Topology and its Applications 275 (2020) 107036

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Intermediate value theorem for simplices for simplicial approximation of fixed points and zeros

Michael N. Vrahatis

Department of Mathematics, University of Patras, GR-26110 Patras, Greece

ARTICLE INFO

Article history:
Received 31 January 2019
Received in revised form 24 April 2019
Accepted 6 June 2019
Available online 13 December 2019

MSC:
primary 55M20, 54H25
secondary 47H10, 65H10, 91A44, 91B50

Keywords:
Sperner covering
Intermediate value theorems
Bolzano theorem
Bolzano-Poincaré-Miranda theorem
Brouwer fixed point theorem
Knaster-Kuratowski-Mazurkiewicz lemma
Sperner lemma
Simplicial approximation
Equilibria
Mathematical economics

ABSTRACT

Two short proofs of a recently proposed intermediate value theorem for simplices are given. The obtained proofs are based on Sperner covering principles. Furthermore, this intermediate value theorem is applied for the localization and approximation of fixed points and zeros of continuous mappings using a simplicial subdivision of a simplex. Also, a theorem for the existence of a Sperner simplex (panchromatic simplex) in the considered simplicial subdivision is proved. In addition, an error estimate is presented.

1. Introduction

The pioneering Bolzano's theorem states the following [8,21]:

Theorem 1.1 *(Bolzano's theorem). If* $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function and if it holds that* $f(a)f(b) < 0$*, then there is at least one* $x \in (a,b)$ *such that* $f(x) = 0$.

This theorem is also called *intermediate value theorem* since it can be easily formulated as follows:

E-mail address: vrahatis@math.upatras.gr.

https://doi.org/10.1016/j.topol.2019.107036

Theorem 1.2 *(Bolzano's intermediate value theorem). If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if y_0 is a real number such that:*

$$\min\{f(a), f(b)\} < y_0 < \max\{f(a), f(b)\},$$

then there is at least one $x_0 \in (a,b)$ such that $f(x_0) = y_0$.

Obviously, Theorem 1.2 can be deduced from Theorem 1.1 by considering the function $g(x) = f(x) - y_0$.

The first proofs of the above theorem, given independently by Bolzano in 1817 [8] and Cauchy in 1821 [11], were crucial in the procedure of *arithmetization of analysis*, which was a research program in the foundations of mathematics during the second half of the 19th century.

A straightforward generalization of Bolzano's theorem to continuous mappings of an n-cube (parallelotope) into $\mathbb{R}^n$ was proposed (without proof) by Poincaré in 1883 and 1884 in his work on the *three body problem* [37,38]. The Poincaré theorem was soon forgotten and it has come to be known as "Miranda's theorem" [33] which partly explains the nomenclature "Poincaré-Miranda theorem" [30] and "Bolzano-Poincaré-Miranda theorem" [6,51,59].

The Bolzano-Poincaré-Miranda theorem states that [33,49,56]:

Theorem 1.3 *(Bolzano-Poincaré-Miranda theorem). Suppose that $P = \big\{x \in \mathbb{R}^n \mid |x_i| < L,\ \text{for}\ 1 \leqslant i \leqslant n\big\}$ and let the mapping $F_n = (f_1, f_2, \ldots, f_n)\colon P \to \mathbb{R}^n$ be continuous on the closure of P such that $F_n(x) \neq \theta^n = (0, 0, \ldots, 0)$ for x on the boundary of P, and*

(a) $f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geqslant 0, \quad \text{for} \quad 1 \leqslant i \leqslant n,$

(b) $f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \leqslant 0, \quad \text{for} \quad 1 \leqslant i \leqslant n.$

Then, there is at least one $x \in P$ such that $F_n(x) = \theta^n$.

