Astron. Astrophys. 300, 579–590 (1995)

ASTRONOMY
AND
ASTROPHYSICS

A numerical study of the influence of the Poynting–Robertson effect on the equilibrium points of the photogravitational restricted three-body problem

II. Out of plane case

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Received 27 July 1994 / Accepted 26 January 1995

Abstract. The photogravitational circular restricted three-body model including the Poynting-Robertson effect is employed to describe the motion of a particle in the vicinity of two massive radiating bodies. The equilibrium points lying out of the orbital plane of the primaries are studied numerically. The exact number of these points is determined by means of the topological degree theory. Subsequently, a modified bisection method is used to compute the positions of the equilibrium. Finally their stability is studied.

Key words: stars: binaries: general – celestial mechanics – methods: numerical

1. Introduction

It is known (Poynting 1903) that small particles approaching luminous celestial bodies are comparably affected by gravitational and light radiation forces.

In space science, the influence of the radiation is connected with the motion and the formation of concentrations of interplanetary and interstellar dust or grains in planetary and binary star systems as well as with the perturbations observed on artificial satellite orbits (Kozai 1961; McCracken & Alexander 1968; Ferraz-Mello 1972; Milani et al 1987; Vokrouhlický 1993, 1994).

The relativistic expression of the total radiation force on a particle has been given by Robertson (1937). He has also stated that a justifiable approximation for this force, immediately rephrasable in classical mechanics nomenclature, can be obtained by considering only linear terms in the ratio of the velocity of the particle over that of light, as follows : If $\mathbf{R}$ is the position vector of the particle P with respect to a radiating source S, $\mathbf{v}$ the corresponding velocity vector and c the velocity of light, then the radiation force on P, to first order in $\mathbf{v}/c$, is

$$\mathbf{F} = F_p \frac{\mathbf{R}}{R} - F_p \frac{\mathbf{v}\cdot\mathbf{R}}{c}\frac{\mathbf{R}}{R} - F_p \frac{\mathbf{v}}{c},$$

where F_p denotes the measure of the radiation pressure force. The first component in the above equation expresses the radiation pressure, while the remaining two forces consist the so called Poynting-Robertson effect.

The above form of the radiation force has been widely used in the bibliography for the study of the motion of a particle with negligible mass in the vicinity of two main bodies of the type sun-planet or the type of a binary star. The so-called photogravitational restricted three-body model is usually used to formulate the problem.

A simplified version of this model, involving central forces only, was first introduced by Radzievskii (1950) and subsequently used by several scientists in studying the existence and stability of equilibrium points (Radzievskii 1953; Perezhogin 1976; Kunitsyn & Perezhogin 1978; Simmons et al 1985; Kunitsyn & Tureshbaev 1985; Ragos & Zagouras 1988a; Kumar & Choudhry 1987; Goździewski et al 1991; Choudhry 1988; Niedzielska 1994), regions of allowed motion (Schuerman 1972; Ragos & Zagouras 1988b) as well as periodic motion (Ragos & Zagouras 1988a; Ragos & Zagouras 1991; Ragos et al 1991).

Extending Radzievskii's model so as to include the two relativistic terms of Robertson's linear approximation Chernikov (1970) and Schuerman (1980) examined the equilibrium points assuming that only one of the primaries radiates. Recently Ragos & Zafiropoulos (1995) have considered the case that both main bodies are luminous. They have studied the libration points lying on the orbital plane of these bodies (coplanar case). In the present paper we deal with the equilibrium points which exist out of that plane. First we derive the system of equations satisfied by these points. Then, using the topological degree theory, we determine the exact number of the solutions of this system.

Finally, the location and stability of the equilibrium positions are studied numerically.

2. Equations of motion

In a rotating, barycentric and dimensionless coordinate system $Oxyz$ (well known from the classical restricted three-body problem), the equations of motion of the particle P are (Ragos and Zafiropoulos (1995); see also Appendix A) :

$$\ddot{x} - 2\dot{y} = x - \frac{Q_1}{r_1^3}(x+\mu) - \frac{Q_2}{r_2^3}(x+\mu-1) - \frac{W_1}{r_1^2}\left[\frac{x+\mu}{r_1^2}\left((x+\mu)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{x}-y\right] - \frac{W_2}{r_2^2}\left[\frac{x+\mu-1}{r_2^2}\left((x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{x}-y\right] \tag{1a}$$

$$\ddot{y} + 2\dot{x} = \left[1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right]y - \frac{W_1}{r_1^2}\left[\frac{y}{r_1^2}\left((x+\mu)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{y}+x+\mu\right] - \frac{W_2}{r_2^2}\left[\frac{y}{r_2^2}\left((x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{y}+x+\mu-1\right] \tag{1b}$$

$$\ddot{z} = \left[-\frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right]z - \frac{W_1}{r_1^2}\left[\frac{z}{r_1^2}\left((x+\mu)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{z}\right] - \frac{W_2}{r_2^2}\left[\frac{z}{r_2^2}\left((x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}\right)+\dot{z}\right], \tag{1c}$$

where

$$Q_1 = q_1(1-\mu), \quad Q_2 = q_2\mu,$$

$$W_1 = \frac{(1-q_1)(1-\mu)}{c_d}, \quad W_2 = \frac{(1-q_2)\mu}{c_d},$$

and

$$r_1 = \sqrt{(x+\mu)^2+y^2+z^2}, \; r_2 = \sqrt{(x+\mu-1)^2+y^2+z^2}.$$

We denote by μ the mass ratio parameter ($0 < \mu \leq 1/2$); by q_i ($q_i \leq 1$) the parameters determining the relationship between the measures of the gravitational and radiation pressure force of the primary $S_i, i = 1, 2$; by c_d the dimensionless velocity of light, i.e. the size of light velocity using as unit the sum of the measures of the velocities of the two main bodies.

3. Existence and location of the out of plane equilibrium points

The position of the out of plane libration points can be determined by setting

$$\ddot{x} = \ddot{y} = \ddot{z} = 0, \qquad \dot{x} = \dot{y} = \dot{z} = 0,$$

and considering $z \neq 0$ in Equations (1). We thus obtain that

$$x - \frac{Q_1}{r_1^3}(x+\mu) - \frac{Q_2}{r_2^3}(x+\mu-1) + \left[\frac{W_1}{r_1^2} + \frac{W_2}{r_2^2}\right]y = 0, \tag{2a}$$

$$\left[1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right]y - \frac{W_1}{r_1^2}(x+\mu) - \frac{W_2}{r_2^2}(x+\mu-1) = 0, \tag{2b}$$

$$\frac{Q_1}{r_1^3} + \frac{Q_2}{r_2^3} = 0. \tag{2c}$$

From Equation (2c) it can be seen that, for the existence of any real solution (x, y, z) of System (2), one of the following conditions is necessary to hold:

$$q_1 q_2 < 0 \qquad \text{or} \qquad q_1 = q_2 = 0. \tag{3}$$

The second condition means that the gravitational attractions balance the corresponding radiation pressure forces, so, the particle moves under the influence of the Poynting-Robertson effect forces. This case will not be considered in this paper.

