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Generalization of the Bolzano theorem for simplices

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ABSTRACT

For an n-dimensional simplex in $\mathbb{R}^n$, a generalization of the Bolzano theorem is given. A proof based on the classical Knaster–Kuratowski–Mazurkiewicz covering lemma is obtained.

1. Introduction

The very important and pioneering Bolzano theorem (also called intermediate value theorem) states that [2,11]:

Bolzano's theorem: *If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if it holds that $f(a)f(b) < 0$, then there is at least one $x \in (a,b)$ such that $f(x) = 0$.*

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Its first proofs, given independently by Bolzano in 1817 [2] and Cauchy in 1821 [5], were crucial in the procedure of arithmetization of analysis (which was a research program in the foundations of mathematics during the second half of the 19th century).

A straightforward generalization of Bolzano's theorem to continuous mappings of an n-cube (parallelotope) into $\mathbb{R}^n$ was proposed without proof by Poincaré in 1883 and 1884 in his work on the three-body problem [23,24]. The Poincaré theorem was soon forgotten and it has come to be known as "Miranda's theorem" [20] which partly explains the nomenclature "Poincaré–Miranda theorem" [18] as well as "Bolzano–Poincaré–Miranda theorem" [33].

The Bolzano–Poincaré–Miranda theorem states that [20,32,34]:

Bolzano–Poincaré–Miranda theorem: *Suppose that* $P = \{x \in \mathbb{R}^n \mid |x_i| < L, \text{ for } 1 \leqslant i \leqslant n\}$ *and let the mapping* $F_n = (f_1, f_2, \dots, f_n)\colon P \to \mathbb{R}^n$ *be continuous on the closure of* P *such that* $F_n(x) \neq \theta^n = (0, 0, \dots, 0)$ *for* x *on the boundary of* P*, and*

(a) $f_i(x_1, x_2, \dots, x_{i-1}, -L, x_{i+1}, \dots, x_n) \geqslant 0, \ \text{for } 1 \leqslant i \leqslant n,$

(b) $f_i(x_1, x_2, \dots, x_{i-1}, +L, x_{i+1}, \dots, x_n) \leqslant 0, \ \text{for } 1 \leqslant i \leqslant n.$

Then, there is at least one $x \in P$ *such that* $F_n(x) = \theta^n$.

Miranda in 1940 [20] showed that this theorem is equivalent to the Brouwer fixed point theorem [4]. It is worth noting that the Bolzano–Poincaré–Miranda theorem is closely related to important theorems in analysis and topology as well as it is an invaluable tool for verified solutions of numerical problems by means of interval arithmetic [14,21,22,27]. For a short proof as well as for a generalization of the Bolzano–Poincaré–Miranda theorem using topological degree theory see [34]. Also, for a generalization of this theorem to an infinite-dimensional setting see [26,27]. Furthermore, for interesting relations between the theorems of Bolzano–Poincaré–Miranda, Borsuk [3], Kantorovich[1] [13] and Smale[2] [30] with respect to the existence of a solution of a system of nonlinear equations, we refer the interested reader to [1].

In the paper at hand, a generalization of the Bolzano theorem for simplices is proposed. The obtained proof is based on the Knaster–Kuratowski–Mazurkiewicz lemma (KKM lemma for short, often called the KKM covering theorem or KKM covering principle). The KKM covering principle simply states that [15]:

KKM covering principle: *A family of* $(n + 1)$ *closed subsets covering an n-dimensional simplex and satisfying the Knaster–Kuratowski–Mazurkiewicz boundary conditions have a non-empty intersection.*

For a mathematical formulation of this statement the reader is referred to Lemma 2.1 below. The KKM lemma constitutes the basis for the proof of many theorems (including the famous Brouwer fixed point theorem). It is worth noting that three pioneering classical results, namely, the Brouwer fixed point theorem [4], the Sperner lemma [31], and the KKM lemma [15] are mutually equivalent in the sense that each one can be deduced from another.

