COMPUTING WITH CERTAINTY INDIVIDUAL MEMBERS OF FAMILIES OF PERIODIC ORBITS OF A GIVEN PERIOD

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Abstract. The accurate computation of families of periodic orbits is very important in the analysis of various celestial mechanics systems. The main difficulty for the computation of a family of periodic orbits of a given period is the determination within a given region of an individual member of this family which corresponds to a periodic orbit. To compute with certainty accurate individual members of a specific family we apply an efficient method using the Poincaré map on a surface of section of the considered problem. This method converges rapidly, within relatively large regions of the initial conditions. It is also independent of the local dynamics near periodic orbits which is especially useful in the case of conservative dynamical systems that possess many periodic orbits, often of the same period, close to each other in phase space. The only computable information required by this method is the signs of various function evaluations carried out during the integration of the equations of motion. This method can be applied to any system of celestial mechanics. In this contribution we apply it to the photogravitational problem.

Key words: periodic orbits, Poincaré map, surface of section, dynamical systems, photogravitational problem, fixed points, zeros of functions, topological degree, generalized bisection method.

1. Introduction

For the analysis of systems of celestial mechanics a central role is played by the families of periodic orbits which are considered as sets of periodic orbits. In general, analytic expressions for evaluating periodic orbits are not available. Also, as it is well known, the traditional techniques for computing families of periodic orbits (symmetric or asymmetric) is a time-consuming procedure. The main difficulty for the computation of a family of periodic orbits of a given period is the determination of an individual member of this family. In general, this individual member can be determined using an equilibrium point of the system under consideration. In the case of symmetric orbits another approach is to create a grid in the (C, x) plane where C is the Jacobian constant (Markellos et al., 1974). In this case an individual member can be determined using a constant value of C.

In this paper, we propose a method to easily compute an individual member of any family, even in cases where the orbit (whether stable or unstable) is asymmetric and/or highly periodic. Our approach is based on the Poincaré map Φ on a surface

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of section. We say that $\mathbf{X} = (x_1, x_2)^\top$ is a *fixed point* or a *periodic orbit* of Φ if $\Phi(\mathbf{X}) = \mathbf{X}$ and a *periodic orbit of period* p if:

$$\mathbf{X} = \Phi^p(\mathbf{X}) \equiv \underbrace{\Phi\Big(\Phi\big(\cdots\big(\Phi\big(\mathbf{X}\big)\big)\cdots\big)\Big)}_{p \text{ times}}. \tag{1}$$

From the above it is evident that the problem of computing an individual member of a family of periodic orbits is equivalent to the problem of evaluating a fixed point of the Poincaré map. Using this approach, it is difficult to find in the literature efficient methods for computing periodic orbits with certainty. Also, traditional iterative schemes such as Newton's method and related classes of algorithms (Ortega and Rheinbolt, 1970; Dennis and Schnabel, 1996) often fail to converge to a specific periodic orbit since their convergence is almost independent of the initial guess. Thus, while there exist several periodic orbits close to each other, which are all desirable for applications, it is difficult for these methods to converge to the specific periodic orbit. Moreover, these methods are affected by the imprecisions of the mapping evaluations. Also, in general, these methods often fail due to the nonexistence of derivatives or poorly behaved partial derivatives (Ortega and Rheinbolt, 1970; Dennis and Schnabel, 1996).

To this end, we use an efficient numerical method for rapidly computing periodic orbits (be they stable or unstable) of any period and to any desired accuracy (Vrahatis, 1995). This method exploits topological degree theory to provide a criterion for the existence of a periodic orbit of an iterate of the mapping within a given region. In particular, the method constructs a polyhedron in such a way that the value of the topological degree of an iterate of the mapping relative to this polyhedron is ± 1, which means that there exists a periodic orbit within this polyhedron. Then it repeatedly subdivides its edges (and/or its diagonals) so that the new polyhedron also retains this property (of the existence of a periodic orbit within its interior) without making any computation of the topological degree. These subdivisions take place iteratively until a periodic orbit is computed to a predetermined accuracy. More details of this method can be found in Vrahatis (1995).

This method becomes especially significant for the computation of high period orbits (stable or unstable) where other more traditional approaches (like Newton's method, etc.) cannot easily distinguish among the closely neighboring periodic orbits. Moreover, this method is particular useful, since the only computable information it requires is the algebraic signs of the components of the mapping. Thus it is not affected by the imprecisions of the mapping evaluations. Recently, this method has been applied successfully to various difficult problems (see for example Drossos et al., 1996; Vrahatis et al., 1996; Vrahatis et al., 1997; Waalkens et al., 1997; Burić and Mudrinić, 1998; Burić et al., 1998).

In this paper, we use this method to compute with certainty individual members of families of periodic orbits of the well-known photogravitational restricted

circular three-body problem, described by Radzievskii (1950). This problem is a very interesting and useful generalization of the classical gravitational restricted three-body problem, especially when the third body is extremely small, which represents the case of interplanetary and interstellar dust where the particles are strongly affected by the radiation pressure force (Wyatt and Whipple, 1950).

