which is applied on an objective function based on the exact periodicity conditions. Thus, the determination of periodic solutions and families of such orbits can be accomplished through unconstrained optimization. In this paper we apply and compare some well-known minimization methods for the solution of this problem. The obtained results are promising.

Keywords: dynamical systems, periodic solutions, families of periodic solutions, Poincaré map, fixed points, optimization methods

1. Introduction

The determination of periodic solutions is one of the main tasks in the study of dynamical systems. Orbits of this kind give significant information about the global behaviour of a system, especially in conservative dynamical systems where these orbits are dense in certain parts of the phase space. In these particular systems periodic solutions form families, i.e. groups with infinite number of members which are represented in the phase space by a continuous line. The study of these families is also very important since they express the evolution of the characteristics of their members within this space and form boundaries between regions of different kinds of motion.

The problem of determining a periodic orbit, which is or is not considered as a member of a family, refers to the solution of systems of nonlinear equations expressing the periodicity conditions of the orbit. The most common way to solve such systems is to apply a rootfinding method. Recently, several rootfinding methods and combinations of them have been tested for the computation of periodic orbits (Drossos et al., 1996; Kalantonis et al., 2001; Perdios et al., 2002; Perdiou et al., 2002). Instead of this consideration, it is also efficient to transform the solution of the above mentioned nonlinear system to a minimization problem. This is

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achieved by deriving a proper objective function from the periodicity conditions. Then, one may optimize this function by applying any optimization algorithm.

In this study, we apply some well-known unconstrained optimization techniques to determine periodic orbits and families in the so-called Robe's problem. First, we utilize them for the calculation of fixed points on a surface of section of the Poincaré map of the corresponding dynamical system and compare their speed of convergence. Then, we compute the families containing some of these points. The paper is organized as follows. In the next section we describe the construction of objective functions by means of the periodicity conditions. In Section 3, we briefly present some well-known unconstrained optimization methods. In Section 4, we apply and compare the effectiveness of these methods for the determination of some periodic solutions of the Robe's problem. We also use some of the determined fixed points as initial conditions for the computation of families of periodic orbits. Finally, in Section 5, we give some concluding remarks.

2. Determination of Periodic Orbits and families

Let us consider a dynamical system expressed by the equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; t),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{F} = (F_1, F_2, \ldots, F_n)$: $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ and t is the independent variable.

A solution $\mathbf{x}$ of this system is periodic of period T, if and only if it satisfies the condition

$$\mathbf{x}(\mathbf{x}_0; t = 0) = \mathbf{x}(\mathbf{x}_0; t = T), \tag{1}$$

where $\mathbf{x}_0$ is the initial point of the orbit at $t = 0$.

Given a guess $\mathbf{x}_0^*$ of the initial point of a periodic orbit, the necessary *corrections* to calculate this point should satisfy the system:

$$\mathbf{x}(\mathbf{x}_0^* + \delta\mathbf{x}_0^*; t = 0) = \mathbf{x}(\mathbf{x}_0^* + \delta\mathbf{x}_0^*; t = T^* + \delta T^*). \tag{2}$$

The classical method to approximate these corrections is to expand into a Taylor series up to first order terms, so that

$$\sum_{j=1}^{n} \frac{\partial x_i}{\partial x_{0j}^*} \delta\, x_{0j}^* + \frac{\partial x_i}{\partial t} \delta T^* = x_i(\mathbf{x}_0^*; t = T^* + \delta T^*) - x_i(\mathbf{x}_0^*; t = 0),$$

$$i = 1, \ldots, n,$$

fix one of the δ-corrections and, then, solve the above system (Goudas, 1961; Shearing, 1960). This procedure should be repeated until (1) is satisfied within a predetermined accuracy.

A more efficient approach is to transform (2) into:

$$f(\mathbf{x}_0^*) = \sum_{i=1}^{n} (x_i(\mathbf{x}_0^*; t = 0) - x_i(\mathbf{x}_0^*; t = T^*))^2$$

by setting $\mathbf{x}_0^* = \mathbf{x}_0^* + \delta\mathbf{x}_0^*$ and $T^* = T^* + \delta T^*$, and, then, minimize the objective function f by using an optimization method. In this way, the overhead of approximating System (2) by its linear terms is avoided.