The KKM lemma has numerous applications in various fields of pure and applied mathematics. In particular, among others, in the field of mathematical economics, the very important and pioneering extension of the KKM lemma due to Shapley[3] [28], customarily called the Knaster–Kuratowski–Mazurkiewicz–Shapley theorem (KKMS theorem for short), constitutes the basis for the proof of many theorems on the existence of solutions in game theory and in the general equilibrium theory of economic analysis. The Shapley's KKMS covering principle simply states that [25,28]:

[1] Nobel Laureate in Economic Sciences in 1975.
[2] Fields Medalist in 1966.
[3] Nobel Laureate in Economic Sciences in 2012.

KKMS covering principle: *Under Shapley's boundary conditions on a family of closed subsets of the unit simplex, the intersection of the subsets that correspond to a "balanced" family is non-empty.*

A mathematical formulation of Shapley's extension of the KKM Lemma can be given as follows [8,10,28,29]:

Knaster–Kuratowski–Mazurkiewicz–Shapley theorem: *Suppose that $\mathcal{N}$ is the family of non-empty subsets of the set $N = \{1, 2, \ldots, n\}$. Let $e^j \in \mathbb{R}^n$ be the unit vector with components $e^j_i = 0$ for $i \in N \backslash \{j\}$ and $e^j_j = 1$. For each $S \in \mathcal{N}$ consider: (a) its normalized characteristic vector $\chi_S = (1/\mathrm{card}\{S\}) \sum_{j \in S} e^j$, where $\mathrm{card}\{S\}$ denotes the number of elements in the set S, and (b) the convex hull $\mathrm{co}\{e^j \mid j \in S\}$ denoted by Δ^S. Let $C_S,\ S \in \mathcal{N}$ be a family of closed subsets of Δ^N, indexed by the members of $\mathcal{N}$, which satisfy the following Shapley's boundary conditions:*

$$\forall\, T \in \mathcal{N}, \quad \Delta^T \subset \bigcup_{S \subset T} C_S.$$

Then, there exists a family $\mathcal{B}$ of members of $\mathcal{N}$ such that $\chi_N \in \mathrm{co}\{\chi_S \mid S \in \mathcal{B}\}$ (called balanced family) for which $\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset$.

It is worth noting that, when $C_S = \emptyset$ for all S for which $\mathrm{card}\{S\} \geqslant 2$, the KKMS theorem reduces to the KKM lemma [10]. Due to its importance, this remarkable theorem has been extended and proved multiple times by several different researchers over decades (i.e., see [6,7,9,12,16,17,19,25]).

2. Generalization of the Bolzano theorem

Notation 2.1. We denote by ϑA the boundary of a set A, by $\mathrm{cl}A$ its closure, by $\mathrm{int}A$ its interior and by $\mathrm{card}\{A\}$ its cardinality (i.e., the number of elements in the set A). Furthermore, we shall frequently use the index sets $N^n = \{0, 1, \ldots, n\}$, $N^n_{\neg 0} = \{1, 2, \ldots, n\}$ and $N^n_{\neg i} = \{0, 1, \ldots, i-1, i+1, \ldots, n\}$. Also, for a given set $I = \{i, j, \ldots, \ell\} \subset N^n$ we denote by $N^n_{\neg I}$ or equivalently by $N^n_{\neg ij \cdots \ell}$ the set $\{k \in N^n \mid k \notin I\}$.

Definition 2.1. For any positive integer n, and for any set of points $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ in some linear space which are affinely independent (i.e., the vectors $\{\upsilon^1 - \upsilon^0, \upsilon^2 - \upsilon^0, \ldots, \upsilon^n - \upsilon^0\}$ are linearly independent) the convex hull $\mathrm{co}\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is called the *n-simplex with vertices* $\upsilon^0, \upsilon^1, \ldots, \upsilon^n$. For each subset of $(m+1)$ elements $\{\omega^0, \omega^1, \ldots, \omega^m\} \subset \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$, the m-simplex $[\omega^0, \omega^1, \ldots, \omega^m]$ is called an *m-face* of $[\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. In particular, 0-faces are vertices and 1-faces are edges. The m-faces are also called *facets* of the n-simplex.