The paper is organized as follows. In the next section, we briefly present the proposed method for computing within a given box individual members of families of periodic orbits of a given period. In Section 3, we briefly present the essential features of the photogravitational problem. Also, in this section we propose a method for the determination with certainty of the total number of the collinear equilibrium points of this problem. In Section 4, we apply the proposed method to the computation of individual members of families of periodic orbits of the photogravitational problem. The paper ends with some concluding remarks.

2. The Characteristic Polyhedron Criterion and Characteristic Bisection

In this section, we briefly present a method based on the *characteristic polyhedron* concept for the computation of periodic orbits. First we implement topological degree theory to give a criterion for the existence of a periodic orbit within a given region of the phase space of the system. Then we construct a characteristic polyhedron containing this orbit. Using a generalized bisection method, we iteratively refine this polyhedron to calculate the orbit within a predetermined accuracy. A detailed description of these procedures can be found in (Vrahatis, 1995).

In general, the problem of finding periodic orbits of period p of dynamical systems in $\mathbb{R}^{n+1}$ amounts to fixing one of the variables, say x_{n+1} =const, and locating points $\mathbf{X}^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$ on an n-dimensional surface of section Σ_{t_0} which satisfy the equation:

$$\Phi^p(\mathbf{X}^\star) = \mathbf{X}^\star, \tag{2}$$

where $\Phi^p = P_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0}$ is the Poincaré map of the system. Obviously, this is equivalent to solving the system:

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}, \tag{3}$$

with $\mathbf{F} = (f_1, f_2, \ldots, f_n) = \Phi^p - I_n$, where I_n is the $n \times n$ identity matrix and $\mathbf{0} = (0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$. For example, consider a conservative dynamical system of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{4}$$

with $\mathbf{x} = (x, \dot{x}) \in \mathbb{R}^2$ and $\mathbf{f} = (f_1, f_2)$ periodic in t with frequency ω. In this case, we can approximate periodic orbits of period p of System (4) by taking as initial conditions of these orbits the points which the orbits intersect the surface of section:

$$\Sigma_{t_0} = \left\{ \Big(x(t_k), \dot{x}(t_k) \Big), \quad \text{with} \quad t_k = t_0 + k\frac{2\pi}{\omega}, \quad k \in \mathbb{N} \right\}, \tag{5}$$

at a finite number of points p. Thus the dynamics can be studied in connection with a Poincaré map $\Phi^p = P_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0}$, constructed by following the solutions of (4) in continuous time.

As it is known, if we have a function $\mathbf{F}$ which is continuous on the closure of a bounded domain $\mathcal{D}$ such that there is not any point x on its boundary for which $\mathbf{F}(\mathbf{X}) = \mathbf{0}$ and the *topological degree of* $\mathbf{F}$ *at* $\mathbf{0}$ *relative to* $\mathcal{D}$ denoted by $\deg[\mathbf{F}, \mathcal{D}, \mathbf{0}]$ is not equal to zero, then there is at least one solution of System (3) within $\mathcal{D}$ (Ortega and Rheinbolt, 1970; Lloyd, 1978). This criterion can be used, in combination with the construction of a suitable n-polyhedron, called the characteristic polyhedron (CP), for the calculation of a solution contained in this region. Briefly, this can be done as follows. Let $\mathcal{M}_n$ be the $2^n \times n$ matrix whose rows are formed by all possible combinations of -1 and 1. Consider now an oriented n-polyhedron Π^n, with vertices $\mathbf{V}_k$, $k = 1, \ldots, 2^n$. If the $2^n \times n$ *matrix of signs associated with* $\mathbf{F}$ *and* Π^n, $\mathcal{S}(\mathbf{F}; \Pi^n)$, whose entries are the vectors:

$$\operatorname{sgn} \mathbf{F}(\mathbf{V}_k) = \Big(\operatorname{sgn} f_1(\mathbf{V}_k), \operatorname{sgn} f_2(\mathbf{V}_k), \ldots, \operatorname{sgn} f_n(\mathbf{V}_k) \Big), \tag{6}$$

is identical to $\mathcal{M}_n$, possibly after some permutations of these rows, then Π_n is called the *characteristic polyhedron relative to* $\mathbf{F}$. Also, if $\mathbf{F}$ is continuous, then, under some suitable assumptions on the boundary of Π^n holds that (Vrahatis and Iordanidis, 1986):

$$\deg[\mathbf{F}, \Pi^n, \mathbf{0}] = \sum_{\mathbf{X} \in \mathbf{F}^{-1}(\mathbf{0}) \cap \overset{\circ}{\Pi}{}^n} \operatorname{sgn} \det J_{\mathbf{F}}(\mathbf{X}) = \pm 1 \neq 0, \tag{7}$$

where $\overset{\circ}{\Pi}{}^n$ determines the interior of Π^n and $\det J_{\mathbf{F}}(\mathbf{X})$ denotes the determinant of the Jacobian matrix at $\mathbf{X}$, which implies the existence of a periodic orbit inside Π^n. For more details on how to construct a CP and locate a desired periodic orbit see in (Vrahatis, 1995; Vrahatis et al., 1996). The characteristic polyhedron can be considered a translation of the Poincaré–Miranda hypercube (Poincaré, 1883; Poincaré, 1884; Miranda, 1940; Vrahatis, 1989; Kulpa, 1997).