Suppose that the initial conditions $\mathbf{x}_0$ and the period T of a member of a family are already known. To find a new member of the family, we have to *predict* proper modifications $\delta\mathbf{x}_0$ and δT of these elements so that the solution with initial point $\mathbf{x}_0^* = \mathbf{x}_0 + \delta\mathbf{x}_0$ to be periodic of period $T^* = T + \delta T$. The new orbit should satisfy:

$$x_i(\mathbf{x}_0 + \delta\mathbf{x}_0; t = 0) - x_i(\mathbf{x}_0 + \delta\mathbf{x}_0; t = T + \delta T) = 0.$$

The Taylor expansion of the left-hand sides of the above equations to first order terms gives

$$\delta x_{0i} + \sum_{j=1}^{n} \frac{\partial x_i}{\partial x_{0j}} \delta x_{0j} + \frac{\partial x_i}{\partial t} \delta T = 0, \; i = 1, \ldots, n. \tag{3}$$

Then, by fixing

$$\sum_{j=1}^{n} \delta x_{0j}^2 = d^2 = const, \tag{4}$$

we are able to approximate these δ-modifications by solving (3) and (4) so that the distance between the initial points of the two periodic solutions remains equal to d.

If this prediction is not satisfactory enough, it can be *corrected* to give a periodic solution with initial point $\mathbf{x}_0^*$ and of period T^*, by minimizing the objective function f given in (3) together with the function

$$h(\mathbf{x}_0) = \mid \sum_{i=1}^{n} (x_{0i}^* - x_{01})^2 - d^2 \mid . \tag{5}$$

The minimization of h ensures that the distance between the initial points of the two periodic solutions will remain equal to d.

3. Optimization Methods

There is a large variety of optimization algorithms for the computation of the minima of an objective function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ (Dennis and Schnabel, 1996; Polak, 1997). Since each of these algorithms possesses advantanges and

disadvantages, it is not always obvious which one is proper for the solution of a specific class of problems. For example, the so called conjugate gradient methods, such as Fletcher-Reeves (FR) and Polak-Ribiere (PR) require storage of order n but they are very sensitive to rounding off errors. On the other hand, the quasi-Newton and variable metric methods, e.g. Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithms, are very stable and they converge superlinearly but they need storage of order n^2. The well known Newton's method enjoys the quadratic convergence but it requires storage of order n^2 and needs the calculation of the Jacobian matrix of f. To avoid this calculation one may utilize Broyden's method which, however, converges superlinearly. In this paper we test the behaviour of BFGS, DFP, Newton and Broyden. These methods are briefly described below.

BFGS AND DFP METHODS

Consider the following set of objective functions:

$$g(x) = (g_1(x), g_2(x), \dots g_n(x)).$$

Minimizing all these functions simultaneously is equivalent to the minimization of

$$f(x) = \sum_{i=1}^{n} g_i^2(x).$$

An efficient algorithm for the optimization (Dennis and Schnabel, 1996) of a function f is the following:

$$x^{k+1} = x^k - \lambda^k A_k \nabla f(x^k), \quad k = 0, 1, 2, \dots,$$

where

$$A_{k+1} = A_k + \frac{r^k (r^k)^\top}{(r^k)^\top q^k} - \frac{A_k q^k (q^k)^\top A_k}{(q^k)^\top A_k q^k} + \gamma (q^k)^\top A_k q^k u^k (u^k)^\top,$$

$$u^k = \frac{r^k}{(r^k)^\top q^k} - \frac{A_k q^k}{(q^k)^\top A_k q^k},$$

$$r^k = x^{k+1} - x^k, \quad q^k = \nabla f(x^{k+1}) - \nabla f(x^k),$$

A_o is an arbitrary symmetric and positive definite matrix, usually taken to be the identity matrix, and λ^k is the optimal length in the direction $p^k = -A_k \nabla f(x^k)$.