In 1940 Miranda [33] showed that the above theorem is equivalent to the traditional Brouwer fixed point theorem [10]. It is worthy to mention that the Bolzano-Poincaré-Miranda theorem is closely related to important theorems in analysis and topology and constitutes an invaluable tool for verified solutions of numerical problems by means of interval arithmetic [25,34,36,44]. For a short proof and a generalization of the Bolzano-Poincaré-Miranda theorem using topological degree theory we refer the interested reader to [56]. In addition, for generalizations with respect to an arbitrary basis of $\mathbb{R}^n$ that eliminate the dependence of the Bolzano-Poincaré-Miranda theorem on the standard basis of $\mathbb{R}^n$ see [15,56]. For generalizations of this theorem to infinite-dimensional settings see, e.g., [6,43,44]. Also, for various interesting relations between the theorems of Bolzano-Poincaré-Miranda, Borsuk [9], Kantorovich[1] [24] and Smale[2] [47] with respect to the existence of a solution of a system of nonlinear equations, we refer the interested reader to [1].

In [59] a generalization of the Bolzano Theorem 1.1 for simplices (intermediate value theorem for simplices) is proposed (cf. Theorem 2.1 below). For an interesting application of this theorem in economic sciences we refer the interested reader to [32]. The obtained proof of this theorem is based on the Knaster-Kuratowski-Mazurkiewicz lemma (KKM lemma for short, often called the KKM covering theorem or KKM covering principle) [26]. For a mathematical formulation of this important lemma the reader is referred to Lemma 2.1 below. The KKM lemma constitutes the basis for the proof of many theorems, including the famous Brouwer fixed point theorem, among others.

[1] Nobel Laureate in Economic Sciences in 1975.
[2] Fields Medalist in 1966.

The KKM lemma has a lot of applications in various fields of pure and applied mathematics. In particular, among others, in the field of mathematical economics, the very important and pioneering extension of the KKM lemma due to Shapley[3] [45], known as Knaster-Kuratowski-Mazurkiewicz-Shapley theorem (KKMS theorem for short), constitutes the basis for the proof of many theorems on the existence of solutions in game theory and in the general equilibrium theory of economic analysis. A mathematical formulation of Shapley's extension of the KKM lemma can be given as follows [18,20,45,46]:

Theorem 1.4 *(Knaster-Kuratowski-Mazurkiewicz-Shapley theorem). Suppose that $\mathcal{N}$ is the family of non-empty subsets of the set $N = \{1, 2, \ldots, n\}$. Let $e^j \in \mathbb{R}^n$ be the unit vector with components $e_i^j = 0$ for $i \in N \backslash \{j\}$ and $e_j^j = 1$. For each $S \in \mathcal{N}$ consider: (a) its normalized characteristic vector $\chi_S = (1/\mathrm{card}\{S\}) \sum_{j \in S} e^j$, where $\mathrm{card}\{S\}$ denotes the number of elements in the set S, and (b) the convex hull $\mathrm{co}\{e^j \mid j \in S\}$ denoted by Δ^S. Let C_S, $S \in \mathcal{N}$ be a family of closed subsets of Δ^N, indexed by the members of $\mathcal{N}$, which satisfy the following Shapley's boundary conditions:*

$$\forall\, T \in \mathcal{N}, \quad \Delta^T \subset \bigcup_{S \subset T} C_S.$$

Then, there exists a family $\mathcal{B}$ of members of $\mathcal{N}$ such that $\chi_N \in \mathrm{co}\{\chi_S \mid S \in \mathcal{B}\}$ (called balanced family) for which $\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset$.

It is worth noting that, when $C_S = \emptyset$ for all S for which $\mathrm{card}\{S\} \geqslant 2$, the KKMS theorem reduces to the KKM lemma [20]. Due to its importance, this remarkable theorem has been extended and proved under different conditions multiple times by several researchers (i.e., see [16,17,19,23,28,29,31,39]).

It is worthy to mention that, the three important and pioneering classical results, namely, the Brouwer fixed point theorem [10], the Sperner lemma [48], and the KKM lemma [26] are mutually equivalent in the sense that each one can be deduced from another.