Next, we describe a generalized bisection method that, in combination with the above mentioned criterion, produces a sequence of characteristic polyhedra of decreasing size always containing the desired solution in order to calculate it within a given accuracy (*characteristic bisection*). This version of bisection does not require the computation of the topological degree at each step to secure its nonzero value, as others do (Kearfott, 1979; Eiger et al., 1984; Vrahatis, 1986; Greene, 1992). It can also be applied to problems with imprecise function values, since it depends only on their signs. The method simply amounts to constructing another refined CP, by bisecting a known one, say Π^n. We compute the midpoint $\mathbf{M}$ of an one-simplex, e.g. $\langle \mathbf{V}_i, \mathbf{V}_j \rangle$, which is one edge of Π^n. Then we obtain another CP, Π^n_*,

by comparing the sign of $\mathbf{F}(\mathbf{M})$ with that of $\mathbf{F}(\mathbf{V}_i)$ and $\mathbf{F}(\mathbf{V}_j)$ and substituting $\mathbf{M}$ for that vertex for which the signs are identical (Vrahatis, 1995, 1988a,b). Then we continue with another edge. The number of iterations ζ required to obtain a refined characteristic polyhedron Π_*^n whose longest edge length, $\Delta(\Pi_*^n)$, satisfies $\Delta(\Pi_*^n) \leqslant \varepsilon$, for some accuracy $\varepsilon \in (0, 1)$, is given by Vrahatis and Iordanidis (1986):

$$\zeta = \left\lceil \log_2 \left(\Delta(\Pi^n) \varepsilon^{-1} \right) \right\rceil, \tag{8}$$

where the notation $\lceil \cdot \rceil$ refers to the smallest integer, which is not less than the real number quoted. Notice that ζ is independent of the dimension n and it has the same computational cost as the bisection in one-dimension which is optimal and possesses asymptotically the best rate of convergence (Sikorski, 1982).

3. The Photogravitational Problem

Let us briefly describe the essential features of the photogravitational problem with the two main bodies having masses $m_1 = (1 - \mu)$ and $m_2 = \mu$ with $\mu \leqslant 1/2$ and radiation pressure parameters q_1 and q_2, respectively, where $q_1, q_2 \leqslant 1$ (in case of $q_i = 1, i = 1, 2$ we have the classical gravitational problem). The equations of motion of the third particle are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x - \frac{Q_1}{r_1^3}(x + \mu) - \frac{Q_2}{r_2^3}(x + \mu - 1), \\ \ddot{y} + 2\dot{x} &= y \left[1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3} \right], \end{aligned} \tag{9}$$

where

$$\mu = \frac{m_2}{m_1 + m_2},$$

$$Q_1 = q_1(1 - \mu), \qquad Q_2 = q_2\mu,$$

$$r_1 = \sqrt{(x + \mu)^2 + y^2}, \qquad r_2 = \sqrt{(x + \mu - 1)^2 + y^2}.$$

The Jacobian integral of the above system is defined by the following equation:

$$C = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}\left(x^2 + y^2\right) - \frac{Q_1}{r_1} - \frac{Q_2}{r_2}. \tag{10}$$

For an extended description of the above problem we refer to Simmons et al. (1985), Ragos and Zagouras (1991) and Markellos et al. (1993).

3.1. DETERMINATION OF THE TOTAL NUMBER OF THE COLLINEAR EQUILIBRIUM POINTS

Next, we briefly describe and apply a method for the determination with certainty of the total number of the collinear equilibrium points of the photogravitational problem. In the literature there are various efficient methods for the determination of collinear equilibrium points (Elipe, 1992). Here we present an alternative approach for the determination of these points. This approach is based on topological degree theory and can be used in any similar or more complicated problem.

First, we note that the number $\mathcal{N}^r$ of collinear equilibrium points of Equation (9) can be considered as the solutions of equation $f(x) = 0$ where the function f can be obtained by setting the right hand side of Equation (9) equal to zero and $y = 0$. Thus, the following equation can be formulated:

$$f(x) = x - \frac{Q_1}{|x+\mu|^3}(x+\mu) - \frac{Q_2}{|x+\mu-1|^3}(x+\mu-1) = 0. \tag{11}$$

In general, in cases where a function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is two times continuously differentiable in $[a,b]$ the total number $\mathcal{N}^r$ of the roots of the equation $f(x) = 0$ can be obtained by the following scheme (Picard, 1892, 1922; Kavvadias and Vrahatis, 1996):

$$\begin{aligned} \mathcal{N}^r \;=\; & -\frac{\gamma}{\pi}\int_a^b \frac{f(x)f''(x) - f'^2(x)}{f^2(x) + \gamma^2 f'^2(x)}\,\mathrm{d}x + \\ & + \frac{1}{\pi}\arctan\left(\frac{\gamma\left[f(a)f'(b) - f(b)f'(a)\right]}{f(a)f(b) + \gamma^2 f'(a)f'(b)}\right), \end{aligned} \tag{12}$$

where γ is a small positive real constant. It was explicitly shown by Picard (Picard, 1892, 1922) that Relation (12) is independent of the value of γ.