The first condition means that the radiation pressure force of just one of the primaries exceeds its gravitational attraction. From Equation (2c) and the definition of r_1, r_2 we have that

$$x = \frac{1}{2} - \mu + \frac{1}{2}\left[1 - \left(\frac{Q_2}{Q_1}\right)^{\frac{2}{3}}\right] r_1^2. \tag{4}$$

Equations (2b) and (4) yield

$$y = \frac{1}{2}\left\{W_1 - W_2 - \left[W_1\left(\frac{Q_2}{Q_1}\right)^{\frac{2}{3}} - W_2\left(\frac{Q_1}{Q_2}\right)^{\frac{2}{3}}\right]\right\} + \frac{1}{2}\left[W_1 - W_2\left(\frac{Q_1}{Q_2}\right)^{\frac{2}{3}}\right]\frac{1}{r_1^2}. \tag{5}$$

Finally, combining Equations (2c),(4),(5) with (2a), we have that the distance of any existing equilibrium point from S_1 must satisfy the equation

$$P(r_1) = a_6 r_1^6 + a_4 r_1^4 + a_2 r_1^2 + a_1 r_1 + a_0 = 0, \tag{6}$$

where

$$a_6 = \frac{1}{2}\left[1 - \left(\frac{Q_2}{Q_1}\right)^{\frac{2}{3}}\right], \quad a_4 = \frac{1}{2} - \mu,$$

$$a_2 = \frac{1}{2}\left[W_1 + W_2\left(\frac{Q_1}{Q_2}\right)^{\frac{2}{3}}\right]\left\{W_1 - W_2 - \left[W_1\left(\frac{Q_2}{Q_1}\right)^{\frac{2}{3}} - W_2\left(\frac{Q_1}{Q_2}\right)^{\frac{2}{3}}\right]\right\},$$

$$a_1 = -Q_1, \qquad a_0 = \frac{1}{2}\left[W_1^2 - W_2^2\left(\frac{Q_1}{Q_2}\right)^{\frac{4}{3}}\right].$$

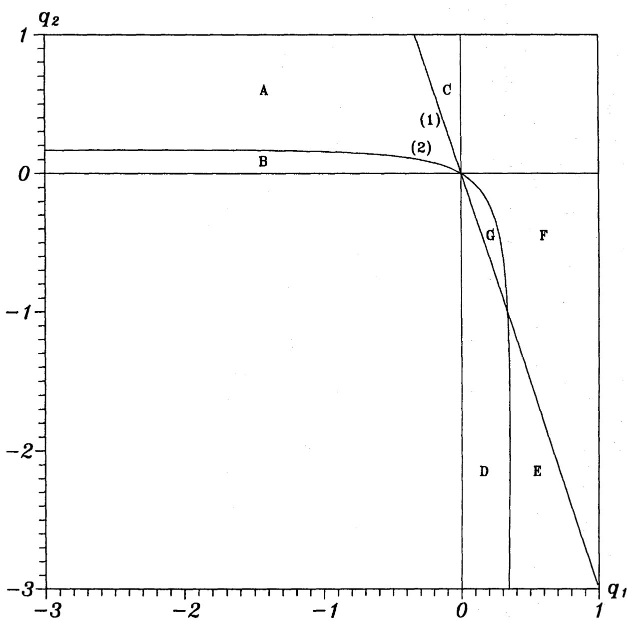

Fig. 1. Binary star Kruger 60. The existence and the number of roots of Equation (6): theoretical study

Any positive root of (6) together with the corresponding value of r_2 obtained from (2c) must fulfill the triangle condition

$$r_1 + r_2 > 1. \tag{7}$$

Then the x, y, z may be derived from Equations (4), (5) and

$$z = \pm\sqrt{r_1^2 - (x+\mu)^2 - y^2}. \tag{8}$$

Equations (8) indicate that the 'out of plane' equilibrium points exist in pairs. The members of each pair are symmetrical with respect to the Oxy plane.

The existence and the number of libration points can be partly determined by studying the behaviour of Equation (6).

The coefficient a_4 of the polynomial $P(r_1)$ is always non-negative and a_1 has a sign opposite to that of q_1. For given values of the mass ratio μ and the light velocity c_d the rest of the coefficients depend on q_1, q_2. We observe that a_6 and a_2 have the same sign. For the typical case of $\mu = 0.25$ and $c_d = 48002.33$ (binary star Kruger 60) we present the curves on which a_6 and a_0 are equal to zero (Figure 1: curves (1) and (2) respectively). These curves determine seven regions on the parts of the plane where $q_1 q_2 < 0$:

A. In this region $a_6 > 0, a_4 > 0, a_2 > 0, a_0 > 0$. Since there is no change of sign in the sequence of a_i, no roots of (6) exist. Hence there are no equilibrium points for $(q_1, q_2) \in A$.
B. Here $a_6 > 0, a_4 > 0, a_2 > 0, a_1 > 0, a_0 < 0$. Since there is just one change of sign in the sequence of a_i and $P(0)P(+\infty) < 0$, there is exactly one positive root of (6). Hence there exists at most (due to Condition (7)) one pair of equilibrium points for $(q_1, q_2) \in B$.
C. In this region $a_6 < 0, a_4 > 0, a_2 < 0, a_1 > 0, a_0 > 0$. There are three changes of sign in the sequence of a_i and $P(0)P(+\infty) < 0$. So there is one or three roots of (6).
D. In this part of the plane $a_6 < 0, a_4 > 0, a_2 < 0, a_1 < 0, a_0 > 0$. The situation here is the same as in C. So, Equation (6) has one or three roots.
E. Here $a_6 > 0, a_4 > 0, a_2 > 0, a_1 < 0, a_0 > 0$. Since there are two changes of sign in the sequence of a_i and $P(0)P(+\infty) > 0$, none or two positive roots of (6) exist. Hence there are either none or at most two pairs of equilibrium points for $(q_1, q_2) \in E$.
F. In this region $a_6 > 0, a_4 > 0, a_2 > 0, a_1 < 0, a_0 < 0$. This case is similar to B. So at most one pair of libration points exists.
G. In this part of the plane $a_6 > 0, a_4 > 0, a_2 > 0, a_1 < 0, a_0 > 0$. As in region E, none or at most two pairs of equilibrium points exist.

Although the above theoretical study gives information about $P(r_1)$, it is not enough to determine the exact number of the roots of Equation (6). For this reason we use a method based on the topological degree and the Kronecker-Picard integral. According to this method the number of roots of the polynomial $P(r_1)$ within an interval (α, β) is :

$$\mathcal{N}^r = -\frac{1}{\pi}\left[\gamma\int_\alpha^\beta \frac{P(r_1)P''(r_1) - P'^2(r_1)}{P^2(r_1) + \gamma^2 P'^2(r_1)} dr_1 \right. \\ \left. - \arctan\left(\frac{\gamma P'(\beta)}{P(\beta)}\right) + \arctan\left(\frac{\gamma P'(\alpha)}{P(\alpha)}\right)\right], \tag{9}$$

where γ is an arbitrary small positive constant. The derivation of Formula (9) is explained in Appendix B.

Once we calculate the exact number $\mathcal{N}^r$ of the solutions ρ_j of Equation (6) within an interval (α, β), we can locate them by subdividing this interval and finding $\mathcal{N}^r$ subintervals (α_j, β_j) for which the well known Bolzano's criterion $P(\alpha_j)P(\beta_j) \leq 0$ is fulfilled. In order to compute the root ρ_j in the interval (α_j, β_j), we utilize a modified version of the bisection method. A short description of this method is presented in Appendix C.

4. Stability of the out of plane equilibrium points

In order to study the stability of the equilibrium, we first linearize Equations (1) around its position (x_0, y_0, z_0). Denoting by ξ, η and ζ the corresponding perturbations along the axes Ox, Oy, Oz, we obtain the following system:

$$\begin{aligned} &(D^2 + A_1 D + A_2)\xi + (A_3 D + A_4)\eta + (A_5 D + A_6)\zeta = 0, \\ &(B_1 D + B_2)\xi + (D^2 + B_3 D + B_4)\eta + (B_5 D + B_6)\zeta = 0, \\ &(C_1 D + C_2)\xi + (C_3 D + C_4)\eta + (D^2 + C_5 D + C_6)\zeta = 0, \end{aligned} \tag{10}$$

where D stands for differentation with respect to time. The coefficients A_i, B_i, C_i, $i = 1, \ldots, 6$ are given in Appendix D. The characteristic equation of the matrix in System (10) is a polynomial of sixth degree :

$$\lambda^6 + c_5\lambda^5 + c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0. \tag{11}$$