Brouwer's theorem is also very important in economic sciences since it can be used to show the existence of equilibria. Nash[4] proved his famous theorem [35] using the Brouwer's fixed point theorem. There are several alternative proofs of Nash's theorem, all using Brouwer's theorem (with different functions) or Kakutani's fixed-point theorem [22] which is a generalization of Brouwer's fixed point theorem for fixed points of multivalued mappings. Furthermore, these fixed point theorems can be used for proving the existence of equilibria due to Arrow[5] and Debreu[6] [7,14,61].

The paper is organized as follows. In the next session two short proofs of the intermediate value theorem for simplices [59] are given. In Section 3 the generalization of the intermediate value theorem for simplices is applied for simplicial approximations of zeros of continuous mappings and an error estimate is presented. The paper ends in Section 4 with a brief synopsis.

2. Intermediate value theorem for simplices

Notation 2.1. We denote by ϑA the boundary of a set A, by $\mathrm{cl}A$ its closure, by $\mathrm{int}A$ its interior, by $\mathrm{card}\{A\}$ its cardinality (i.e., the number of elements in the set A) and by $\mathrm{co}A$ its convex hull (i.e., the set of all finite convex combinations from A).

[3] Nobel Laureate in Economic Sciences in 2012.
[4] Nobel Laureate in Economic Sciences in 1994.
[5] Nobel Laureate in Economic Sciences in 1972.
[6] Nobel Laureate in Economic Sciences in 1983.

Notation 2.2. We shall frequently use the index sets $N^n = \{0, 1, \ldots, n\}$, $N^n_{\neg 0} = \{1, 2, \ldots, n\}$ and $N^n_{\neg i} = \{0, 1, \ldots, i-1, i+1, \ldots, n\}$. Also, for a given set $I = \{i, j, \ldots, \ell\} \subset N^n$ we denote by $N^n_{\neg I}$ or equivalently by $N^n_{\neg ij\ldots\ell}$ the set $\{k \in N^n \mid k \notin I\}$.

Definition 2.1. For any positive integer n, and for any set of points $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ in some linear space which are affinely independent (i.e., the vectors $\{\upsilon^1 - \upsilon^0, \upsilon^2 - \upsilon^0, \ldots, \upsilon^n - \upsilon^0\}$ are linearly independent) the convex hull $\mathrm{co}\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is called the *n-simplex with vertices* $\upsilon^0, \upsilon^1, \ldots, \upsilon^n$. For each subset of $(m+1)$ elements $\{\omega^0, \omega^1, \ldots, \omega^m\} \subset \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$, the m-simplex $[\omega^0, \omega^1, \ldots, \omega^m]$ is called an *m-face* of $[\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. In particular, 0-faces are vertices and 1-faces are edges. The m-faces are also called *facets* of the n-simplex. An m-face of the n-simplex is called the *carrier* of a point p if p lies on this m-face and not on any sub-face of this m-face.

Notation 2.3. We denote the n-simplex with set of vertices $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ by $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. Also, we denote the $(n-1)$-simplex that determines the i-th $(n-1)$-face of σ^n by $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$. Furthermore, for a given index set $I = \{i, j, \ldots, \ell\} \subset N^n$ with cardinality $\mathrm{card}\{I\} = \kappa$, we denote by $\sigma^n_{\neg I}$ or equivalently by $\sigma^n_{\neg ij\ldots\ell}$ the $(n-\kappa)$-face of σ^n with vertices $\upsilon^m, m \in N^n_{\neg I}$.

Lemma 2.1 *(Knaster-Kuratowski-Mazurkiewicz (KKM) lemma). Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*

(b) *For each* $\emptyset \neq I \subset N^n$ *it holds that* $\bigcap_{i \in I} \sigma^n_{\neg i} \subset \bigcup_{j \in N^n_{\neg I}} C_j$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Definition 2.2. A covering satisfying the conditions in the KKM Lemma 2.1 is called a *KKM covering*.