Now, since the denominator of Equation (11) vanishes at the points $x = -\mu$ and $x = 1-\mu$ we apply (12) to the subintervals $[a, -\mu-\delta]$, $[-\mu+\delta, 1-\mu-\delta]$ and $[1-\mu+\delta, b]$ of the interval $[a,b]$, where δ is a small positive real constant proportional to the relative machine precision. Notice that within the above considered intervals the required assumptions, regarding the function f, are fulfilled since the points where the function f is not continuously differentiable are avoided. Obviously, for the numerical computation of the integral of Equation (12) we can use any numerical integration technique. Thus, by restricting our study in the case of $q_1 = 0.5$, $q_2 = 1$ and $\mu = 0.01214$ which corresponds to the Earth–Moon system and using Romberg integration (Press et al., 1992) (where the corresponding derivatives of Equation (12) have been computed using central finite differences) we have concluded that the total number of roots of Equatin (11) is three. Once we know with certainty the exact number of the collinear equilibrium points we can easily compute them by applying any well-known rootfinding method. Thus, by applying the traditional bisection method and using the above values q_1, q_2 and μ we have

computed the following coordinates of the collinear equilibrium points, L_1, L_2 and L_3:

$$x_{L_1} = 0.72672723, \qquad x_{L_2} = 1.11687297, \qquad x_{L_3} = -0.79984490.$$

3.2. STABILITY OF A PERIODIC ORBIT AND INVARIANT CURVES

In order to compute the stability of a periodic orbit, we can integrate the equations of motion, for the three-dimensional problem, simultaneously with the equations of variation for the whole period. Thus, we can compute the indexes a_h, b_h, c_h, d_h and a_v, b_v, c_v, d_v, respectively (Hénon, 1973; Markellos, 1976). In the case of asymmetric periodic orbits, an orbit is considered *stable* if both $|s_h| < 1$ and $|s_v| < 1$ hold where:

$$s_h = \frac{a_h + d_h}{2}, \qquad s_v = \frac{a_v + d_v}{2}. \tag{13}$$

If only one of the above inequalities hold then the orbit is considered *horizontally* or *vertically stable*, respectively (Zagouras and Markellos, 1977; Markellos, 1978). In the case of symmetric periodic orbits we have:

$$a_h = d_h \quad \text{and} \quad a_v = d_v.$$

Thus

$$s_h = a_h \quad \text{and} \quad s_v = a_v.$$

Finally, if $|s_h| = 1$ or $|s_v| = 1$ then the stability of the orbit is considered *critical*.

It is well known that the invariant curves of a system play an important role in its analysis since they bound the motion of the third particle within their interior. In order to determine the invariant curves of the photogravitational problem, we set $y = 0$ and $\dot{y} > 0$ in Equation (10) and we obtain (Hénon, 1966):

$$\dot{x}^2 < C + x^2 + \frac{2Q_1}{r_1} + \frac{2Q_2}{r_2}.$$

Thus, using the following equation:

$$\dot{x} = \pm \left(C + x^2 + \frac{2Q_1}{r_1} + \frac{2Q_2}{r_2} \right)^{1/2}, \tag{14}$$

where

$$r_1^2 = (x + \mu)^2 \quad \text{and} \quad r_2^2 = (x + \mu - 1)^2,$$

we can obtain the corresponding invariant curves.

4. Results

Let us apply the proposed method to the photogravitational problem described by Equation (9). To produce the surface of section of the problem, we take successive sections of an orbit with the straight line $y = 0$, along the positive direction of the flow ($\dot{y} > 0$). In particular, we have taken the initial conditions $(x, 0, \dot{x}, \dot{y})$ where the value of $\dot{y}$ is computed using Equation (10) for a given value of C. Each section can be depicted as a point in the $(x, \dot{x})$ plane. The transition from one section to another can be considered as a transformation in the $(x, \dot{x})$ plane. Notice that, this particular transformation is well defined only if the Jacobian constant has a specific value. It is evident that a periodic orbit of period p intersects the x-axis $2p$ times. Obviously, in the simple case where $p = 1$ the orbit will be represented in the $(x, \dot{x})$ plane by a single point and thereupon a p periodic orbit is represented by p points.

To compute successively the intersection points with the surface of section, we can choose a value of the Jacobian constant C and by keeping this value fixed we can integrate numerically Equation (9), using for example the Bulirsch–Stoer algorithm with proper adaptive step-size control (Stoer and Bulirsch, 1980; Press et al., 1992). These points are exhibited in Figure 1(a)–(c) for several initial conditions with arbitrarily chosen Jacobian constants $C_{L_2} = -2.31058003$, $C_{L_1} = -1.95810456$ and $C_{L_3} = -1.90743262$, respectively.

Furthermore, in Figure 1(a)–(c) we plot the corresponding invariant curves using Relation (14). From these figures we observe that the motion of the third particle is bounded within regions which are changed as the Jacobian constant C varies. The area of these regions expands as the value of C is increased. This can be easily observed in Figure 1(a)–(c). More specifically, in Figure 1(a), where the surface of section for C_{L_2} is depicted, we observe that the motion is restricted in four separate regions. As the value of C is increased these regions tend to be unified. In Figure 1(b), where the surface of section for C_{L_1} is plotted, the four separate regions of Figure 1(a) have become three and, finally, in Figure 1(c) where the surface of section for C_{L_3} is depicted, there are only two separate regions of bounded motion. Our experience is that, by further increasing the value of C these regions are unified.