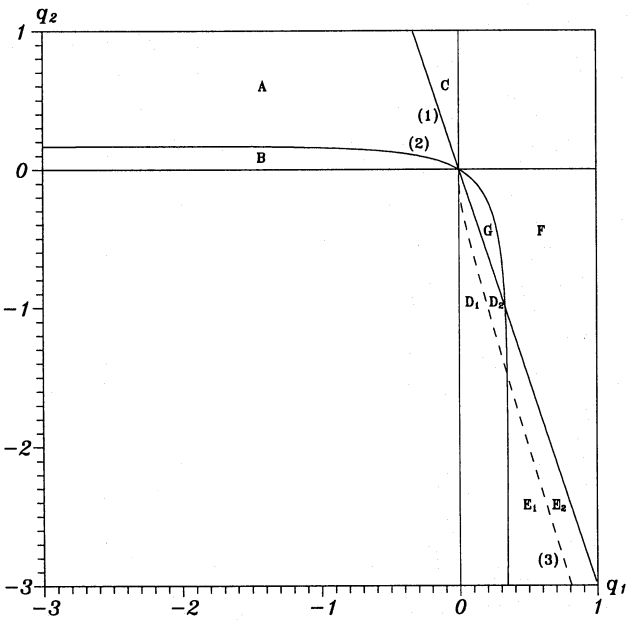

Fig. 2. Binary star Kruger 60. The existence and the number of roots of Equation (6): numerical study

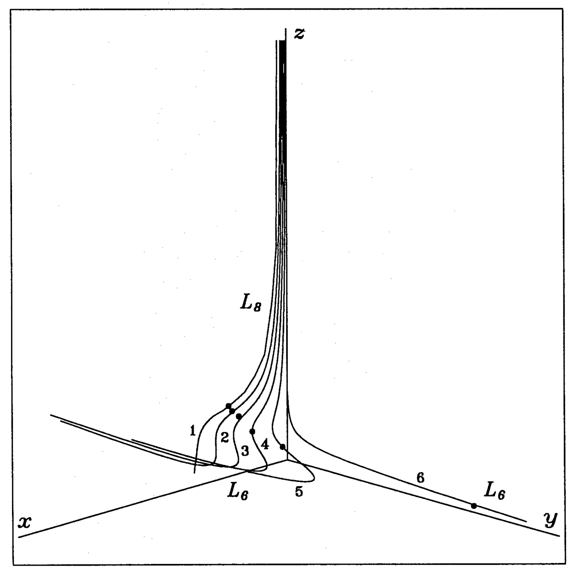

Fig. 3. Binary star Kruger 60. A three-dimensional representation of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

The coefficients c_i, $i = 0, \ldots, 5$ are also given in Appendix D. The obtained eigenvalues determine the stability or instability of the respective point.

5. Numerical results

The topological degree method for the computation of the exact number of the roots of Equation (6) has verified the analytical results regarding the regions A, B and F. For the rest of them it gave the followings:

In C there is only one root of Equation (6), while for $(q_1, q_2) \in$G it has two roots.

Region D is divided into two subregions, D_1 and D_2, giving one and three roots respectively (Figure 2). Region E is also divided into two subregions. For $(q_1, q_2) \in E_1$ there are no roots of Equation (6), while if these parameters belong to E_2 there exist two roots. The boundary between D_1, D_2 and E_1, E_2 is the curve (3) which includes the points (q_1, q_2) where the derivative $P'(r_1)$ has a double root.

The above mentioned regions depend on the value of mass ratio. As μ increases, B and C get larger, G and F get smaller, while D_1 and E_1 increase at the expense of D_2 and E_2 respectively.

However, in certain cases, due to Condition (7), not all the roots of Equation (6) give pairs of equilibrium points.

In this paper we present results regarding two binary stars, namely Kruger 60 ($\mu = 0.25$, $c_d = 48002.33$) and BD-8°4352 ($\mu = 0.33333$, $c_d = 12561.56$). We have calculated the out of plane libration points for q_1 equal to 1, 0.8, 0.6, 0.4, 0.2, 0.01, −0.01, −0.05, −0.15, −0.25, −0.35, −0.45 and $-3.5 \leq q_2 \leq 1$. The stability of these points is also studied.

5.1. Existence and location

Firstly, we refer to the binary star Kruger 60.

For all considered positive values of q_1 there are intervals of q_2 of the form $[a, b]$, $a, b < 0$ for which there exist equilibrium points. For q_1 =1, 0.8, 0.6, 0.4, 0.2 and q_2 within a certain subinterval $[c, b]$, $c > a$, there is only one pair of equilibrium points. We name L_6 the point with positive $z-$coordinate and L_7 the other one. If $q_2 \in (a, c]$ a second pair of such points appears, L_8 and L_9, which have relatively smaller x and y coordinates and larger value of z. When $q_2 = a$, L_6 and L_7 coincide with L_8 and L_9 respectively. When $q_1 = 0.01$ the picture is the same except that L_8 and L_9 appear for all $q_2 \in [a, b]$ while L_6 and L_7 exist only for $q_1 \in (a, c]$ (Figures 3,4,5).

For these values of q_1, the $x-$coordinate of any existing equilibrium point is positive.

When $q_1 = -0.01, -0.05, -0.15, -0.25$ there is a range of q_2 of the form $[a, 1]$, $a > 0$, for which one pair of equilibrium points exists, L_6 and L_7 (Figures 6,7,8). Due to Condition (7) there are no libration points for those (q_1, q_2) within region B. Consequently, for every $q_2 > 0$ the values of q_1 for which equlibrium points exist are bounded by curve (1) (Figure 2). Thus, obviously, for $q_1 = -0.35, -0.45$ there are no such points.

For the considered negative values of q_1 the $x-$coordinate of any existing equilibrium point is negative while the $y-$coordinate is positive.

We observe that as q_1 decreases the respective ranges of q_2, namely $[a, b]$ when $q_1 > 0$ and $[a, 1]$ when $q_1 < 0$, get smaller. Figures 3 and 6 depict the equilibrium positions in three dimensions for the above mentioned values of q_1 and q_2.

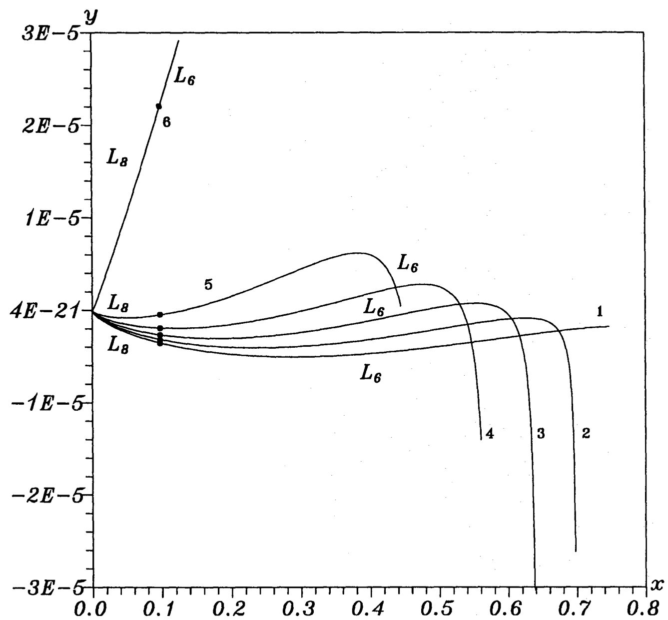

Fig. 4a. Binary star Kruger 60. The variation of $y-$ versus $x-$coordinate of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

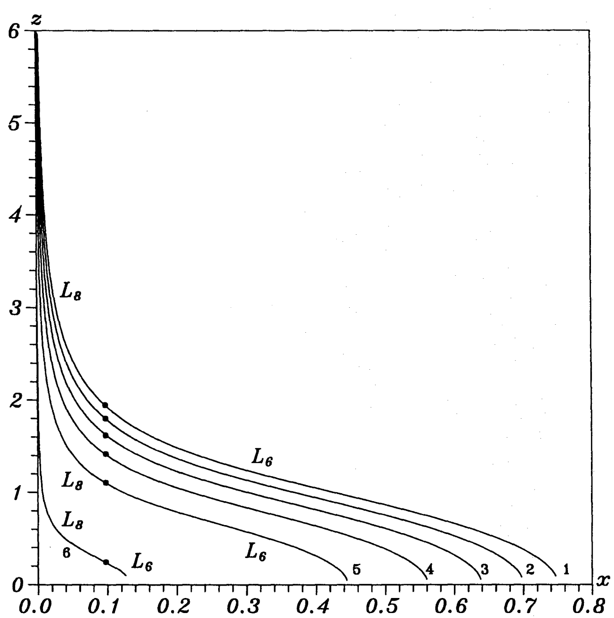

Fig. 4b. Binary star Kruger 60. The variation of $z-$ versus $x-$coordinate of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

