Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 184 (2007) 93–103
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Existence and computation of short-run equilibria in economic geography

N.G. Pavlidis [a,b], M.N. Vrahatis [a,b,*], P. Mossay [c,1]

[a] Computational Intelligence Laboratory (CI Lab), Department of Mathematics, University of Patras, GR-26110 Patras, Greece
[b] University of Patras Artificial Intelligence, Research Center (UPAIRC), University of Patras, GR-26110 Patras, Greece
[c] Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, E-03080 Alicante, Spain

Abstract

The new economic geography literature provides a general equilibrium framework that explains the emergence of economic agglomerations as a trade-off between increasing returns at the firm level and transportation costs related to the shipment of goods. The existence and uniqueness of short-run equilibria of this model has been shown for the case of two regions. The proposed approach employs the differential evolution algorithm to obtain estimates of the Lipschitz constant and the infinity norm of the function along the boundary and utilizes these values to investigate the existence of solutions of a function, and the computational burden of computing the topological degree of this function. This approach is employed to investigate the existence of short-run equilibria for more than two regions using fixed point and topological degree theory, as well as, the differential evolution algorithm. Irrespective of parameter settings the criteria from topological degree theory suggest that the model can have equilibria. The differential evolution algorithm identified such equilibria and for none of the parameter settings that were considered more than one equilibria were detected. The experimental results obtained also indicate that the computation of such equilibria has an exponential worst-case lower bound complexity, as the model yields a function that is neither contractive, nor nonexpanding. Finally, the computation of the topological degree to identify the number of equilibria also has a very high computational cost.

Keywords: Economic geography; Spatial economy; Short-run equilibria; Fixed points; Topological degree

1. Introduction

New Economic Geography has emerged from the long-existing need to explain concentrations of economic activity. The distinction between the manufacturing sector and farm belts, the existence of cities, and the role of industry clusters, are issues that come within the scope of the New Economic Geography. The literature in the

* Corresponding author.
E-mail addresses: npav@math.upatras.gr (N.G. Pavlidis), vrahatis@math.upatras.gr (M.N. Vrahatis), mossay@merlin.fae.ua.es (P. Mossay).
[1] The author acknowledges financial support from the Spanish Ministerio de Ciencia y Tecnología, under ERDF Project SEJ2004-08011ECON.

0096-3003/$ - see front matter
doi:10.1016/j.amc.2005.11.177

field provides a general equilibrium framework in which agglomerations of manufacturing activity emerge due to the trade-off between increasing returns at the firm level, and transportation costs [1]. We consider a standard new economic geography model involving a finite number of regions [2,3], and two sectors in the economy: the agricultural sector and the manufacturing sector. In this framework, economic equilibria refer to the allocation of the factors of production (in the model the manufacturing labor force), and the set of prices, that arise by the optimal behavior of firms and consumers and yield market clearing. Short-run equilibria of the model are characterized by the fixed distribution of labor across regions, while in long-run equilibria the spatial adjustment of workers is allowed. The present paper focuses on issues relating to the existence and the computational complexity of locating short-run equilibria, which correspond to fixed points of a function.

Computing fixed points, or equivalently, solving systems of nonlinear equations has long been a topic of great interest for researchers in the field of mathematics, engineering, economics, and many other professions. Numerous problems such as finding an equilibrium, a zero point, or a fixed point, can be formulated as the problem of finding a solution to an equation of the form $F(x) = p$ in an appropriate space. Topological degree theory provides means for examining this solution set and obtaining information about the existence of solutions, their number and their nature. The Lipschitz constant of the function, as well as the value of the infinity norm along the boundary of its domain, also provide information about the existence of solutions to the aforementioned problem. These values are also employed to determine the computational cost of computing the topological degree. The proposed approach relies on the approximation of the Lipschitz constant and the infinity norm along the boundary using an evolutionary optimization algorithm, namely the differential evolution algorithm. In particular, we estimate the modulus of continuity of the function, which serves as a lower bound for the Lipschitz constant. The differential evolution algorithm is employed as the approximation of both the modulus of continuity and the infinity norm involve the minimization of a non-differentiable objective function. This approach is applied to investigate the existence, as well as, the computational complexity of locating short-run equilibria, which correspond to fixed points of a function.

The rest of the paper is organized as follows: Section 2 outlines the basic notions associated with economic geography and presents the model. Section 3 presents background material. Specifically, Section 3.1 outlines the basic notions of topological degree theory and provides references to efficient methods for its computation. In Section 3.2 algorithms for computing fixed points and their computational complexity, are discussed. The differential evolution algorithm, is presented in Section 3.3. Section 4 provides a detailed exposition of the proposed approach. It describes criteria, based on the value of the Lipschitz constant and the infinity norm along the boundary of the function, for the existence of solutions and discusses the corresponding complexity of computing the topological degree. In Section 5 the obtained experimental results are discussed, and the paper ends with conclusions.

2. Economic geography

Lately the increasing interest in the field of economic geography has attracted numerous scientists from various disciplines ranging from economics to regional science and geography. There is no doubt that the building of the European Union and the several policy issues which come along have contributed to boost interest in the field. Space has always been a concern in economics. If mainstream economics has rather neglected it during the past, it is not so much because economists have been uninterested in the subject, but rather because it has remained intractable for a long time. Modeling tools that had been developed to analyze industrial organization, international trade, and economic growth, have allowed us to overcome technical problems arising when dealing with *imperfect competition in a general equilibrium framework* [1,2]. The literature in the field of New Economic Geography provides a general equilibrium framework explaining the emergence of economic agglomerations as a trade-off between increasing returns at the firm level and transportation costs related to the shipment of goods. This means that this literature provides economic motives for agglomeration rather than assuming that some regions are more productive than others ex ante. Recent work in the field has incorporated additional economic concerns into the analysis (e.g. capital, welfare analysis, expectations), while at the same time, an empirical literature testing the theory with data has emerged.