In Figure 2 we exhibit a magnification of the region enclosed in box A of Figure 1(a). We can easily distinguish that there are several periodic orbits of various periods. For example, we can observe that the points marked by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$ determine a periodic orbit of period 5. Also in this figure we observe that around this period–5 periodic orbit there are other periodic orbits of higher periods.

To compute the periodic point of the period–5 orbit using the proposed characteristic bisection method, we use a small box surrounding a point of the orbit, say for example, the point $\mathbf{P}_1$ and by proper successive refinements of this box we calculate the desired point. For example, by taking the box $[-3.8, -3.2] \times [0, 0.1]$ we have computed the included periodic point $\mathbf{P}_1 = (-3.60825836, 0.05089673)^\top$.

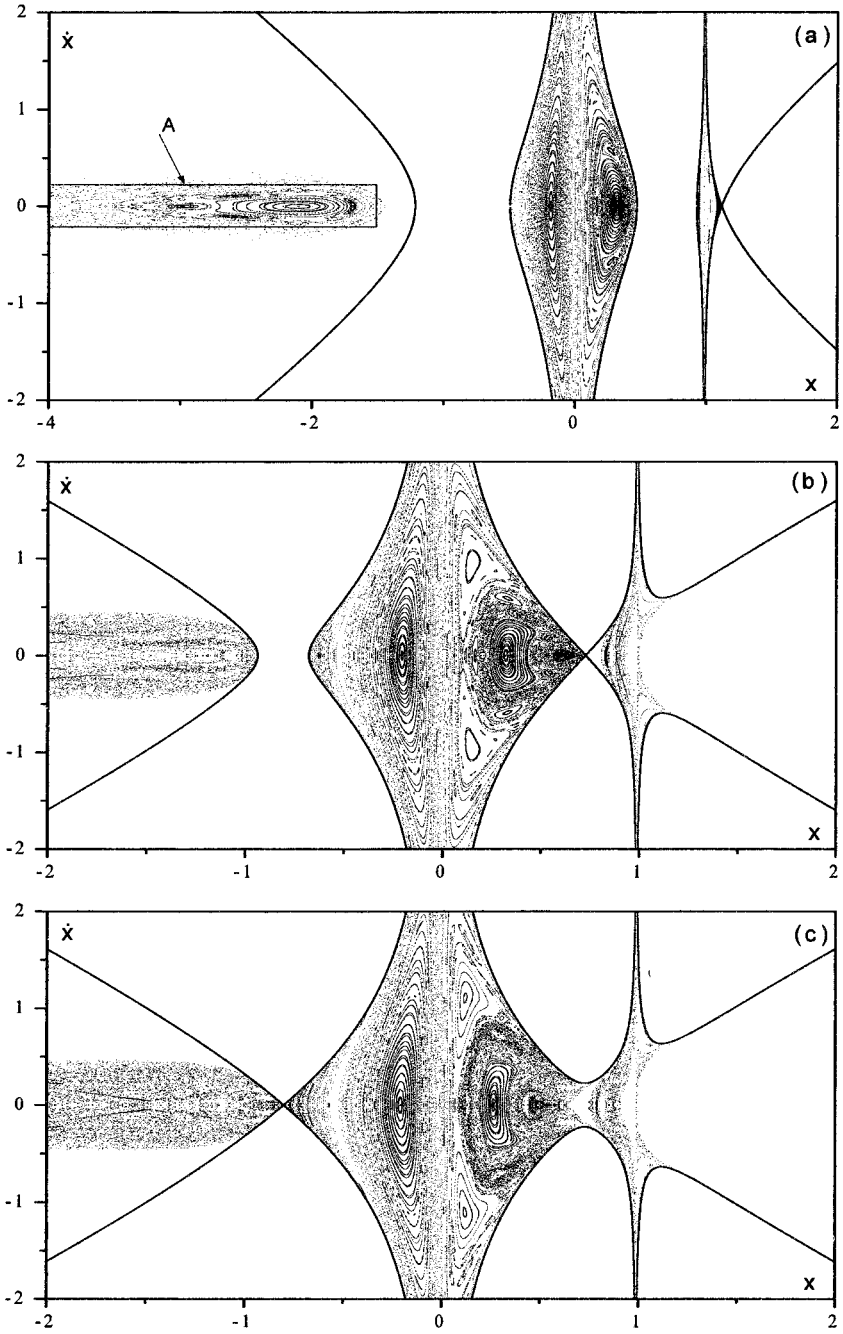

Figure 1. Surface of section points and the corresponding invariant curves of System (9) with parameters $q_1 = 0.5$, $q_2 = 1$, $\mu = 0.01214$, for (a) $C_{L_2} = -2.31058003$, (b) $C_{L_1} = -1.95810456$ and (c) $C_{L_3} = -1.90743262$.