Notation 2.2. We denote the n-simplex with set of vertices $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ by $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. Also, we denote the $(n-1)$-simplex that determines the i-th $(n-1)$-face of σ^n by $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$. Furthermore, for a given index set $I = \{i, j, \ldots, \ell\} \subset N^n$ with cardinality $\mathrm{card}\{I\} = \kappa$, we denote by $\sigma^n_{\neg I}$ or equivalently by $\sigma^n_{\neg ij \cdots \ell}$ the $(n-\kappa)$-face of σ^n with vertices $\upsilon^m, m \in N^n_{\neg I}$.

Lemma 2.1 *(KKM lemma). Let $C_i,\ i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ and

(b) if for each $\emptyset \neq I \subset N^n$ it holds that $\bigcap_{i \in I} \sigma^n_{\neg i} \subset \bigcup_{j \in N^n_{\neg I}} C_j$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Definition 2.2. A covering satisfying the conditions in the KKM Lemma 2.1 is called a *KKM covering*.

Observation 2.1. The KKM covering can be given as follows:

Each facet of σ^n is covered by the sets that correspond to the vertices spanning that facet.

Thus, the vertex υ^i is covered by the closed subset C_i, the edge $[\upsilon^i, \upsilon^j]$ is covered by $C_i \cup C_j$ while the facet $[\upsilon^i, \upsilon^j, \ldots, \upsilon^\ell]$ is covered by $C_i \cup C_j \cup \cdots \cup C_\ell$ for each index set $\{i, j, \ldots, \ell\} \subset N^n$.

Definition 2.3. Let ψ be a real number, and let us set

$$\operatorname{sgn} \psi = \begin{cases} -1, & \text{if} \quad \psi < 0, \\ \ \ 0, & \text{if} \quad \psi = 0, \\ \ \ 1, & \text{if} \quad \psi > 0. \end{cases}$$

Then, for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ the *sign* of a, denoted $\operatorname{sgn} a$, is defined as follows:

$$\operatorname{sgn} a = (\operatorname{sgn} a_1, \operatorname{sgn} a_2, \ldots, \operatorname{sgn} a_n).$$

Next, we give the proposed generalization of Bolzano's theorem for simplices.

Theorem 2.1 *(Generalization of the Bolzano theorem). Assume that $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is an n-simplex in $\mathbb{R}^n$. Let $F_n = (f_1, f_2, \ldots, f_n)\colon \sigma^n \to \mathbb{R}^n$ be a continuous function such that $f_j(\upsilon^i) \neq 0$, $\forall\, j \in N^n_{\neg 0}$, $i \in N^n$ and $\theta^n = (0, 0, \ldots, 0) \notin F_n(\vartheta\sigma^n)$. Assume that the vertices υ^i, $i \in N^n$ are reordered such that the following hypotheses are fulfilled:*

$$(a) \quad \operatorname{sgn} f_j(\upsilon^j)\, \operatorname{sgn} f_j(x) = -1, \quad \forall\, x \in \sigma^n_{\neg j}, \quad j \in N^n_{\neg 0}, \tag{1}$$

$$(b) \quad \operatorname{sgn} F_n(\upsilon^0) \neq \operatorname{sgn} F_n(x), \quad \forall\, x \in \sigma^n_{\neg 0}. \tag{2}$$

Then, there is at least one $x \in \operatorname{int} \sigma^n$ such that $F_n(x) = \theta^n$.

Proof. Due to hypotheses (1) and (2) it is evident that the following holds:

$$\operatorname{sgn} F_n(\upsilon^i) \neq \operatorname{sgn} F_n(x), \quad \forall\, x \in \sigma^n_{\neg i}, \quad i \in N^n. \tag{3}$$

By virtue of conditions (1) it is obvious that for the vertex υ^0 the following relations are also fulfilled:

$$\operatorname{sgn} f_j(\upsilon^0) \operatorname{sgn} f_j(\upsilon^j) = -1, \quad \forall\, j \in N^n_{\neg 0}, \tag{4}$$

as well as it is evident that for the vertices υ^i, $i \in N^n$ it holds that:

$$\operatorname{sgn} F_n(\upsilon^i) \neq \operatorname{sgn} F_n(\upsilon^\ell), \quad \forall\, i, \ell \in N^n, \quad i \neq \ell. \tag{5}$$