The main ingredients of the new economic geography are transportation costs, and the interaction of market size with increasing returns which creates a cumulative process in which larger markets attract additional economic activity. A central feature of the models in the literature is that a higher taste for product variety, a larger share of the manufacturing expenditure, and lower transportation costs, favor the emergence of agglomerations (e.g. spatial concentration of economic activities). We consider a standard new economic geography model involving a finite number of regions, see [2,3]. This model can be viewed as an extension of Krugman's core-periphery model [1] to the case of a spatial economy consisting of N regions. As in Krugman's original work, there are two sectors in the economy. The agricultural sector employs farmers and produces a single homogeneous good under constant returns to scale. The manufacturing sector employs workers and produces a differentiated good, giving rise to manufacturing varieties. Consumers (workers and farmers) buy the agricultural good on a perfectly competitive national market and manufacturing varieties on monopolistically competitive regional markets. In addition, transporting manufacturing varieties from their production location to where they are consumed, is costly. Economic equilibria define economic allocations and prices derived from optimal behaviors of firms and consumers that are compatible with market clearing. On the one hand, short-run equilibria are obtained under the assumption of no spatial adjustment. These short-run equilibria are thus viewed as implicitly determined by some given spatial distribution of labor. On the other hand, long-run equilibria refer to steady states of a spatial economy where workers are allowed to adjust their location over time. In the case of a spatial economy consisting of two regions, a short-run equilibrium has been shown to exist and to be unique [4], while the number and stability of steady states have also been studied [2]. However, for the case of three or more regions, no analytical result concerning short- or long-run equilibria has been derived so far.

In this paper we focus on short-run equilibria. Regions are denoted by $i = 1,\dots,N$. Assume a spatial distribution of labor L_i across these regions. The proportion of the labor force in region i is denoted by $\lambda_i = L_i/\sum_{j=1}^{N} L_j$. The variables of the model are y_i, θ_i, and w_i respectively the income, the manufacturing price index and the manufacturing wage in region i. The system of equations defining short-run equilibria of the spatial economy can be written in the following reduced form:

$$\begin{aligned} y_i &= (1-\mu)/N + \mu\lambda_i w_i, \\ \theta_i &= \left\{ \sum_{j=1}^{N} \lambda_j w_j^{(1-\sigma)} \exp\left[-\tau(\sigma-1)d(i,j)\right] \right\}^{-1/(\sigma-1)}, \\ w_i &= \left\{ \sum_{j=1}^{N} y_j \theta_j^{(\sigma-1)} \exp\left[-\tau(\sigma-1)d(i,j)\right] \right\}^{1/\sigma}, \end{aligned} \tag{1}$$

where

$d(i,j)$ distance between locations i and j
$\sigma > 1$ elasticity of substitution among manufacturing varieties
$\mu \in (0,1)$ share of manufacturing expenditure
$\tau > 0$ transportation cost per unit of distance for manufacturing goods

The final equation of the system in Eq. (1) corresponds to the level of nominal wages at which manufacturing in region i breaks even. Throughout the remaining paper we refer to this equation as $f_i(w)$ to avoid confusing notation when referring to the fixed point problem $f = w$, or the zero point problem $f - w = 0$.

3. Background material

3.1. The topological degree and its computation

This subsection is devoted to a brief presentation of topological degree theory to determine the exact number of zeros of a system of nonlinear transcendental equations. This is achieved by computing the value of the topological degree using Kronecker's integral [5] on Picard's extension [6,7]. Assume a function $F_n = (f_1, f_2, \dots, f_n) : \overline{\mathscr{D}_n} \subset \mathbb{R}^n \to \mathbb{R}^n$, which is defined and twice continuously differentiable in an open and bounded domain $\mathscr{D}_n$ of $\mathbb{R}^n$ with boundary $\vartheta\mathscr{D}_n$. Further suppose that the zeros of the equation $F_n(x) = p$, where $p \in \mathbb{R}^n$ is a given vector, are not located on $\vartheta\mathscr{D}_n$, and that they are simple, i.e. the determinant, $\det J_{F_n}$, of the Jacobian matrix of F_n at these points is non-zero.

Definition. The topological degree of F_n at p relative to $\mathscr{D}_n$ is denoted by $\deg[F_n, \mathscr{D}_n, p]$ and is defined by the following sum:

$$\deg[F_n, \mathscr{D}_n, p] = \sum_{x \in F_n^{-1}(p) \cap \mathscr{D}_n} \operatorname{sgn}(\det J_{F_n}(x)), \tag{2}$$

where $\operatorname{sgn}(\psi)$ stands for the three valued sign function.

The topological degree is invariant under changes of the vector p in the sense that, for any $q \in \mathbb{R}^n$, it holds that: $\deg[F_n, \mathscr{D}_n, p] \equiv \deg[F_n - q, \mathscr{D}_n, p - q]$, where $F_n - q$ denotes the mapping $F_n(x) - q$, $x \in \mathscr{D}_n$ [8, p. 157]. We thus consider only the case where the topological degree is defined at the point $\Theta_n = (0, \ldots, 0)$ in $\mathbb{R}^n$.