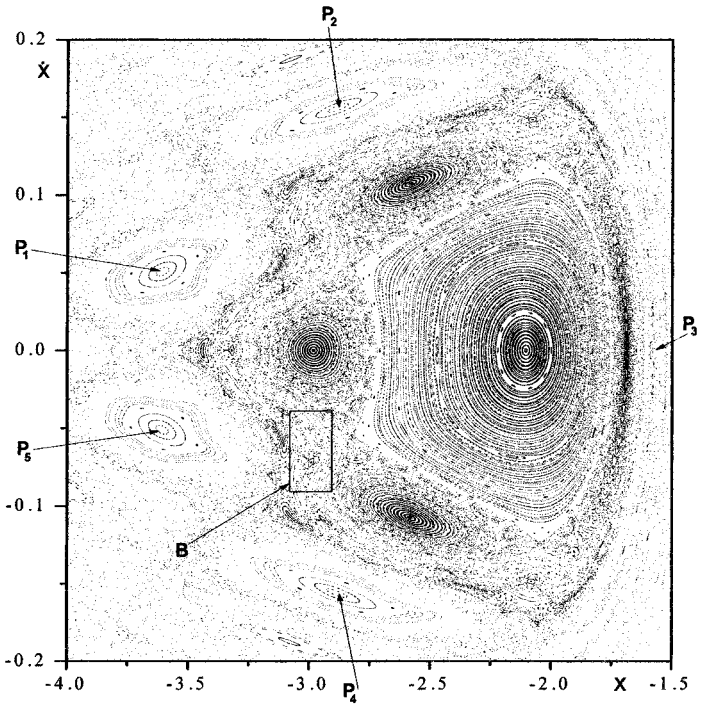

Figure 2. Magnification of box *A* of Figure 1.

In the case of the unstable periodic point we use also a box around a region where a periodic point of the desired periodic orbit is expected to exist.

After one periodic point of the orbit has been computed, the method can be applied to obtain easily all the other points of the same period to the same accuracy. More specifically, the method checks whether each mapping iteration gives a periodic point (of the same period) to the predetermined accuracy. If so, the method continues with the next iteration, otherwise it applies the process of subdivisions to a smaller box which contains the approximate periodic point. The vertices of this small box can be easily selected by permuting the components of the approximate periodic point. Using this approach we have computed all the periodic points of this orbit with coordinates:

$$\begin{aligned}
\mathbf{P}_2 &= (-2.88487215, 0.15609264)^\top, \\
\mathbf{P}_3 &= (-1.57785783, 0.00000000)^\top, \\
\mathbf{P}_4 &= (-2.88487215, -0.15609264)^\top, \\
\mathbf{P}_5 &= (-3.60825831, -0.05089673)^\top.
\end{aligned}$$

Note that, from the sequence in which these points are created on the $(x, \dot{x})$ plane, we are able to infer the rotation number of this orbit. In general, periodic orbits can be identified by their *winding* or *rotation number* σ, which is defined as follows:

$$\sigma = \frac{n_1}{n_2}, \qquad n_1, n_2 \in \mathbb{N}, \tag{15}$$

where n_1, n_2, are two positive integers which indicate that the orbit has produced n_2 points, by rotating counterclockwise around the origin n_1 times (Greene, 1979; Vrahatis, 1995). In particular for the above periodic orbit of period 5 the value of the rotation number is

$$\sigma = \frac{n_1}{n_2} = \frac{4}{5},$$

indicating that the orbit has produced $n_2 = 5$ points by rotating counterclockwise around the origin $n_1 = 4$ times. The values of rotation numbers for all the periodic orbits given here are exhibited in Table I. These values have been computed utilizing a simple angle counting procedure which we have created for this purpose.

Magnifying box B of Figure 2, we observe, in Figure 3, the existence of a periodic orbit of period 13 that is marked by $\mathbf{O}_{13}$. This orbit is surrounded by a group of four islands indicating the existence of another periodic orbit of period 52. Also, in this picture we can distinguish four islands around the period–35 periodic orbit, which is marked by $\mathbf{O}_{35}$. It is evident that, the period of one of these islands is $p = 35 \times 4 = 140$. It is worth noticing that this particular periodic orbit of period 140, required 108 rotations around the origin. Thus, in this case the value of the rotation number of this orbit is $\sigma = 108/140$.

We have also examined the effect of small perturbations to the initial conditions of unstable periodic orbits. For instance, by perturbing with a value $\delta_x = 10^{-5}$ the x coordinate of the unstable periodic orbit of period 4 (listed in Table I) and marked by $\mathbf{U}_4$ in Figure 4(a) we observe that the iterations of the perturbed orbit diffuse away from the point $\mathbf{U}_4$ surrounding also the stable orbit of period 4. The same phenomenon occurs if we perturb any other unstable periodic orbit of Table I as we can observe for example, in Figure 4(b), where 15 000 points are depicted of the perturbed unstable periodic orbit of period 9 ($\mathbf{U}_9$). Its x coordinate has been correspondingly perturbed with the value $\delta_x = 10^{-5}$.

In Table I we exhibit several fixed points of periodic orbits of period p on the Poincaré surface of section for System (9) using the value of the Jacobian constant $C_{L_2} = -2.31058003$. Also, in this table we give the corresponding rotation numbers σ as well as the elapsed CPU time t in seconds on a Personal Computer (Pentium III, Xeon at 550 MHz) required for their computation with an accuracy of $\varepsilon = 10^{-8}$. In addition, in this table we give the stability indexes s_h, s_v as they are defined in Relations (13) as well as the symmetry identification of the corresponding orbit.