Observation 2.1. Property (b) of a KKM covering can be stated also as follows:

> *Each face of any dimension of the simplex σ^n is covered by the sets that correspond to the vertices spanning that face. Thus, the vertex υ^i is covered by the closed subset C_i, the edge $[\upsilon^i, \upsilon^j]$ is covered by $C_i \cup C_j$ while the face $[\upsilon^i, \upsilon^j, \ldots, \upsilon^\ell]$ is covered by $C_i \cup C_j \cup \cdots \cup C_\ell$ for each index set $\{i, j, \ldots, \ell\} \subset N^n$.*

Definition 2.3. ([2, p. 4]). A system (family) of subsets of a set A whose union is A is called a *covering* of A. The *order* of a finite system of sets is the greatest integer k for which the system has k elements with nonempty intersection. A system of sets is said to be *simple* if every two elements of the system are distinct. A covering is called an *ε-covering* if the finite system of sets of this covering are of diameter less than $\varepsilon > 0$.

A similar to KKM covering principle was proposed by Sperner [48] (see also [2, p. 162], [3, p. 378]):

Lemma 2.2 *(Sperner covering principle). Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*

(b) $\sigma^n_{\neg i} \cap C_i = \emptyset, \quad \forall\, i \in N^n$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Remark 2.1. Obviously, Lemma 2.2 is similar to KKM Lemma 2.1 in which the hypothesis (b) is replaced by the hypothesis $\sigma^n_{\neg i} \cap C_i = \emptyset, \ \forall\, i \in N^n$.

A similar result is the following [13,42,50]:

Lemma 2.3 *(Sperner covering principle). Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*

(b) $\sigma^n_{\neg i} \subset C_i, \quad \forall\, i \in N^n$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Remark 2.2. Obviously, Lemma 2.3 is similar to KKM Lemma 2.1 in which the hypothesis (b) is replaced by the hypothesis $\sigma^n_{\neg i} \subset C_i, \ \forall\, i \in N^n$. Lemma 2.2 and Lemma 2.3 are equivalent in the sense that each one can be deduced from another. A short proof of the equivalence of Lemma 2.1 and Lemma 2.3 was given in [13].

Definition 2.4. Let ψ be a real number, and let us set

$$\operatorname{sgn}\psi = \begin{cases} -1, & \text{if} \quad \psi < 0, \\ \ \ 0, & \text{if} \quad \psi = 0, \\ \ \ 1, & \text{if} \quad \psi > 0. \end{cases}$$

Then, for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ the *sign* of a, denoted $\operatorname{sgn} a$, is defined as follows:

$$\operatorname{sgn} a = (\operatorname{sgn} a_1, \operatorname{sgn} a_2, \ldots, \operatorname{sgn} a_n)\,.$$

Next, we give short proofs of the generalization of the intermediate value theorem for simplices [59]. The obtained proofs are based on Lemma 2.2 and Lemma 2.3 correspondingly.

Theorem 2.1 *(Intermediate value theorem for simplices [59]). Assume that $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is an n-simplex in $\mathbb{R}^n$. Let $F_n = (f_1, f_2, \ldots, f_n)\colon \sigma^n \to \mathbb{R}^n$ be a continuous function such that $f_j(\upsilon^i) \neq 0, \ \forall\, j \in N^n_{\neg 0} = \{1, 2, \ldots, n\}$, $i \in N^n = \{0, 1, \ldots, n\}$ and $\theta^n = (0, 0, \ldots, 0) \notin F_n(\vartheta\sigma^n)$ (i.e., F_n does not vanish on the boundary $\vartheta\sigma^n$ of σ^n). Assume that the vertices υ^i, $i \in N^n$ are reordered such that the following hypotheses are fulfilled:*

$$\operatorname{sgn} f_j(\upsilon^j) \operatorname{sgn} f_j(x) = -1, \quad \forall\, x \in \sigma^n_{\neg j}, \ \ j \in N^n_{\neg 0}, \tag{1}$$

$$\operatorname{sgn} F_n(\upsilon^0) \neq \operatorname{sgn} F_n(x), \quad \forall\, x \in \sigma^n_{\neg 0}, \tag{2}$$

where $\operatorname{sgn} F_n(x) = \big(\operatorname{sgn} f_1(x), \operatorname{sgn} f_2(x), \ldots, \operatorname{sgn} f_n(x)\big)$ and $\sigma^n_{\neg i}$ denotes the face opposite to vertex υ^i. Then, there is at least one point $x \in \operatorname{int}\sigma^n$ such that $F_n(x) = \theta^n$.