The topological degree $\deg[F_n, \mathscr{D}_n, \Theta_n]$ can be represented by the Kronecker integral which is defined as

$$\deg[F_n, \mathscr{D}_n, \Theta_n] = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int\int_{\vartheta\mathscr{D}_n} \cdots \int \frac{\sum_{i=1}^{n} A_i \, \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1} \, \mathrm{d}x_{i+1} \cdots \mathrm{d}x_n}{(f_1^2 + f_2^2 + \cdots + f_n^2)^{n/2}}, \tag{3}$$

where A_i denote the following determinants:

$$A_i = (-1)^{n(i-1)} \det \begin{bmatrix} F_n & \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_{i-1}} & \frac{\partial F_n}{\partial x_{i+1}} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}, \tag{4}$$

where, $\frac{\partial F_n}{\partial x_k} = (\frac{\partial f_1}{\partial x_k}, \frac{\partial f_2}{\partial x_k}, \ldots, \frac{\partial f_n}{\partial x_k})$ is the kth column of the determinant $\det J_{F_n}$ of the Jacobian matrix J_{F_n}. The topological degree can be generalized when the function is only continuous [8]. In this case, Kronecker's theorem [8] states that $F_n(x) = \Theta_n$ has at least one zero in $\mathscr{D}_n$ if $\deg[F_n, \mathscr{D}_n, \Theta_n] \neq 0$.

Eq. (2) states that $\deg[F_n, \mathscr{D}_n, \Theta_n]$ is equal to the number of zeros of $F_n(x) = \Theta_n$ that give positive determinant of the Jacobian matrix minus the number of zeros that give negative determinant of the Jacobian matrix. The total number $\mathscr{N}^r$ of zeros of $F_n(x) = \Theta_n$ would therefore be equal to the value of $\deg[F_n, \mathscr{D}_n, \Theta_n]$ if the determinant of the Jacobian matrix at all these zeros yielded the same sign. Recall that all the zeros of $F_n(x) = \Theta_n$ are assumed to be simple. To this end, Picard proposed the following extension of the function F_n and the domain $\mathscr{D}_n$:

$$F_{n+1} = (f_1, \ldots, f_n, f_{n+1}) : \mathscr{D}_{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

where $f_{n+1} = y \det J_{F_n}$, and $\mathscr{D}_{n+1}$ is the direct product of the domain $\mathscr{D}_n$ with an arbitrary interval of the real y-axis containing the point $y = 0$. The zeros of the following system of equations:

$$\begin{aligned} &f_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, \ldots, n, \\ &y \det J_{F_n}(x_1, x_2, \ldots, x_n) = 0 \end{aligned} \tag{5}$$

are, therefore, identical with the zeros of $F_n(x) = \Theta_n$ provided that $y = 0$. Furthermore, the determinant of the Jacobian matrix of (5) is equal to $[\det J_{F_n}(x)]^2$ which is always nonnegative, and positive at the simple zeros. We may thus conclude:

Theorem [7]. *The total number $\mathscr{N}^r$ of zeros of $F_n(x) = \Theta_n$ is given by*

$$\mathscr{N}^r = \deg[F_{n+1}, \mathscr{D}_{n+1}, \Theta_{n+1}], \tag{6}$$

under the hypotheses that F_n is twice continuously differentiable and that all the zeros are simple and lie in the strict interior of $\mathscr{D}_{n+1}$.

Several methods for the computation of the topological degree have been proposed [5,9,10]. These methods are based on Stenger's method that is an almost optimal complexity algorithm for a number of classes of functions [10].

3.2. Fixed points and their computation

The development of fixed point algorithms has been an intensive research area since 1922 when Banach in his famous dissertation proposed the simple iteration algorithm [8,10,11]

$$x^{k+1} = F_n(x^k), \quad k = 0, 1, \ldots \tag{7}$$

which for a function $F_n : \mathcal{D}_n \subset \mathbb{R}^n \to \mathbb{R}^n$ which is contractive on a closed set $\mathcal{Q}_n \subset \mathcal{D}_n$ and that $F_n(\mathcal{Q}_n) \subset \mathcal{Q}_n$, converges to the unique fixed point $x^* \in \mathcal{Q}_n$ for any arbitrary starting point $x^0 \in \mathcal{Q}_n$ [8]. For Lipschitz functions F_n with constant $L < 1$ and large dimension n the iteration algorithm (7) is optimal [10,12,13] and requires $v = \lceil \log_2(\varepsilon^{-1})/\log_2(L^{-1}) \rceil$ function evaluations to compute the approximation $\tilde{x}$ of the fixed point x^* such that $\|\tilde{x} - x^*\|_2 \leqslant \varepsilon \|x^*\|_2$ (see [10]).

Numerous algorithms have been proposed since Banach's algorithm, including homotopy continuation, simplicial and Newton type methods [14–17]. The latter algorithms, for Lipschitz functions with constant $L > 1$ with respect to the infinity norm, exhibit exponential complexity in the worst case when computing ε-*residual approximation* $\tilde{x} : \|F_n(\tilde{x}) - \tilde{x}\|_\infty \leqslant \varepsilon$, (that is the computed approximation $\tilde{x}$ satisfies the residual criterion $\|F_n(\tilde{x}) - \tilde{x}\|_\infty \leqslant \varepsilon$ where $\varepsilon > 0$) and that the lower bound on the complexity is also exponential [18]. On the other hand, the computation of fixed points of a function that is contractive ($L < 1$) is a commonly encountered problem in numerical computation (for nonlinear problems and large scale linear systems). Moreover, fixed points of contractive, or nonexpanding ($L = 1$), functions appear in numerous fields including economics, game theory (especially ergodic games), meromorphic functions, nonlinear differential equations and dynamical systems. In the study of dynamical systems with two degrees of freedom, such fixed point (periodic orbits) problems model conservative or dissipative systems depending on whether the mapping is area-preserving or area-contracting, respectively (see [19–27]).