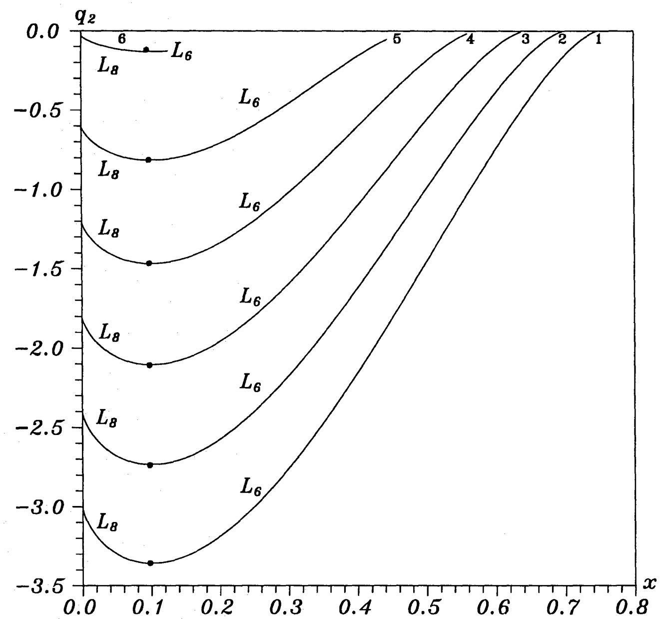

Fig. 5. Binary star Kruger 60. The variation of $x-$coordinate of equilibrium points L_6, L_8 (separated by •) versus q_2 for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01

Comparing the results obtained for Kruger 60 with the ones for BD-8°4352 (Figures 9–14) we see that the whole situation appears similar. The observed differences are the following:

For positive q_1, the intervals of q_2 for which equilibrium points exist increase with the mass ratio μ. The opposite is true when q_1 is negative.

Besides when q_1 is positive, as μ increases, the range of variation of the $x-$coordinate of the libration points is more narrow while this of the $y-$coordinate is wider. The opposite happens when q_1 is negative.

5.2. Stability

For the investigation of the stability of the out of plane libration points it is necessary to study the solutions of Equation (11). This has been carried out numerically ranging the parameters q_1 and q_2 from −3.5 to 1 with step 0.001.

For the above mentioned binaries it has been found that none of the equilibrium positions is stable. It is known that in the classical photogravitational restricted three-body problem the out of plane equilibrium points are conditionally stable. Apparently, the presence of the Poynting-Robertson effect ruins this stability.

6. Concluding remarks

The study of the out of plane equilibrium points presented in this article leads to the following conclusions:

1. The equilibrium points appear in pairs. The members of these pairs are symmetrical with respect to the orbital plane of the primaries.

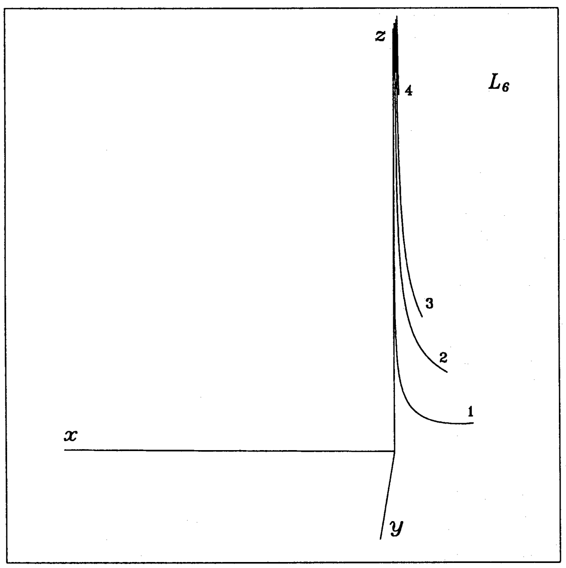

Fig. 6. Binary star Kruger 60. A three-dimensional representation of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$ and $0 < q_2 \leq 1$

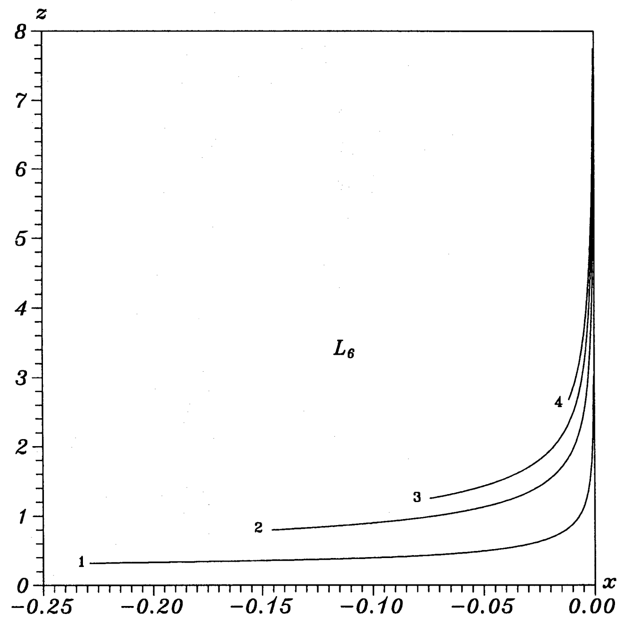

Fig. 7b. Binary star Kruger 60. The variation of $z-$ versus $x-$coordinate of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$ and $0 < q_2 \leq 1$

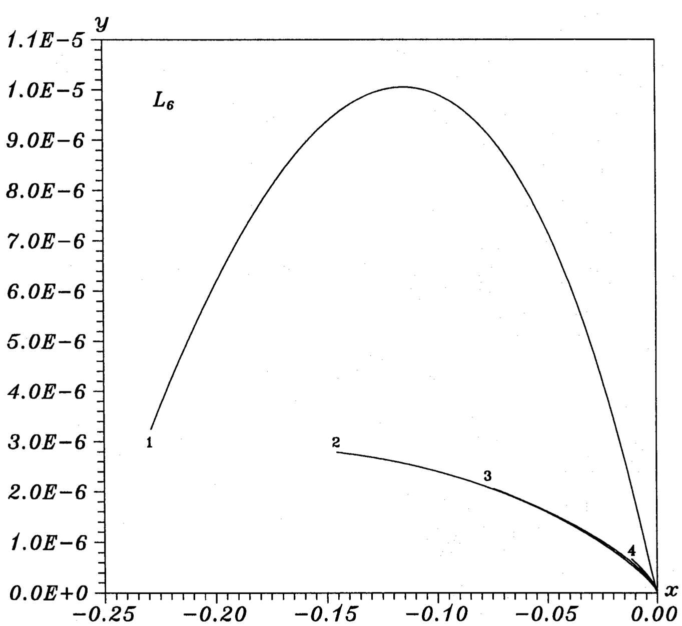

Fig. 7a. Binary star Kruger 60. The variation of $y-$ versus $x-$coordinate of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$ and $0 < q_2 \leq 1$

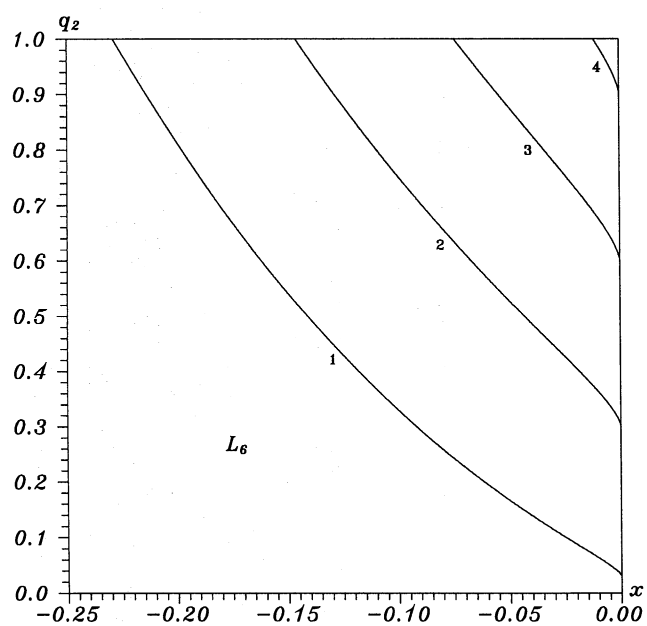

Fig. 8. Binary star Kruger 60. The variation of $x-$coordinate of equilibrium point L_6 versus q_2 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$