Furthermore, it is obvious that for all $j \in N^n_{\neg 0}$ the j-th component $\operatorname{sgn} f_j(\upsilon^i)$ of $\operatorname{sgn} F_n(\upsilon^i) = (\operatorname{sgn} f_1(\upsilon^i), \operatorname{sgn} f_2(\upsilon^i), \ldots, \operatorname{sgn} f_n(\upsilon^i))$ is not the same for all the vertices υ^i, $i \in N^n$. Therefore, for the following sets S_{f_j} we have that:

$$S_{f_j} = \bigl\{x \in \operatorname{int} \sigma^n \;\big|\; f_j(x) = 0\bigr\} \neq \emptyset, \quad \forall\, j \in N^n_{\neg 0}. \tag{6}$$

For each one of the vertices υ^j, $j \in N^n_{\neg 0}$ we consider the corresponding closed set:

$$C_j = \operatorname{cl}\bigl\{x \in \sigma^n \;\big|\; \operatorname{sgn} f_j(x) = \operatorname{sgn} f_j(\upsilon^j)\bigr\}, \quad j \in N^n_{\neg 0}, \tag{7}$$

as well as we consider the following closed set:

$$C_0 = \mathrm{cl}\{x \in \sigma^n \mid \operatorname{sgn} F_n(x) = \operatorname{sgn} F_n(\upsilon^0)\}. \tag{8}$$

Since by hypothesis we have $f_j(\upsilon^i) \neq 0$, $\forall\, j \in N^n_{\neg 0}$, $i \in N^n$, it is obvious that the following holds:

$$\operatorname{int} C_i \neq \emptyset, \quad \forall\, i \in N^n. \tag{9}$$

Furthermore, it is evident that the following relation is valid:

$$\operatorname{int} C_0 \bigcap \left\{ \bigcup_{j \in N^n_{\neg 0}} \operatorname{int} C_j \right\} = \emptyset. \tag{10}$$

The above sets C_i, $i \in N^n$ are well defined according to the hypotheses of Lemma 2.1, since, the vertex υ^i, $i \in N^n$ is covered by the corresponding closed set C_i. That is $\upsilon^i \subset C_i$, $i \in N^n$. In particular, this corresponds to the case of Lemma 2.1 where $\mathrm{card}\{I\} = n$. This is so, because, if we consider that $I = N^n_{\neg i}$ for any $i \in N^n$, then we obtain $N^n_{\neg I} = \{i\}$ and consequently we have that:

$$\upsilon^i = \bigcap_{j \in N^n_{\neg i}} \sigma^n_{\neg j} \;\subset\; C_i, \quad i \in N^n. \tag{11}$$

Let us denote by x^* any $x \in \sigma^n$ such that $f_j(x^*) \neq 0$, $\forall\, j \in N^n_{\neg 0}$. It is obvious that for these points x^* the number of values that the function $\operatorname{sgn} F_n(x^*) = (\operatorname{sgn} f_1(x^*), \operatorname{sgn} f_2(x^*), \ldots, \operatorname{sgn} f_n(x^*))$ can obtain is 2^n. Let us consider that the function values of $\operatorname{sgn} F_n(x^*)$ form a set with cardinality $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\}$. Due to relation (8) for any $x^* \in \operatorname{int} C_0$ we obtain that $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 1$. On the other hand, due to relation (7), for any $x^* \in \operatorname{int} C_j$, $j \in N^n_{\neg 0}$, we have $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 2^{n-1}$. Therefore, for any $x^* \in \cup_{j \in N^n_{\neg 0}} \operatorname{int} C_j$ we obtain $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = \sum_{\ell=1}^{n} 2^{n-\ell}$ or, equivalently, $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 2^n - 1$. Thus, due to relations (9) and (10) we have that for any $x^* \in \cup_{i \in N^n} \operatorname{int} C_i$, it holds that $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 2^n$. Therefore, we conclude that the following is valid:

$$\sigma^n = \bigcup_{i \in N^n} C_i\,. \tag{12}$$

Based on the above approach, since for any $x^* \in \cup_{j \in N^n_{\neg 0}} \operatorname{int} C_j$ we obtain that $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 2^n - 1$ and since by hypothesis we have that $\theta^n \notin F_n(\vartheta\, \sigma^n)$, due to relation (2) it is easy to conclude that the 0-th face $\sigma^n_{\neg 0}$ of σ^n is covered by $\cup_{j \in N^n_{\neg 0}} C_j$. Furthermore, due to relations (1) and (7), for any $x^* \in \sigma^n_{\neg i}$, $i \in N^n_{\neg 0}$, we have that $\mathrm{card}\{\operatorname{sgn} F_n(x^*)\} = 2^{n-1}$. On the other hand, it is evident that for any $x^* \in \cup_{j \in N^n_{\neg i}} \operatorname{int} C_j$ the function $\operatorname{sgn} F_n(x^*)$ can obtain 2^{n-1} values, and in particular $\sum_{\ell=2}^{n} 2^{n-\ell} = 2^{n-1} - 1$ values in the case where $j \neq 0$ and one additional value in the case when $j = 0$. Therefore, we conclude that the following holds:

$$\sigma^n_{\neg i} \subset \bigcup_{j \in N^n_{\neg i}} C_j\,, \quad \forall\, i \in N^n. \tag{13}$$

Obviously, the above result corresponds to the case of Lemma 2.1 where $\mathrm{card}\{I\} = 1$.

Let us denote by $\sigma^n_{\neg ij}$ the j-th $(n-2)$-face of $\sigma^n_{\neg i}$. It is evident that $\sigma^n_{\neg ij}$ coincides with the i-th $(n-2)$-face of $\sigma^n_{\neg j}$ denoted by $\sigma^n_{\neg ji}$. Thus, it holds that $\sigma^n_{\neg ij} = \sigma^n_{\neg ji} = \sigma^n_{\neg i} \cap \sigma^n_{\neg j}$. Due to relations (3), (12) and (13), it is evident that $\sigma^n_{\neg ij} \subset \cup_{m \in N^n_{\neg ij}} C_m$. Therefore, we conclude that $\sigma^n_{\neg i} \cap \sigma^n_{\neg j} \subset \cup_{m \in N^n_{\neg ij}} C_m$. Using the same approach as that outlined above, we can easily obtain that for each index set $\{i, j, \ldots, \ell\} \subset N^n$ with cardinality κ, $1 < \kappa < n$, and for the corresponding $(n-\kappa)$-face $\sigma^n_{\neg ij\cdots\ell}$ of σ^n it holds that:

$$\sigma^n_{\neg ij\cdots\ell} = \bigcap_{m\in\{i,j,\ldots,\ell\}\subset N^n} \sigma^n_{\neg m}\,. \tag{14}$$

On the other hand we can also easily obtain that:

$$\sigma^n_{\neg ij\cdots\ell} \subset \bigcup_{m\in N^n_{\neg ij\cdots\ell}} C_m\,. \tag{15}$$

Thus, by taking into consideration relations (11), (13), (14) and (15) we conclude that for each $\emptyset \neq I \subset N^n$ and $N^n_{\neg I} = \{k \in N^n \mid k \notin I\}$ the following is valid:

$$\bigcap_{i\in I} \sigma^n_{\neg i} \subset \bigcup_{j\in N^n_{\neg I}} C_j\,. \tag{16}$$

By taking into consideration relations (12), (13) and (16), by virtue of Lemma 2.1 we obtain that $\cap_{i\in N^n} C_i \neq \emptyset$. Consequently, due to the continuity of F_n, it also holds that $\cap_{i\in N^n} \vartheta C_i \neq \emptyset$. Therefore, for the following solution set S_{F_n} it holds that:

$$S_{F_n} = \left\{x \in \operatorname{int}\sigma^n \;\middle|\; f_i(x) = 0,\ \forall\, i \in N^n_{\neg 0}\right\} \neq \emptyset.$$

Thus, the theorem is proved. □

Remark 2.1. For $n = 1$, the proposed Theorem 2.1 clearly reduces to the Bolzano theorem. For this reason, Theorem 2.1 was named "generalization of the Bolzano theorem for simplices".

3. Synopsis

A generalization of the Bolzano theorem for simplices is given. The obtained proof is stemmed from the very important and pioneering Knaster–Kuratowski–Mazurkiewicz covering principle.

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