TABLE I

Fixed points of periodic orbits of period p on the Poincaré surface of section for System (9) using Jacobian constant $C_{L_2} = -2.31058003$ and their rotation number σ; CPU time t in seconds required for their computation within accuracy $\varepsilon = 10^{-8}$; stability indexes s_h, s_v and symmetry identification *Sym.* ('*S*' denotes symmetry while '*A*' denotes asymmetry)

p	Fixed point	σ	t	s_h	s_v	Sym
1	(0.93717344, 0.00000000)	1	1	0.924068	0.924723	*S*
1	(1.11150338, 0.00000000)	1	1	−0.225994	0.367130	*S*
1	(−2.10504200, 0.00000000)	1	1	−0.320791	−0.341787	*S*
1	(−0.18431881, 0.00000000)	1	1	0.836302	0.836922	*S*
1	(0.32311727, 0.00000000)	1	1	−0.737989	−0.748571	*S*
3	(0.16581940, 0.00000000)	2/3	3	0.954434	0.999713	*S*
3	(0.43782960, 0.00000000)	2/3	2	1.049516	0.997775	*S*
4	(−2.98416804, 0.00000000)	3/4	1	0.665597	0.999959	*S*
4	(−2.82606292, 0.05524204)	3/4	2	1.601737	0.980688	*A*
5	(−1.57785783, 0.00000000)	4/5	2	0.421337	0.999965	*S*
5	(−3.65008738, 0.00000000)	4/5	2	3.916594	0.966065	*S*
7	(−2.47322591, 0.00000000)	5/7	3	0.989496	0.989421	*S*
7	(−1.84329896, 0.00000000)	5/7	4	1.010509	0.988807	*S*
7	(0.09075069, 0.00000000)	5/7	8	0.988499	0.995896	*S*
7	(0.46640390, 0.00000000)	5/7	7	1.011501	0.995794	*S*
9	(−3.44436456, 0.00000000)	7/9	6	1.015720	0.975509	*S*
9	(−1.62423304, 0.00000000)	7/9	5	1.753103	0.975804	*S*
11	(−2.62914789, 0.00000000)	8/11	5	0.967173	0.969919	*S*
11	(−1.78051126, 0.00000000)	8/11	5	1.032966	0.969198	*S*
13	(−3.32141447, 0.00000000)	10/13	5	−0.074102	0.957174	*S*
13	(−1.63525729, 0.00000000)	10/13	8	2.452893	0.947995	*S*
17	(−3.27000405, 0.00000000)	13/17	6	−0.858363	0.912913	*S*
17	(−1.64026111, 0.00000000)	13/17	8	3.259311	0.901168	*S*
19	(−2.71020057, 0.00000000)	14/19	12	0.911912	0.870039	*S*
19	(−1.75442473, 0.00000000)	14/19	13	1.088462	0.869368	*S*
22	(−3.36990585, 0.00000000)	17/22	6	0.570365	0.869416	*S*
22	(−3.05346495, −0.07823604)	17/22	13	1.852898	0.868464	*A*
23	(−2.71984195, 0.00000000)	17/23	11	0.806463	0.775248	*S*
23	(−1.75154707, 0.00000000)	17/23	19	1.194401	0.774477	*S*
25	(−1.59026031, 0.00000000)	20/25	11	0.915142	1.000485	*S*
25	(−1.56733859, 0.00000000)	20/25	12	1.082528	1.000320	*S*
32	(−1.72895845, 0.00000000)	24/32	13	0.899478	0.966856	*S*
32	(−3.16205619, 0.00000000)	24/32	14	1.093301	0.970232	*S*
35	(−3.35824891, 0.00000000)	27/35	9	−0.074700	0.675224	*S*
35	(−1.63329154, 0.00000000)	27/35	15	−2.703773	0.674002	*S*

TABLE I
(*continued*)

p	Fixed point	σ	t	s_h	s_v	Sym
36	(−2.75989136, 0.00000000)	27/36	18	0.103799	0.820277	*S*
36	(−3.20216680, 0.00000000)	27/36	18	1.793506	0.832127	*S*
52	(−2.99324919, 0.06747215)	40/52	20	0.837158	0.370871	*A*
52	(−3.33276884, 0.00000000)	40/52	20	1.085981	0.377574	*S*
140	(−3.03248329, 0.05557997)	108/140	63	0.163394	−0.984141	*A*
140	(−3.35744012, 0.00000000)	108/140	85	1.190881	−0.984347	*S*

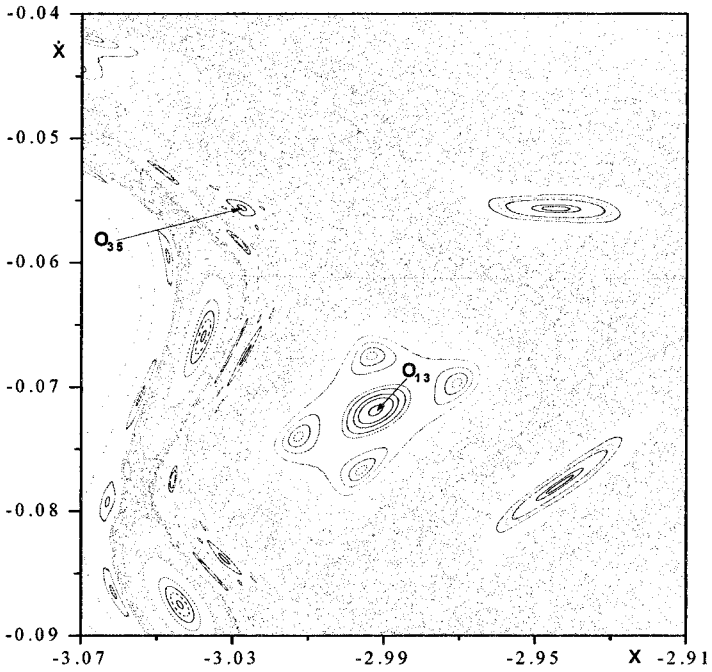

Figure 3. Magnification of box *B* of Figure 2.