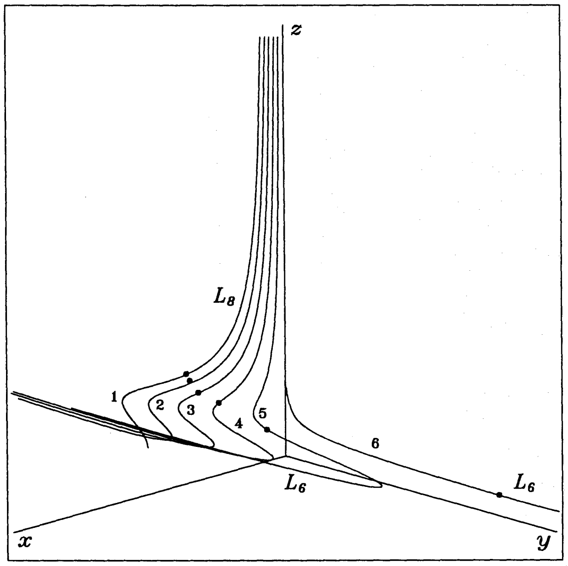

Fig. 9. Binary star BD-8°4352. A three-dimensional representation of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

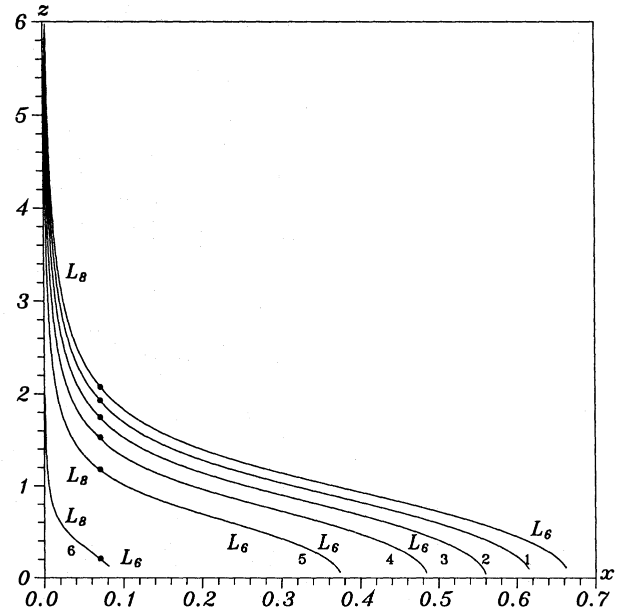

Fig. 10b. Binary star BD-8°4352. The variation of $z-$ versus $x-$coordinate of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

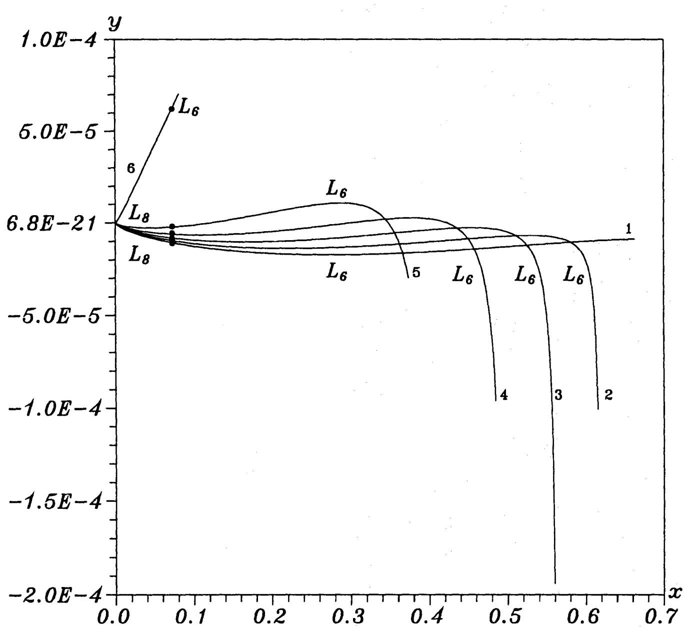

Fig. 10a. Binary star BD-8°4352. The variation of $y-$ versus $x-$coordinate of equilibrium points L_6, L_8 (separated by •) for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01 and $-3.5 \leq q_2 < 0$

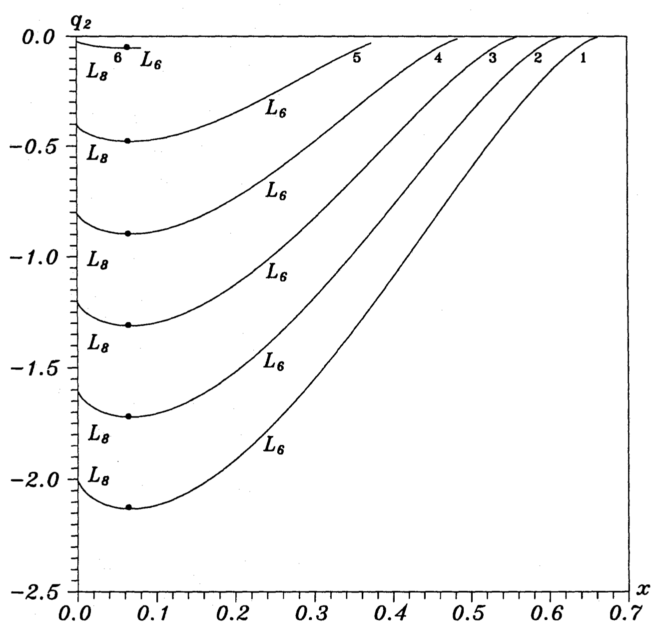

Fig. 11. Binary star BD-8°4352. The variation of $x-$coordinate of equilibrium points L_6, L_8 (separated by •) versus q_2 for (1) q_1=1, (2) q_1=0.8, (3) q_1=0.6, (4) q_1=0.4, (5) q_1=0.2, (6) q_1=0.01

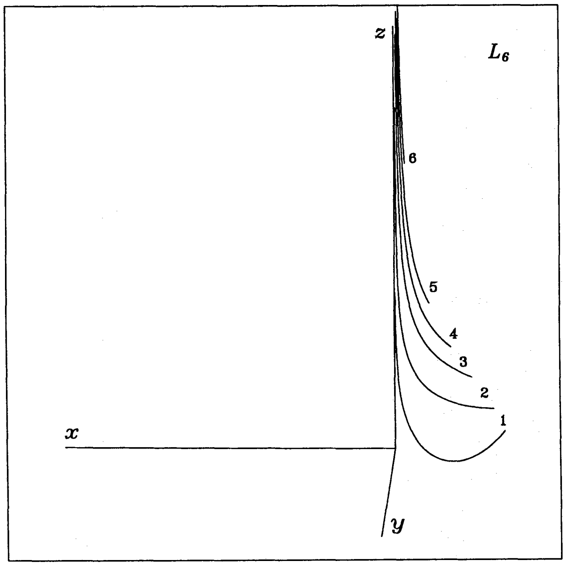

Fig. 12. Binary star BD-8°4352. A three-dimensional representation of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$, (5) $q_1 = -0.35$, (6) $q_1 = -0.45$ and $0 < q_2 \leq 1$