Several algorithms for approximating a fixed point, x^*, of a Lipschitz function that is contractive or nonexpanding with respect to the second norm have been developed [10,28,29]. An efficient method for the computation of fixed points is the interior ellipsoid method [10,28,30]. In the nonexpanding case and moderate dimensions n the interior ellipsoid algorithm is optimal [28]. This algorithm requires $v = cn\log(\varepsilon^{-1})$ function evaluations to compute an ε-residual approximation $\tilde{x} : \|F_n(\tilde{x}) - \tilde{x}\|_2 \leqslant \varepsilon$. Notice that the worst-case complexity of computing an ε-*absolute approximation* $\tilde{x} : \|\tilde{x} - x^*\|_2 \leqslant \varepsilon$ for the nonexpanding case is infinite [10]. This means that there exists no algorithm based on function evaluations that solves this problem for all functions in this class. For the contractive case the interior ellipsoid algorithm computes $\tilde{x} : \|\tilde{x} - x^*\| \leqslant \varepsilon$ within $v = cn(\log(\varepsilon^{-1}) + \log((1 - L)^{-1}))$ function evaluations. A recently proposed algorithm named PFix for approximating a fixed point of a function F_n, where F_n has arbitrary dimensionality, is defined on a rectangular domain, and is Lipschitz continuous with respect to the infinity norm with constant one, has been presented in [31]. This algorithm computes an approximation that satisfies the residual error criterion, and can also compute an approximation satisfying the absolute error criterion when the Lipschitz constant is less than one. Furthermore it is a recursive algorithm, in that it uses solutions to an n-dimensional problem to compute a solution to an $(n+1)$-dimensional problem.

3.3. Differential evolution algorithm

The differential evolution algorithm (DE) [32,33] is a population-based stochastic optimization algorithm that exploits a population of potential solutions, called *individuals*, to probe the search space. New individuals are generated by the combination of randomly chosen individuals from the existing population. An operation which is called *mutation* in the context of DE. Specifically, for each individual x_g^k, $k = 1, \ldots, NP$, where g denotes the index of the current generation, a new individual v_g^i, called mutant individual, is generated according to one of the equations below:

$$v_g^i = x_g^{\text{best}} + \mu(x_g^{r_1} - x_g^{r_2}), \tag{8}$$

$$v_g^i = x_g^{r_1} + \mu(x_g^{r_2} - x_g^{r_3}), \tag{9}$$

$$v_g^i = x_g^i + \mu(x_g^{\text{best}} - x_g^i) + \mu(x_g^{r_1} - x_g^{r_2}), \tag{10}$$

$$v_g^i = x_g^{\text{best}} + \mu(x_g^{r_1} - x_g^{r_2}) + \mu(x_g^{r_3} - x_g^{r_4}), \tag{11}$$

$$v_g^i = x_g^{r_1} + \mu(x_g^{r_2} - x_g^{r_3}) + \mu(x_g^{r_4} - x_g^{r_5}), \tag{12}$$

where, x_g^{best} is the best member of generation g; $\mu > 0$ is a real parameter, called mutation constant, which controls the amplification of the difference between two individuals so as to avoid the stagnation of the search process; and $r_1, r_2, r_3, r_4, r_5 \in \{1, 2, \ldots, k-1, k+1, \ldots, NP\}$, are mutually different random integers, different from the running index k. To further increase the diversity of the mutant individuals, they are combined with other predetermined individuals – the *target* individuals – through an operation known as *recombination* – to produce *trial* individuals. At the recombination stage, for each component $l\,(l = 1, 2, \ldots, D)$ of the mutant individual v_g^k, a random real number r in the interval $[0, 1]$ is drawn and compared to the recombination constant, ρ. If $r \leqslant \rho$, then the lth component of the trial individual u_g^k is set equal to the lth component of the mutant individual v_g^k. Otherwise, the lth component of the target vector, x_g^k, becomes the lth component of the trial vector. After the completion of recombination the trial individuals are subjected to *selection*. Each trial individual, u_g^k, is accepted for the next generation, if and only if, it yields a reduction in the value of the error function relative to the individual of the previous generation, x_g^k. Otherwise, x_g^k, is retained at the next generation.

4. The proposed approach

An approach for the investigation of the existence of roots requires the value of the Lipschitz constant of the function, as well as, the infinity norm of the function along the boundary of $\mathscr{D}_n$ [10,34,35]. These values are also relevant for the estimation of the computational burden, in terms of required function evaluations, for the computation of the topological degree of the function. The value of the topological degree, under certain conditions, provides information regarding the existence of roots. Moreover, by properly extending the function and the corresponding domain, the value of the topological degree gives the number of simple roots of the function, within the interior of $\mathscr{D}_n$.