For $\gamma = 1$ we obtain *BFGS* method, while for $\gamma = 0$ we get *DFP*.

NEWTON'S AND BROYDEN'S METHODS

Consider the following set of equations:

$$g(x) = 0, \tag{6}$$

where $g = (g_1, \ldots, g_n)$ and $x = (x_1, \ldots, x_n)$. Then, solving System (6) is equivalent to the minimization of the objective function

$$f(x) = \frac{1}{2} gg$$

The *Newton* step for the above system is:

$$x^{k+1} = x^k - J_k^{-1} g(x^k), \; k = 0, 1, 2, \ldots$$

where J_k is the Jacobian matrix of g evaluated at x_k. This step is a descent direction for f because:

$$\nabla f^k(-J_k^{-1} g(x^k)) = (g(x^k) J_k)(-J_k^{-1} g(x^k)) = -g(x^k) g(x^k).$$

The *Broyden* step for solving System (6) is:

$$x^{k+1} = x^k + s^k, \quad k = 0, 1, 2, \ldots,$$

where

$$s^k = B_k^{-1} g(x^k), \; B_{k+1} = B_k + \frac{((g(x^{k+1}) - g(x^k)) - B_k s^k)(s^k)^\top}{(s^k)^\top s^k},$$

and B_0 is usually taken to be equal to the identity matrix. In this method, the matrix B is used as an approximation of the Jacobian matrix J.

In all cases, a line-search technique can be additionally utilized in order to ensure the convergence of these methods. More specifically, if p^k is a descent direction of the objective function at the current iterate x^k, namely $\nabla f(x^k)^\top p^k < 0$, the line-search consists in determining a step-size $\lambda_k > 0$ along p^k, so that the next iterate $x^{k+1} = x^k + \lambda^k p^k$ satisfies the strong Wolfe conditions (Wolfe, 1969; Wolfe, 1971):

$$f(x^{k+1}) \leq f(x^k) + c_1 \lambda^k \nabla f(x^k)^\top p^k,$$

$$\nabla f(x^{k+1})^\top p^k \geq c_2 \nabla f(x^k)^\top p^k. \tag{7}$$

where c_1, c_2 are constants such that $0 < c_1 < c_2 < 1$.

4. Applications

The above mentioned methods have been applied to several dynamical systems for the computation of periodic orbits and families. In the following we present some results related to a type of restricted three-body problem which was introduced by Robe (1977). Recently, the interest of many researchers has been attracted by this problem (see Perdios, 1985; Giordano et al., 1991; Hallan and Rana, 2001, among others). This dynamical system describes the behaviour of a small solid sphere of infinitesimal mass and of density ρ moving under the gravitational influence of a body, consisting from a spherical rigid shell m_1 which is filled with a homogeneous incompressible fluid of mass m_* and density ρ_*, and a mass point m_2 which is positioned outside this shell. We assume that m_2 moves on a circular orbit around m_1. Then, if we express the elements of the problem in a rotating dimensionless coordinate system Ox_1x_2, whose Ox_1-axis always contains the two primaries with O being their center of mass, the motion of the small particle is described by the system:

$$\begin{aligned} &\dot{x}_1 = x_3, && \dot{x}_2 = x_4, \\ &\dot{x}_3 = 2x_4 + V_{x_1}, && \dot{x}_4 = -2x_3 + V_{x_2}, \end{aligned} \tag{8}$$

where

$$V = \frac{1}{2}(x_1^2 + x_2^2) + \frac{\mu}{\sqrt{(x_1 + \mu - 1)^2 + x_2^2}} - \frac{K}{2}[(x_1 + \mu)^2 + x_2^2]$$

and

$$\mu = \frac{m_2}{m_1 + m_* + m_2}, k = \frac{4\pi}{3}\left(\frac{\rho_*}{m_1 + m_* + m_2}\right)\left(1 - \frac{\rho_*}{\rho}\right).$$

The Jacobi integral of this dynamical system is:

$$\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - V = C,$$

where C is the Jacobi constant. This integral can be used, among others, for the determination of the zero velocity curves which bound the regions of the space of coordinates where the motion of the small particle is permitted.