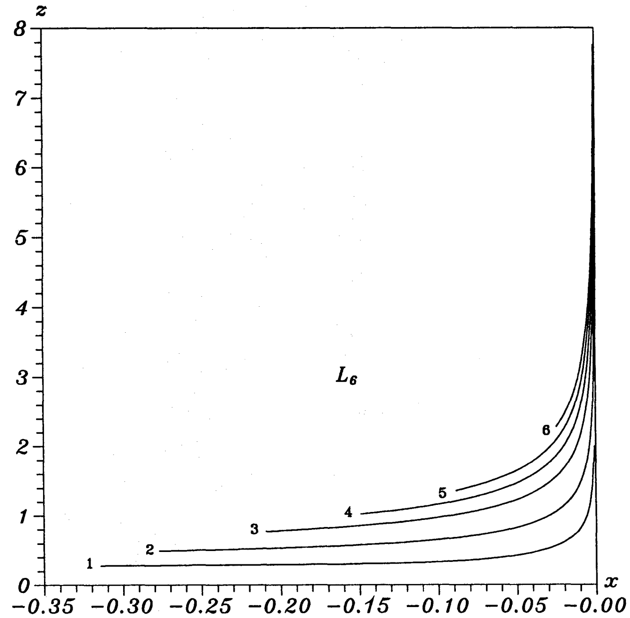

Fig. 13b. Binary star BD-8°4352. The variation of $z-$ versus $x-$coordinate of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$, (5) $q_1 = -0.35$, (6) $q_1 = -0.45$ and $0 < q_2 \leq 1$

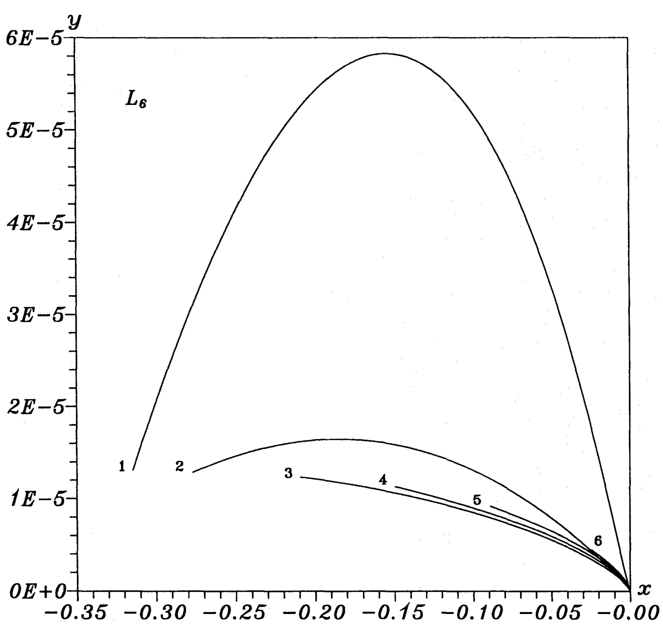

Fig. 13a. Binary star BD-8°4352. The variation of $y-$ versus $x-$coordinate of equilibrium point L_6 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$, (5) $q_1 = -0.35$, (6) $q_1 = -0.45$ and $0 < q_2 \leq 1$

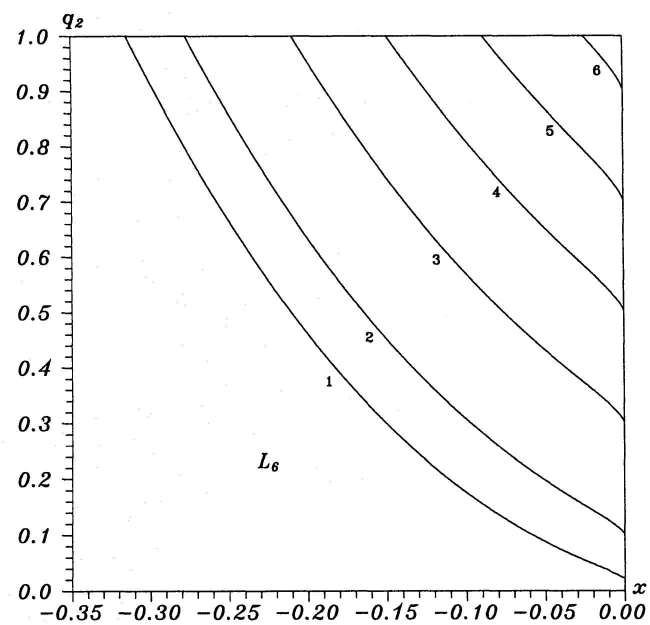

Fig. 14. Binary star BD-8°4352. The variation of $x-$coordinate of equilibrium point L_6 versus q_2 for (1) $q_1 = -0.01$, (2) $q_1 = -0.05$, (3) $q_1 = -0.15$, (4) $q_1 = -0.25$, (5) $q_1 = -0.35$, (6) $q_1 = -0.45$

2. When the radiation pressure force of the more massive primary does not exceed the gravitational one there are at most two pairs of such points. In the opposite case there is at most one pair.
3. The influence of the Poynting-Robertson effect keeps, in general, the equilibrium points slightly off the $Oxz-$plane.
4. Both pairs of libration points are unstable.

7. Discussion

In this article we study the number, the location and the stability of the "out of plane" equilibrium points for particles moving in the vicinity of two massive bodies which emit light radiation. As it is known, such points do not appear if only gravitational forces are considered. The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc.

The existence of radiation perceptibly influences the characteristics of motion for particles with appropriate masses and cross-sections. Comparing previous contributions with the present one it is obvious that the results about the "out of plane" equilibium positions depend also widely on the order of approximation of the forces introduced:

Consider that the radiation pressure force is enough to describe the radiation influnce. Then, provided that this influnce of just one of the main bodies dominates the gravitalional one, there is one or two pairs of equilibrium points, symmetrical with respect to the orbital plane and lying on a plane perpenticular to the above mentioned one and containing the primaries. For particles of certain physical properties these positions are linearly stable for a specific range of the mass ratio of the primaries.

In our model, which, additionally, takes into account the aberrational deceleration due to the Poynting-Robertson effect, these points deviate from the plane on which they used to lie in the previous consideration. Moreover, for the cases studied, these positions are unstable and, as indicated, not significantly affected by the masses and orbital characteristics of the primaries. So, the out of plane equilibrium points in binary star systems should not be expected to gather particles susceptible to radiation influence. We note that similar results were obtained for the "coplanar" libration positions in a previous communication of ours (Ragos & Zafiropoulos 1995). Consequently, small as the two additional forces are, their influence is remarkable.

Appendix A: derivation of Eq. (1)

In an inertial frame $OXYZ$ whose origin is the mass centre of two radiating primaries S_1, S_2 with coordinates $(s_1, 0, 0)$, $(s_2, 0, 0)$, the total acceleration of the particle P will be :

$$\ddot{\mathbf{R}} = -G\frac{q_1 m_1}{R_1^3}\mathbf{R}_1 - G\frac{q_2 m_2}{R_2^3}\mathbf{R}_2 -(1-q_1)G\frac{m_1}{R_1^2}\left[\frac{\dot{\mathbf{R}}_1\cdot\mathbf{R}_1}{cR_1}\frac{\mathbf{R}_1}{R_1}+\frac{\dot{\mathbf{R}}_1}{c}\right] -(1-q_2)G\frac{m_2}{R_2^2}\left[\frac{\dot{\mathbf{R}}_2\cdot\mathbf{R}_2}{cR_2}\frac{\mathbf{R}_2}{R_2}+\frac{\dot{\mathbf{R}}_2}{c}\right], \quad \text{(A1)}$$

where m_i is the mass of S_i ($m_1 \geq m_2$), $q_i = 1 - F_{pi}/F_{gi}$ expresses the relation between the radiation pressure and the gravitational force due to S_i and $R_i = \sqrt{(X - s_i)^2 + Y^2 + Z^2}$ ($i = 1, 2$). Obviously $q_i \leq 1$.