Consider the class $\mathscr{F}$ of Lipschitz functions with Lipschitz constant L, defined on the n-dimensional unit hypercube $\mathscr{C}$,

$$F_n = (f_1, f_2, \ldots, f_n) : \mathscr{C} \subset \mathbb{R}^n \to \mathbb{R}^n,$$

such that for every $F_n \in \mathscr{F}$ we have

$$\|F_n(x)\|_\infty \geqslant \delta > 0$$

for all $x \in \vartheta\mathscr{C}$, where $\vartheta\mathscr{C}$ denotes the boundary of $\mathscr{C}$. Then, the following existence criteria hold [10]:

$$\left.\begin{array}{ll} \text{(i)} & \text{if } L/(2\delta) \geqslant 4, \text{ then the function } F_n \text{ may have zeros in } \mathscr{C}; \\ \text{(ii)} & \text{if } L/(2\delta) < 1, \text{ then the function } F_n \text{ does not have any zeros in } \mathscr{C}. \end{array}\right\} \tag{13}$$

The case $1 \leqslant L/(2\delta) < 4$ remains an open problem [10].

Furthermore, using the values of L and δ, Boult and Sikorski proved in [34] that the topological degree can be computed using

$$A = \left(\left\lfloor \frac{L}{2\delta} + 1 \right\rfloor + 1\right)^n - \left(\left\lfloor \frac{L}{2\delta} + 1 \right\rfloor - 1\right)^n, \tag{14}$$

function evaluations for every $F_n \in \mathscr{F}$, $n \geqslant 2$. This can be achieved through an algorithm due to Kearfott [36], with cost given by $A(c + (n^2/2)(n-1)!)$, where c is the cost of each function evaluation, while the cost of elementary arithmetic operations and comparisons is unity. Thus, for a small n, e.g. $n \leqslant 5$, and a small value of $L/(2\delta)$, e.g. $L/(2\delta) \leqslant 9$, the degree can be computed in time at most $10^5(c + 300)$. For large n and/or large $L/(2\delta)$ the problem is intractable [10,34]. The Lipschitz constant can also be used for the determination of the complexity of the fixed point problem, in the case of the residual error criterion, in the class of functions $F_n : \mathscr{C} \to \mathscr{C}$, satisfying the Lipschitz condition with Lipschitz constant $L > 1$ [10,18].

In numerous cases, the parameters δ and L are not a priori known, and their estimation is a computationally heavy task [35,37]. In this paper, DE is used to estimate the parameter δ and the Lipschitz constant, L, of a function, to infer conclusions regarding the existence of roots and estimate the computational burden, in terms of the required function evaluations, for the computation of the topological degree. DE is a population-based optimization method, that requires function values solely. Consequently, in general, neither derivatives nor an analytic representation of the function is needed.

As previously mentioned, the values of main interest that have to be computed, are

$$\delta = \min_{x \in \vartheta\mathscr{D}_n} \|F_n(x)\|_\infty \tag{15}$$

and

$$L = \max_{\substack{x \neq y \\ x,y \in \mathscr{D}_n}} \frac{\|F_n(x) - F_n(y)\|_\infty}{\|x - y\|_\infty}, \tag{16}$$

where $\mathscr{D}_n$ is an n-dimensional polyhedron. The value L can be estimated by performing the maximization of the corresponding fraction on x, with y being randomly selected at each evaluation of the fraction, following a uniform distribution. Alternatively, one can estimate the *modulus of continuity of* F_n *on* $\mathscr{D}_n$

$$\omega(F_n, t) = \sup\{\|F_n(x) - F_n(y)\|, \text{ for } x, y \in \mathscr{D}_n, \text{ and } \|x - y\| \leqslant t\}. \tag{17}$$

Note that if F_n is Lipschitz for some real number L and for all $x, y \in \mathscr{D}_n$ then we immediately have

$$\omega(F_n, t) \leqslant Lt. \tag{18}$$

In the proposed approach, DE is employed to compute δ, through consecutive optimization on the boundary of the domain under consideration. Moreover, L is estimated by approximating the modulus of continuity, Eq. (17). In general DE has proved to be considerably noise-tolerant. The procedure for the estimation of the modulus of continuity can be considered, in regions of small size, as a noisy optimization procedure with noise proportional to the value of the function.

Obviously, the estimates δ' and L', of δ and L, respectively, which are obtained through the aforementioned procedures, will differ from their actual values. Since the value of δ is computed through a minimization procedure with a prespecified accuracy, where δ is the global minimum, it holds that $\delta' \geqslant \delta$. Similarly, the obtained estimate, L', of the Lipschitz constant, will be $L' \leqslant L$ (as shown in Eq. (18)), i.e., there will always be an underestimation of L and an overestimation of δ. However, if

$$\frac{L'}{2\delta'} \geqslant 4$$

for the computed values in the unit hypercube, then, the function may have roots (c.f. criterion in (13)), since

$$4 \leqslant \frac{L'}{2\delta'} \leqslant \frac{L}{2\delta}.$$

Thus, the proposed approach can provide valuable information regarding the existence of roots of functions with unknown Lipschitz constant and infinity norm along the boundary. We recall that in the case of $L'/(2\delta') < 4$, conclusions cannot be derived. Furthermore, the quantity

$$\hat{A} = \left(\left\lfloor \frac{L'}{2\delta'} + 1 \right\rfloor + 1 \right)^n - \left(\left\lfloor \frac{L'}{2\delta'} + 1 \right\rfloor - 1 \right)^n \tag{19}$$

constitutes a lower bound on the number of function evaluations, A, required for the computation of the topological degree, as defined in Eq. (14), since A is non-decreasing with respect to $L/(2\delta)$.