The localization of periodic orbits of this problem can be based on the construction of Poincaré surfaces of section. In Figure 1 we present such a surface together with the projection of the zero velocity curves for map of System (8) for $\mu = 0.5$, $K = 0.2$ and $C = 1.2$. The construction of this surface has been achieved by fixing the values of the Jacobi integral and $x_2 = 0$, then, by integrating System (8) for several initial conditions and calculating successive intersections of the corresponding solutions with $O-x_1x_3$ plane (Kalantonis et al., 2001). A magnification of a region of this surface is given in Figure 2. In this magnification we can distinguish several fixed points which correspond to periodic solutions.

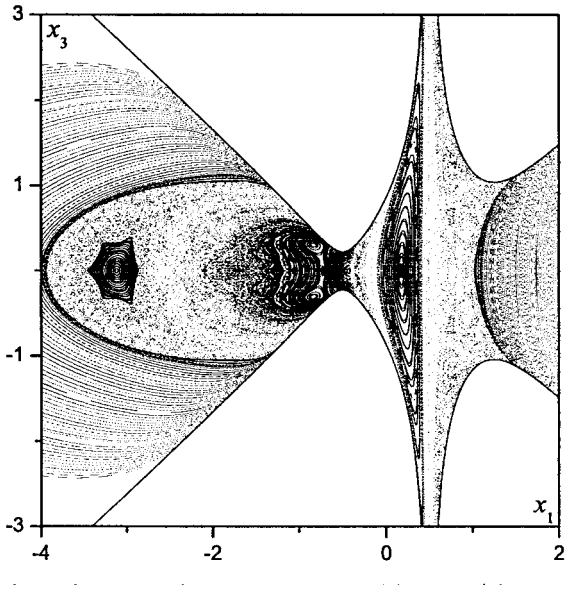

Figure 1. A surface of section of a Poincaré map of System (8) for $\mu = 0.5$, $K = 0.2$ and $C = 1.2$.

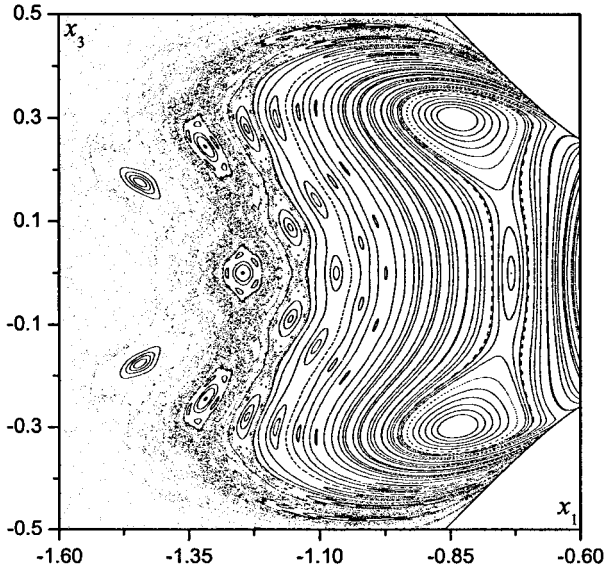

Figure 2. A magnification of a specific region of the surface of section presented in Figure 1.

TABLE I

Fixed points of periodic orbits of several periods on the Poincaré surface of section given in Figure 2 together with the time (in msecs) required for their computation: t_1 corresponds to *DFP*, t_2 to *BFGS*, t_3 to *Newton* and t_4 to *Broyden*

p	Fixed Point (x_1, x_3)		t_1	t_2	t_3	t_4	Stability
1	(–3.12086833,	0.00000000)	30	28	9	7	*S*
1	(–0.73265839,	0.00000000)	12	13	5	4	*S*
1	(0.17354678,	0.00000000)	29	34	5	6	*S*
2	(–0.83732621,	0.30053937)	37	36	9	10	*S*
2	(–0.74343532,	0.12226822)	67	39	7	9	*U*
5	(–1.24572111,	0.00000000)	128	121	25	33	*S*
5	(–1.25536828,	0.11460568)	121	121	27	34	*U*
7	(–1.06780854,	0.00000000)	297	254	47	36	*S*
7	(–1.07545470,	0.07023601)	284	225	41	49	*U*
7	(–1.80720005,	0.00000000)	361	288	56	82	*S*
7	(–1.73575887,	0.02598722)	255	255	65	65	*U*
9	(–1.02098781,	–0.05884154)	306	237	60	53	*S*
9	(–1.01336326,	0.00000000)	342	284	53	59	*U*