Considering now that the primaries rotate in circular orbits around O under the influence of their mutual gravitational attraction, we use a new coordinate system $Oxyz$ which follows them. The bodies are positioned on the $Ox-$axis, the $Oxy-$plane coincides with OXY and the $Oz-$axis with OZ. Furthermore, we transform the units of mass, distance, time so that this system will be dimensionless : the distance between the two primaries as well as the sum of their masses are assumed to be equal to unity and the unit of time is such that their angular velocity is 1. The Gaussian constant G will also be 1. Since, in this frame, the sum of the velocities of the primaries is equal to one, we measure the velocity of light using as unit this quantity. If μ denotes the mass of S_2, the mass of S_1 will be $1 - \mu$ and their positions on Ox $1 - \mu$ and $-\mu$, respectively.

In the above described reference system, the acceleration acting on P can be expressed, in terms of the Coriolis Theorem, as follows :

$$\ddot{\mathbf{r}} = -\frac{1-\mu}{r_1^3}\mathbf{r}_1 - \frac{\mu}{r_2^3}\mathbf{r}_2 +\frac{(1-\mu)(1-q_1)}{r_1^2}\left[\left(1-\frac{(\dot{\mathbf{r}}_1+\mathbf{k}\times\mathbf{r}_1)\cdot\mathbf{r}_1}{c_d r_1}\right)\frac{\mathbf{r}_1}{r_1}+\frac{\dot{\mathbf{r}}_1+\mathbf{k}\times\mathbf{r}_1}{c_d}\right] +\frac{\mu(1-q_2)}{r_2^2}\left[\left(1-\frac{(\dot{\mathbf{r}}_2+\mathbf{k}\times\mathbf{r}_2)\cdot\mathbf{r}_2}{c_d r_2}\right)\frac{\mathbf{r}_2}{r_2}+\frac{\dot{\mathbf{r}}_2+\mathbf{k}\times\mathbf{r}_2}{c_d}\right] +2\mathbf{k}\times\dot{\mathbf{r}}+\mathbf{k}\times(\mathbf{k}\times\mathbf{r}), \quad \text{(A2)}$$

where $\mathbf{k}$ is the unit vector along the $Oz-$axis, $\mathbf{r}$ is the position vector of P and $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$, $r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}$.

Appendix B: derivation of Eq. (9)

Definition. Suppose that a function $\mathbf{F}_n = (f_1, f_2, \dots, f_n) : \mathscr{D}_n \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is defined and twice continuously differentiable in an open and bounded domain $\mathscr{D}_n$ with boundary $b(\mathscr{D}_n)$. Suppose also that the roots of the equation

$$\mathbf{F}_n(\mathbf{x}) = \Theta_n, \qquad \Theta_n = (0, 0, \dots, 0), \quad \text{(B1)}$$

are located in $\mathscr{D}_n$ and are simple i.e. the Jacobian determinant $J_{\mathbf{F}_n}$ of $\mathbf{F}_n$ at these roots is non-zero. Then the *topological degree of* $\mathbf{F}_n$ *at* Θ_n *relative to* $\mathscr{D}_n$ is defined by (Alexandroff & Hopf 1935; Ortega & Rheinbolt 1970)

$$\deg[\mathbf{F}_n, \mathscr{D}_n, \Theta_n] = \sum_{\mathbf{x}\in\mathbf{F}_n^{-1}(\Theta_n)} \operatorname{sgn} J_{\mathbf{F}_n}(\mathbf{x}), \quad \text{(B2)}$$

where by sgn we denote the well known sign function.

The above definition means that the topological degree is equal to the number of the simple roots of (1) which give positive $J_{\mathbf{F}_n}$ minus the number of those which give negative $J_{\mathbf{F}_n}$.

The topological degree can be represented by the Kronecker integral (Kronecker 1895):

$$\deg\,[\mathbf{F}_n, \mathscr{D}_n, \Theta_n] = = \frac{1}{\Omega_n} \int\limits_{b(\mathscr{D}_n)} \frac{\sum_{i=1}^{n} D_i dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n}{(f_1^2 + f_2^2 + \ldots + f_n^2)^{n/2}}, \tag{B3}$$

where

$$D_i = (-1)^{n(i-1)} \left| \mathbf{F}_n \ \frac{\partial \mathbf{F}_n}{\partial x_1} \cdots \frac{\partial \mathbf{F}_n}{\partial x_{i-1}} \ \frac{\partial \mathbf{F}_n}{\partial x_{i+1}} \cdots \frac{\partial \mathbf{F}_n}{\partial x_n} \right| \tag{B4}$$

and Ω_n is the surface of a unit sphere in $\mathbb{R}^n$ i.e. $\Omega_n = 2\pi^{n/2}/\Gamma\left(\frac{n}{2}\right)$.

Picard (1892; 1922) considered, instead of $\mathbf{F}_n$, the following function:

$$\mathbf{F}_{n+1} = (f_1, \ldots, f_n, f_{n+1}) : \mathscr{D}_{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \tag{B5}$$

where $f_{n+1} = x_{n+1} J_{\mathbf{F}_n}$ and $\mathscr{D}_{n+1}$ is the direct product of the domain $\mathscr{D}_n$ with an arbitrary interval of the $x_{n+1}-$axis containing the point $x_{n+1} = 0$ (see also Hoenders & Slump 1983). Then the system :

$$\begin{aligned} &f_i(x_1, x_2, \ldots, x_n) = 0, \qquad i = 1, 2, \ldots, n, \\ &f_{n+1} = x_{n+1} J_{\mathbf{F}_n}(x_1, x_2, \ldots, x_n) = 0, \end{aligned} \tag{B6}$$

has the same simple roots with Equation (B1), provided that $x_{n+1} = 0$. Besides the Jacobian of (B6) is equal to $(J_{\mathbf{F}_n})^2$ which is always positive. Thus we conclude that the total number of the roots of (B1) can be given by $\mathscr{N} = \deg[\mathbf{F}_{n+1}, \mathscr{D}_{n+1}, \Theta_{n+1}]$.

We consider now the problem of calculating the total number of simple roots of

$$f_1(x_1) = 0, \tag{B7}$$

where $f_1 : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable in this interval; α and β are arbitrarily chosen so that $f_1(\alpha) f_1(\beta) \neq 0$. According to Picard's extension we define the function $\mathbf{F}_2 = (f_1, f_2) : \mathscr{P} \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the corresponding system

$$\begin{aligned} &f_1(x_1, x_2) = f_1(x_1) = 0, \\ &f_2(x_1, x_2) = x_2 f_1'(x_1) = 0, \end{aligned} \tag{B8}$$

where $\mathscr{P}$ is the rectangular parallelepiped in the $(x_1, x_2)-$plane $[\alpha, \beta] \times [-\gamma, \gamma]$, with γ an arbitrary small positive constant. Since the roots are simple, which means $f_1'(x) \neq 0$ for $x \in f_1^{-1}(0)$, it is easily seen that Equation (B7) and System (B8) have the same roots in $\mathscr{P}$. Also, $J_{\mathbf{F}_2} = (f_1')^2$. So the total number of simple zeros $\mathscr{N}^r$ of the function f_1 in (α, β) can be given by

$$\mathscr{N}^r = \deg[\mathbf{F}_2, \mathscr{P}, \Theta_2].$$

Now, Equation (B3), for $n = 2$, yields

$$\deg[\mathbf{F}_2, \mathscr{P}, \Theta_2] = \frac{1}{2\pi} \oint_{b(\mathscr{P})} \frac{D_1 dx_2 + D_2 dx_1}{f_1^2 + f_2^2},$$

where D_1 and D_2 are defined as in Relation (B4) and the integration is performed along $b(\mathscr{P})$. By replacing D_1, D_2 in the above integral we get

$$\deg[\mathbf{F}_2, \mathscr{P}, \Theta_2] = \frac{1}{2\pi} \oint_{b(\mathscr{P})} \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}. \tag{B9}$$

Moreover, since $d \arctan\left(\frac{f_2}{f_1}\right) = \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$, we obtain the following :

$$\begin{aligned} \deg[\mathbf{F}_2, \mathscr{P}, \Theta_2] &= \frac{1}{2\pi} \oint\limits_{b(\mathscr{P})} \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2} \\ &= \frac{1}{2\pi} \oint\limits_{b(\mathscr{P})} d \arctan\left(\frac{f_2}{f_1}\right). \end{aligned} \tag{B10}$$

Performing the integration in Equation (B10) we finally get

$$\mathscr{N}^r = -\frac{1}{\pi}\left[\gamma \int_\alpha^\beta \frac{f_1(x) f_1''(x) - f_1'^2(x)}{f_1^2(x) + \gamma^2 f_1'^2(x)} dx - \arctan\left(\frac{\gamma f_1'(\beta)}{f_1(\beta)}\right) + \arctan\left(\frac{\gamma f_1'(\alpha)}{f_1(\alpha)}\right)\right]. \tag{B11}$$

It has been explicitly shown by Picard that $\mathscr{N}^r$ is independent of the value of γ and need not be computed within high accuracy, because it is known a priori that the number of roots has to be an integer.