5. Presentation of experimental results

To obtain an estimate for the modulus of continuity (which in turn is used to obtain a lower bound of the Lipschitz constant, L) and also to approximate the infinity norm along the boundary, δ, of the function we employ the DE algorithm. The domain of the function F_N of Eq. (1) is determined by the requirement of the economic geography model to have the sum of nominal wages across regions constant

$$\sum_{j=1}^{N} \lambda_j x_j^i = 1,$$

where, x^i denotes the ith DE individual, and x_j^i, $j = 1, \ldots, N$ stands for the nominal manufacturing wage at region j, w_j. As DE is an unconstrained optimization algorithm, we employ the following normalization to the individuals, x^i, to evaluate F_N

$$x_p^i = \|x^i\| \bigg/ \sum_{j=1}^{m_i} \|\lambda_j x_j^i\|,$$

where m_i is the number of regions. Note that this normalization is used only when evaluating the individuals and not to constraint the population of the algorithm in the domain of the function. If the normalized individuals, x_p^i, replace the original individuals, x^i, our experience suggests that the diversity of the population decreases drastically which can cause the premature convergence of the algorithm. To evaluate the fitness function for the case of computing the modulus of continuity, for each DE individual (x^i), 1000 random points within a fixed range, $t = 0.1$, from x^i (using as a distance metric the infinity norm) are generated. The fitness function for each individual, x^i, is an estimate of the negative of the modulus of continuity in the neighborhood of the individual

$$-\max_{r^j}\{\|F(x^i) - F(r^j)\|_\infty\}, \quad j = 1, \ldots, 1000,$$

where r^j, $j = 1, \ldots, 1000$ represents a random point within the specified distance from the individual x^i. The global minimum of this fitness function corresponds to minus the modulus of continuity.

At a first step, we investigated whether the Lipschitz constant of the function F_N is lower than unity for a number of parameter settings. Employing the DE algorithm to compute the modulus of continuity for a number of parameter settings, and for three to five regions equidistributed along the unit circle, the obtained results for all instances of the problem yielded a lower bound on the Lipschitz constant substantially larger than one. As previously mentioned, fixed point algorithms for Lipschitz functions with constant $L > 1$ with respect to the infinity norm, exhibit exponential complexity in the worst case when computing *ε-residual approximation*. Next we investigate whether the criteria from topological degree theory can provide information concerning the existence of equilibria and estimate the computational complexity associated with the computation of the degree for this type of functions. Since short-run equilibria correspond to fixed points of Eq. (1), while the topological degree of a function provides information concerning the number of simple zeros of a function in a domain, we transform the fixed point problem to a root finding problem through the following manipulation:

$$G_N = (f_i(w) - w_i, \ldots, f_N(w) - w_N), \tag{20}$$

where $f_i(w)$ and w_i are as in Eq. (1). To obtain a lower bound for the Lipschitz constant we estimate through DE the modulus of continuity $\omega'(G_N, t)$ for fixed $t = 0.1$, and from Eq. (18) we have $L' \geqslant \omega'(G_N, t)/t$. In Tables 1–3 the results obtained for the case of three to five regions are summarized. Note that due to space limitations, all entries in the tables are rounded to three decimal places.

In all the conducted numerical experiments the ratio $L'/2\delta'$ exceeded the value of four, indicating that the function under consideration may have roots (Eq. (13)). The existence of zeros of the function for different parameter settings has been verified through their computation using the DE algorithm. A point was considered a short-run equilibrium if $\|G_N\|_1 \leqslant 10^{-6}$. Due to space limitations, only a small sample of the parameter settings tested and the corresponding short-run equilibria are reported in Tables 4–6. For each parameter setting the DE algorithm was executed ten times. It is important to note that DE did not locate more than one equilibria for all the parameter settings considered, indicating that the fixed point of the short-run economic geography model might be unique.

As illustrated in Tables 1–3 the ratio $L'/(2\delta')$ assumes large values sometimes even exceeding the threshold of nine. Hence, the computation of the number of short-run equilibria of the economic geography model through the computation of the topological degree of the function G_N is a computationally hard task, as sug-

Table 1
Estimation for the infinity norm along the boundary and the Lipschitz constant for the three-region model

Parameter setting						δ'	L'
μ	σ	τ	λ_1	λ_2	λ_3		
0.513	4.288	2.043	0.245	0.462	0.292	2.558	32.990
0.840	4.386	2.538	0.477	0.267	0.254	2.538	36.835
0.304	4.021	2.870	0.382	0.166	0.450	2.071	48.522
0.121	1.503	2.637	0.350	0.361	0.288	2.413	11.963
0.449	1.680	2.171	0.651	0.140	0.208	1.840	18.343

Table 2
Estimation for the infinity norm along the boundary and the Lipschitz constant for the four-region model

Parameter setting							δ'	L'
μ	σ	τ	λ_1	λ_2	λ_3	λ_4		
0.709	3.606	1.404	0.262	0.404	0.261	0.070	2.150	13.500
0.735	3.224	2.744	0.238	0.155	0.339	0.266	2.128	45.349
0.765	4.536	2.013	0.204	0.471	0.212	0.111	2.245	35.344
0.625	2.526	2.370	0.239	0.083	0.323	0.353	2.022	36.120
0.071	3.883	1.815	0.334	0.298	0.021	0.344	2.040	20.109

Table 3
Estimation for the infinity norm along the boundary and the Lipschitz constant for the five-region model