TABLE II

Some members of the family containing the periodic solution presented in the second entry of Table I

N	x_1	x_4	C	T
1	–0.50359696	0.00602327	1.24998696	5.04339668
2	–0.73265839	0.37042827	1.20000000	5.05567615
3	–1.10468168	0.90274199	0.95542709	5.11918795
4	–1.50260874	1.41625999	0.55034448	5.23655000
5	–1.94790671	1.94297237	0.00842447	5.41293149

We have calculated some of these points by utilizing DFP, BFGS, Newton's and Broyden's methods. The results together with the computational time used by each of these methods are presented in Table I. In all test runs the initial conditions used are accurate to 2 decimal digits. We have seen that all methods succeed to converge to the desired solution within accuracy $\varepsilon \leq 10^{-8}$.

We have also calculated, via the proposed predictor-corrector scheme and the above mentioned methods, the families containing these solutions. In Tables II and

TABLE III

Some members of the family containing the periodic solution presented in the third entry of Table I

N	x_1	x_4	C	T
1	–1.99622722	3.19281348	–6.25626942	4.18301976
2	–0.49238116	1.49938880	–0.99806184	2.90158513
3	0.17354678	1.34261366	1.20000000	0.97085450
4	0.47279680	4.09470114	20.02807328	0.03440645
5	0.49409593	8.98618121	88.66971825	0.00375675

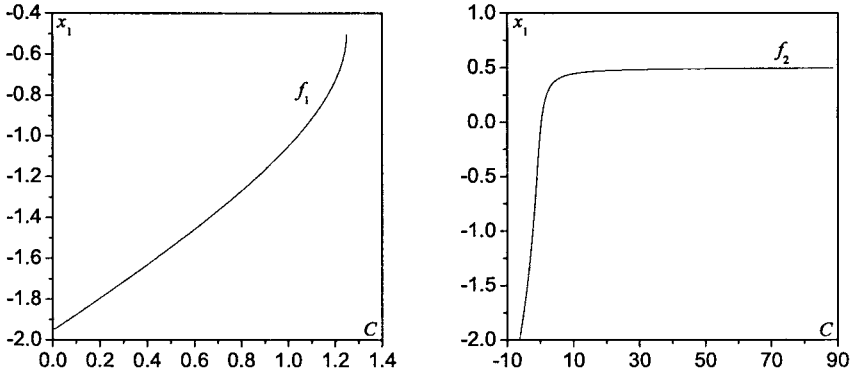

Figure 3. Characteristic curves of the families containing the second and third periodic solutions presented in Table I.

III we give some members of the families containing the second and third solutions appearing in Table I. The characteristic curves (C, x_1) of these families are shown in Figure 3. Both families consist of plane and Ox_1-symmetric orbits.

5. Conclusions

In this contribution, the problem of the computation of periodic solutions of dynamical systems is treated as an unconstrained optimization problem. More specifically, the set of nonlinear equations expressing the periodicity conditions is transformed to an objective function whose minima correspond to the periodic orbits of the considered system. The minimization of this function can be accomplished through any optimization technique. Here, we use the well-known DFP and BGFS as well as two techniques based on Newton's and Broyden's methods. In all cases, a line-search technique is additionally utilized to ensure the convergence of

these methods. This procedure is applied to the computation of periodic orbits and families of the so-called Robe's problem.

We note that all the above mentioned methods are efficient. A comparison between the computational speed of these methods shows that, as expected, Newton's method is superior to the others. The speed of Broyden's method is close to that of Newton's while the convergence behaviour and speed of BFGS are better than those of DFP.

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