The Kronecker-Picard integral can also be applied for the determination of the total number of multiple roots (Davidoglou 1901; Tzitzéica 1901; Hoenders & Slump 1992). However the multiple roots of f_1 can be found by the method described above by considering the roots of its derivatives.

Appendix C: description of the bisection method

The modified bisection method used to solve Equation (6) is described by the scheme

$$\rho_j^{i+1} = \rho_j^i + \operatorname{sgn} P(\rho_j^0) \operatorname{sgn} P(\rho_j^i) h_j / 2^{i+1}, \ i = 0, 1, \ldots, \tag{C1}$$

with $\rho_j^0 = \alpha_j$ and $h_j = \beta_j - \alpha_j$ (Vrahatis, 1988). The above sequence converges with certainty if the Bolzano's criterion holds. If the number of roots in an interval is even, Algorithm (C1) converges to the root ρ_j provided that, for some ρ_j^i, $i = 1, 2, \ldots$, the following holds:

$$\operatorname{sgn} P(\rho_j^0) \operatorname{sgn} P(\rho_j^i) = -1.$$

The number of iterations ν, required to obtain an approximation ρ_j^* of the root such that $|\rho_j - \rho_j^*| \leq \varepsilon$, for some $\varepsilon \in (0, 1)$, is given by

$$\nu = \lceil \log_2(h_j \varepsilon^{-1}) \rceil,$$

where $\lceil \cdot \rceil$ denotes the smallest integer not less than the real number quoted.

The bisection method always converges within the given interval and it is a global convergence method. Moreover it is "optimal", i.e. it possesses asymptotically the best possible rate of convergence (Sikorski 1982). Also, we know in advance the number of iterations required in order to attain an approximate root to a predetermined accuracy. Finally, Scheme (C1) requires only the algebraic signs of the function values to be computed, thus it can be applied to problems with imprecise function values.

Appendix D: coefficients of Sys. (10) – Eq. (11)

The coefficients of System (10) are :

$$A_1 = \frac{W_1}{r_{10}^4}(x_0+\mu)^2 + \frac{W_2}{r_{20}^4}(x_0+\mu-1)^2 + \frac{W_1}{r_{10}^2} + \frac{W_2}{r_{20}^2},$$

$$A_2 = -1 - 3\left[\frac{Q_1}{r_{10}^5}(x_0+\mu)^2 + \frac{Q_2}{r_{20}^5}(x_0+\mu-1)^2\right] +2\left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] y_0,$$

$$A_3 = -2 + \left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] y_0,$$

$$A_4 = -3\left[\frac{Q_1}{r_{10}^5}(x_0+\mu) + \frac{Q_2}{r_{20}^5}(x_0+\mu-1)\right] y_0 -\frac{W_1}{r_{10}^2} - \frac{W_2}{r_{20}^2} + 2\left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] y_0^2,$$

$$A_5 = \left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] z_0,$$

$$A_6 = -3\left[\frac{Q_1}{r_{10}^5}(x_0+\mu) + \frac{Q_2}{r_{20}^5}(x_0+\mu-1)\right] z_0 +2\left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] y_0 z_0,$$

$$B_1 = 2 + \left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] y_0,$$

$$B_2 = -3\left[\frac{Q_1}{r_{10}^5}(x_0+\mu) + \frac{Q_2}{r_{20}^5}(x_0+\mu-1)\right] y_0 +\frac{W_1}{r_{10}^2} + \frac{W_2}{r_{20}^2} - 2\left[\frac{W_1}{r_{10}^4}(x_0+\mu)^2 + \frac{W_2}{r_{20}^4}(x_0+\mu-1)^2\right],$$

$$B_3 = \left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] y_0^2 + \frac{W_1}{r_{10}^2} + \frac{W_2}{r_{20}^2},$$

$$B_4 = -1 - 3\left[\frac{Q_1}{r_{10}^5} + \frac{Q_2}{r_{20}^5}\right] y_0^2 -2\left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] y_0,$$

$$B_5 = \left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] y_0 z_0,$$

$$B_6 = -3\left[\frac{Q_1}{r_{10}^5} + \frac{Q_2}{r_{20}^5}\right] y_0 z_0 -2\left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] z_0,$$

$$C_1 = \left[\frac{W_1}{r_{10}^4}(x_0+\mu) + \frac{W_2}{r_{20}^4}(x_0+\mu-1)\right] z_0,$$

$$C_2 = -3\left[\frac{Q_1}{r_{10}^5}(x_0+\mu) + \frac{Q_2}{r_{20}^5}(x_0+\mu-1)\right] z_0,$$

$$C_3 = \left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] y_0 z_0,$$

$$C_4 = -3\left[\frac{Q_1}{r_{10}^5} + \frac{Q_2}{r_{20}^5}\right] y_0 z_0,$$

$$C_5 = \left[\frac{W_1}{r_{10}^4} + \frac{W_2}{r_{20}^4}\right] z_0^2 + \frac{W_1}{r_{10}^2} + \frac{W_2}{r_{20}^2},$$

$$C_6 = -3\left[\frac{Q_1}{r_{10}^5} + \frac{Q_2}{r_{20}^5}\right] z_0^2.$$

where :

$$r_{10}=\sqrt{(x_0+\mu)^2+y_0^2+z_0^2},\; r_{20}=\sqrt{(x_0+\mu-1)^2+y_0^2+z_0^2}.$$

The coefficients of Equation (11) are :

$$c_5 = A_1 + B_3 + C_5,$$

$$c_4 = A_2 - A_3B_1 + A_1B_3 + B_4 - A_5C_1 - B_5C_3 + A_1C_5 +B_3C_5 + C_6,$$

$$c_3 = -A_4B_1 - A_3B_2 + A_2B_3 + A_1B_4 - A_6C_1 -A_5B_3C_1 + A_3B_5C_1 - A_5C_2 +A_5B_1C_3 - A_1B_5C_3 - B_6C_3 - B_5C_4 + A_2C_5 -A_3B_1C_5 + A_1B_3C_5 + B_4C_5 + A_1C_6 + B_3C_6,$$

$$c_2 = -A_4B_2 + A_2B_4 - A_6B_3C_1 - A_5B_4C_1 + A_4B_5C_1 + A_3B_6C_1 - A_6C_2 - A_5B_3C_2 + A_3B_5C_2 + A_6B_1C_3 + A_5B_2C_3 - A_2B_5C_3 - A_1B_6C_3 + A_5B_1C_4 - B_6C_4 - A_1B_5C_4 - A_4B_1C_5 - A_3B_2C_5 + A_2B_3C_5 + A_1B_4C_5 + A_2C_6 - A_3B_1C_6 + A_1B_3C_6 + B_4C_6,$$

$$c_1 = -A_6B_4C_1 + A_4B_6C_1 - A_6B_3C_2 - A_5B_4C_2 + A_4B_5C_2 + A_3B_6C_2 + A_6B_2C_3 - A_2B_6C_3 + A_6B_1C_4 + A_5B_2C_4 - A_2B_5C_4 - A_1B_6C_4 - A_4B_2C_5 + A_2B_4C_5 - A_4B_1C_6 - A_3B_2C_6 + A_2B_3C_6 + A_1B_4C_6,$$

$$c_0 = -A_6B_4C_2 + A_4B_6C_2 + A_6B_2C_4 - A_2B_6C_4 - A_4B_2C_6 + A_2B_4C_6.$$

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