Parameter setting								δ'	L'
μ	σ	τ	λ_1	λ_2	λ_3	λ_4	λ_5		
0.851	2.512	1.889	0.192	0.207	0.182	0.353	0.066	2.143	27.925
0.423	3.299	2.284	0.147	0.221	0.098	0.124	0.410	2.218	43.522
0.889	3.358	2.373	0.128	0.283	0.023	0.181	0.386	2.043	45.992
0.490	1.912	2.649	0.249	0.246	0.214	0.141	0.149	2.274	17.106

Table 4
Short-run equilibria for the three-region model

Parameter settings						Equilibrium point		
μ	σ	τ	λ_1	λ_2	λ_3	w_1	w_2	w_3
0.277	1.230	2.841	0.203	0.367	0.428	1.053	0.995	0.978
0.803	2.781	2.289	0.357	0.316	0.325	0.932	1.053	1.022
0.907	1.492	2.611	0.458	0.095	0.445	1.021	0.814	1.017
0.719	4.679	2.178	0.353	0.403	0.242	0.942	0.825	1.374
0.898	1.303	2.135	0.257	0.393	0.348	0.934	1.038	1.005
0.060	3.524	2.380	0.357	0.316	0.325	0.931	1.054	1.022
0.776	4.383	2.509	0.357	0.316	0.325	0.931	1.054	1.022
0.348	3.115	2.737	0.353	0.403	0.242	0.942	0.825	1.373
0.798	1.976	2.433	0.458	0.095	0.445	0.883	2.035	0.897
0.803	3.575	2.55	0.458	0.095	0.445	0.728	3.470	0.748
0.871	1.600	2.402	0.357	0.316	0.325	1.000	0.999	0.999

Table 5
Short-run equilibria for the four-region model

Parameter settings							Equilibrium point			
μ	σ	τ	λ_1	λ_2	λ_3	λ_4	w_1	w_2	w_3	w_4
0.984	4.886	2.678	0.211	0.261	0.272	0.254	1.184	0.955	0.917	0.981
0.757	2.152	2.570	0.155	0.161	0.311	0.372	1.379	1.389	0.867	0.783
0.325	2.487	2.572	0.211	0.261	0.272	0.254	1.175	0.958	0.919	0.983
0.903	2.493	2.688	0.155	0.161	0.311	0.372	1.442	1.442	0.847	0.751
0.931	2.323	2.785	0.087	0.322	0.185	0.403	1.600	0.954	1.132	0.844
0.066	1.294	2.712	0.211	0.261	0.272	0.254	1.033	0.996	0.975	1.002
0.610	4.691	2.576	0.155	0.161	0.311	0.372	1.612	1.550	0.802	0.671
0.682	2.844	2.597	0.155	0.161	0.311	0.372	1.581	1.535	0.809	0.685
0.720	2.305	2.168	0.103	0.337	0.141	0.418	1.508	0.929	1.311	0.826
0.769	3.451	2.027	0.155	0.161	0.311	0.372	1.563	1.527	0.813	0.693

gested by the lower bound on the number of function evaluations (Eq. (14)) that has been suggested by Boult and Sikorski [34].

Table 6
Short-run equilibria for the 5-region model

Parameter settings								Equilibrium point				
μ	σ	τ	λ_1	λ_2	λ_3	λ_4	λ_5	w_1	w_2	w_3	w_4	w_5
0.839	4.458	2.778	0.154	0.163	0.231	0.214	0.236	1.296	1.224	0.864	0.931	0.847
0.965	3.398	2.112	0.282	0.145	0.176	0.148	0.246	0.837	1.148	1.154	1.167	0.887
0.980	3.962	2.426	0.282	0.145	0.176	0.148	0.246	0.750	1.289	1.148	1.288	0.834

6. Conclusions

The new economic geography literature provides a general equilibrium framework that explains the emergence of economic agglomerations. The existence and uniqueness of short-run equilibria of this model has been shown for the case of two regions. In this paper we applied criteria from the theory of fixed points and that of topological degree to investigate the existence and the computational complexity of computing short-run equilibria of a model of a spatial economy consisting of three to five regions. The criteria employed make use of the Lipschitz constant, or alternatively the modulus of continuity, and the infinity norm along the boundary of the domain, of the function. The proposed approach employs the differential evolution algorithm to obtain an estimate of these quantities as their approximation involves the minimization of non-differentiable objective functions. The obtained experimental results for a number of different parameter settings of the model suggest that the function is neither contractive, nor nonexpanding. The complexity of computing ε-residual approximations to fixed points of Lipschitz functions with constant $L > 1$, with respect to the infinity norm, is exponential in the worst case. For all the parameter settings tested, the existence criteria from topological degree theory state the function can have zeros in the domain under consideration. However, the estimated Lipschitz constants and infinity norms along the boundary indicate that the computation of the topological degree for this type of problems is computationally very demanding. The application of the differential evolution algorithm to obtain short-run equilibria suggests that such points exist. For each parameter setting a number of executions of the algorithm were performed to investigate whether more than one equilibria can be located. This was not the case for all the considered parameter settings, suggesting that the equilibrium might be unique for a spatial economy with more than two regions.