5. Epilogue

In this paper, we have proposed a method for computing with certainty individual members of families of periodic orbits of a given period within a given box using the Poincaré map on a surface of section. We have used this method to compute individual members of families of periodic orbits of the photogravitational restricted circular three-body problem and we have succeeded in computing symmetric and asymmetric periodic orbits of various periods and stability characteristics.

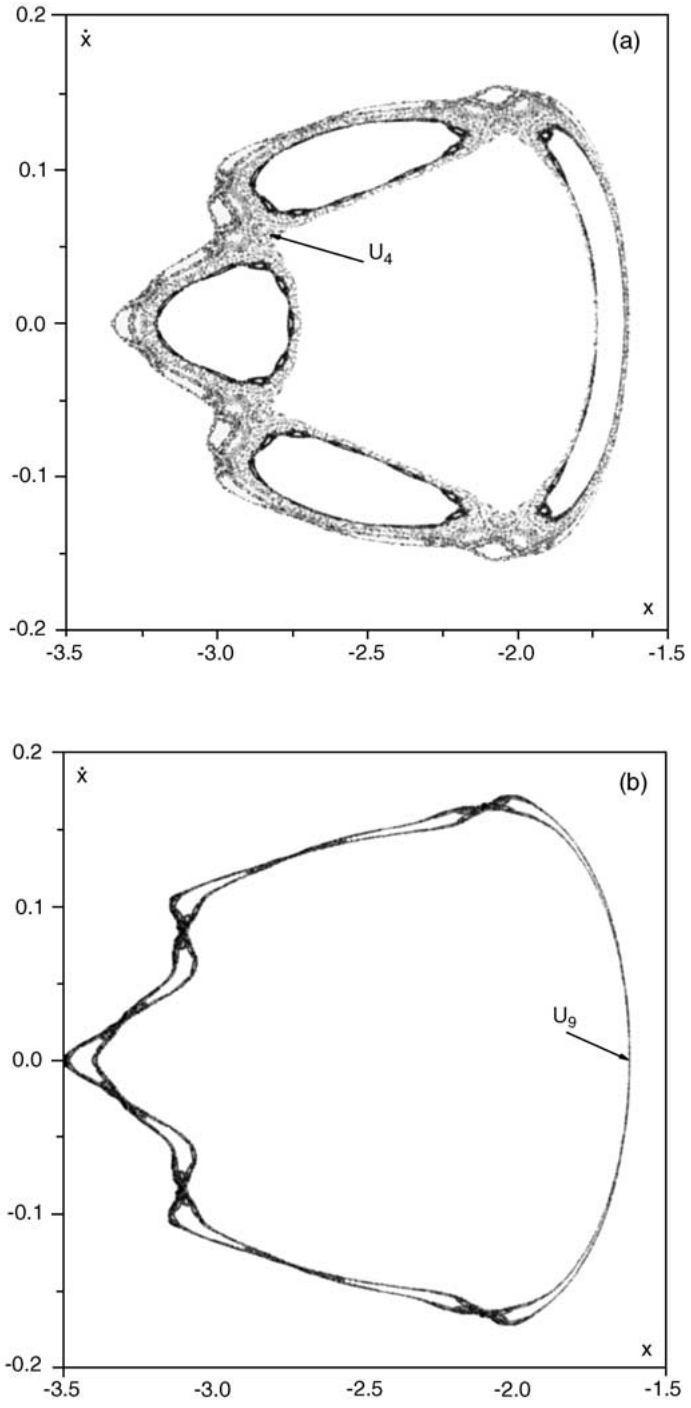

Figure 4. Perturbations with the value $\delta_x = 10^{-5}$ of the x coordinate of two unstable periodic obits, (a) 15 000 points of the perturbed periodic point $\mathbf{U}_4$ of period–4 periodic orbit and (b) 15 000 points of the perturbed periodic point $\mathbf{U}_9$ of period–9 periodic orbit.

According to the proposed method, we have used a box around a region where a periodic point of the desired periodic orbit is expected to exist and by proper successive refinements of this box we have calculated the periodic point included. Thus, this method becomes especially significant for the computation of high period orbits (stable or unstable) where other more traditional approaches cannot easily distinguish among periodic orbits which are close neighbours. The only information needed for this refinement is the algebraic signs of various function evaluations on the vertices of the considered box and thus it is not affected by the imprecisions of the mapping evaluations. The only limitation of the proposed method is that it is dependent upon the integration method which is used to calculate the algebraic signs required.

After one periodic point of the orbit has been computed, the method can easily obtain all the other points of the same period to the same accuracy. From the sequence in which these points are created on the $(x, \dot{x})$ plane, we have also identified the rotation number of the corresponding orbit.

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