References

[1] P. Krugman, Increasing returns and economic geography, The Journal of Political Economy 99 (3) (1991) 483–499.
[2] M. Fujita, P. Krugman, A. Venables, The Spatial Economy. Cities, Regions and International Trade, MIT Press, 1999.
[3] P. Mossay, The structure of new economic geography models, in: L. Yee (Ed.), Contemporary Issues in Urban and Regional Economic, Nova Publishers, 2005.
[4] P. Mossay, The core-periphery model: A note on the existence and uniqueness of short-run equilibrium, Journal of Urban Economics 59 (3) (2006) 389–393.
[5] F. Stenger, Computing the topological degree of a mapping in $\mathbb{R}^n$, Numerische Mathematik 25 (1975) 23–38.
[6] B.J. Hoenders, C.H. Slump, On the calculation of the exact number of zeros of a set of equations, Computing 30 (1983) 137–147.
[7] E. Picard, Traité d'analyse, third ed., Gauthier-Villars, Paris, 1922 (Chapter 4.7).
[8] J.M. Ortega, W.C. Rheinbolt, Iterative Solution of on Linear Equations in Several Variables, Academic Press, New York, 1970.
[9] B. Mourrain, M.N. Vrahatis, J.C. Yakoubsohn, On the complexity of isolating real roots and computing with certainty the topological degree, Journal of Complexity 18 (2) (2002) 612–640.
[10] K. Sikorski, Optimal Solution of Nonlinear Equations, Oxford Press, New York, 2001.
[11] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae 3 (1922) 133–181.
[12] A. Nemirovsky, On the complexity of Krylov's information when solving linear operator equations, Journal of Complexity 7 (1991) 121–130.
[13] A. Nemirovsky, D.B. Yudin, Problem Complexity and Method Efficiency in Optimization, Wiley, New York, 1983.
[14] G. Allgower, K. Georg, Numerical Continuation Methods, Springer-Verlag, New York, Berlin, 1990.
[15] B.C. Eaves, Homotopies for computation of fixed points, Mathematical Programming 3 (1972) 1–22.
[16] B.C. Eaves, R. Saigal, Homotopies for computation of fixed points on unbounded regions, Mathematical Programming 3 (1972) 225–237.
[17] H. Scarf, The approximation of fixed points of a continuous mapping, SIAM Journal on Applied Mathematics 15 (1967) 1328–1343.

[18] M.D. Hirsch, C. Papadimitriou, S. Vavasis, Exponential lower bounds for finding Brouwer fixed points, Journal of Complexity 5 (1989) 379–416.
[19] G.D. Birkhoff, Dynamical systems with two degrees of freedom, Transactions of the American Mathematical Society 18 (1917) 199–300.
[20] T. Bountis, R.H.G. Helleman, On the stability of periodic orbits of two-dimensional mappings, Journal of Mathematical Physics 22 (1981) 1867–1877.
[21] J.M. Greene, A method for determining a stochastic transition, Journal of Mathematical Physics 20 (1979) 1183–1201.
[22] J. Howland, R. Vaillancourt, Attractive cycles in the iteration of meromorphic functions, Numerische Mathematik 46 (1985) 323–337.
[23] C. Polymilis, G. Servizi, Ch. Skokos, G. Turchetti, M.N. Vrahatis, Topological degree theory and local analysis of area preserving maps, Chaos 13 (2003) 94–104.
[24] M.N. Vrahatis, An efficient method for locating and computing periodic orbits of nonlinear mappings, Journal of Computational Physics 119 (1995) 105–119.
[25] M.N. Vrahatis, T.C. Bountis, M. Kollmann, Periodic orbits and invariant surfaces of 4-D nonlinear mappings, International Journal of Bifurcation and Chaos 6 (1996) 1425–1437.
[26] M.N. Vrahatis, H. Isliker, T.C. Bountis, Structure and breakdown of invariant tori in a 4-D mapping model of accelerator dynamics, International Journal of Bifurcation and Chaos 7 (1997) 2707–2722.
[27] M.N. Vrahatis, A.E. Perdiou, V.S. Kalantonis, E.A. Perdios, K. Papadakis, R. Prosmiti, S.C. Farantos, Application of the characteristic bisection method for locating and computing periodic orbits in molecular systems, Computer Physics Communications 138 (2001) 53–68.
[28] Z. Huang, L.G. Khachiyan, K. Sikorski, Approximating fixed points of weakly contracting mappings, Journal of Complexity 15 (1999) 200–213.
[29] K. Sikorski, C.W. Tsay, H. Woźniakowski, An ellipsoid algorithm for the computation of fixed points, Journal of Complexity 9 (1993) 181–200.
[30] S.P. Tarasov, L.G. Khachiyan, I.I. Erlikh, The method of inscribed ellipsoids, Soviet Mathematics Doklady 37 (1988) 226–230.
[31] S. Shellman, K. Sikorski, A recursive algorithm for the infinity-norm fixed point problem, Journal of Complexity 19 (6) (2003) 799–834.
[32] R. Storn, K. Price, Differential evolution-a simple and efficient adaptive scheme for global optimization over continuous spaces, Journal of Global Optimization 11 (1997) 341–359.
[33] K.V. Price, R.M. Storn, J.A. Lampinen, Differential Evolution: A Practical Approach to Global Optimization, Springer, Berlin/Heidelberg, Germany, 2005.
[34] T. Boult, K. Sikorski, Complexity of computing topological degree of Lipschitz functions in n dimensions, Journal of Complexity 2 (1986) 44–59.
[35] K.E. Parsopoulos, M.N. Vrahatis, Investigating the existence of function roots using particle swarm optimization, in: Proceedings of the IEEE Congress on Evolutionary Computation, 2003, pp. 1448–1455.
[36] R.B. Kearfott, An efficient degree-computation method for a generalized method of bisection, Numerische Mathematik 32 (1979) 109–127.
[37] R. Horst, H. Tuy, Global Optimization – Deterministic Approaches, Springer, New York, 1996.