Proof based on Lemma 2.2. Due to hypotheses (1) and (2) it is evident that the following holds:

$$\operatorname{sgn} F_n(\upsilon^i) \neq \operatorname{sgn} F_n(x), \quad \forall\, x \in \sigma^n_{\neg i}, \ \ i \in N^n. \tag{3}$$

By virtue of conditions (1) it is obvious that for the vertex υ^0 the following relations are also fulfilled:

$$\operatorname{sgn} f_j(\upsilon^0)\operatorname{sgn} f_j(\upsilon^j) = -1, \quad \forall\, j \in N^n_{\neg 0}\,, \tag{4}$$

as well as it is evident that for the vertices υ^i, $i \in N^n$ it holds that:

$$\operatorname{sgn} F_n(\upsilon^i) \neq \operatorname{sgn} F_n(\upsilon^\ell), \quad \forall\, i, \ell \in N^n,\;\; i \neq \ell. \tag{5}$$

Furthermore, it is obvious that for all $j \in N^n_{\neg 0}$ the j-th component $\operatorname{sgn} f_j(\upsilon^i)$ of $\operatorname{sgn} F_n(\upsilon^i) = (\operatorname{sgn} f_1(\upsilon^i),$ $\operatorname{sgn} f_2(\upsilon^i), \ldots, \operatorname{sgn} f_n(\upsilon^i))$ is not the same for all the vertices υ^i, $i \in N^n$. Therefore, for the following sets S_{f_j} we have that:

$$S_{f_j} = \bigl\{x \in \operatorname{int}\sigma^n \;\big|\; f_j(x) = 0\bigr\} \neq \emptyset, \quad \forall\, j \in N^n_{\neg 0}. \tag{6}$$

For each one of the vertices υ^j, $j \in N^n_{\neg 0}$ we correspondingly consider the following closed sets:

$$C_j = \operatorname{cl}\bigl\{x \in \sigma^n \;\big|\; \operatorname{sgn} f_j(x) = \operatorname{sgn} f_j(\upsilon^j)\bigr\}\,, \quad j \in N^n_{\neg 0}, \tag{7}$$

while for the vertex υ^0 we consider the following closed set:

$$C_0 = \operatorname{cl}\bigl\{x \in \sigma^n \;\big|\; \operatorname{sgn} F_n(x) = \operatorname{sgn} F_n(\upsilon^0)\bigr\}\,. \tag{8}$$

Since by hypothesis we have $f_j(\upsilon^i) \neq 0,\ \forall\, j \in N^n_{\neg 0},\ i \in N^n$, it is obvious that the following holds:

$$\operatorname{int} C_i \neq \emptyset, \quad \forall\, i \in N^n. \tag{9}$$

Furthermore, it is evident that the following relation is valid:

$$\operatorname{int} C_0 \bigcap \left\{ \bigcup_{j \in N^n_{\neg 0}} \operatorname{int} C_j \right\} = \emptyset. \tag{10}$$

Due to hypotheses (1) and (2) and relations (7), and (8) it is obvious that the faces $\sigma^n_{\neg i}$, $i \in N^n$ have no points in common with the corresponding closed sets C_i. That is

$$\sigma^n_{\neg i} \cap C_i = \emptyset, \quad \forall\, i \in N^n\,. \tag{11}$$

Let us denote by x^* any $x \in \sigma^n$ such that $f_j(x^*) \neq 0,\ \forall\, j \in N^n_{\neg 0}$. It is obvious that for these points x^* the number of values that the function $\operatorname{sgn} F_n(x^*) = (\operatorname{sgn} f_1(x^*), \operatorname{sgn} f_2(x^*), \ldots, \operatorname{sgn} f_n(x^*))$ can obtain is 2^n. Let us consider that the function values of $\operatorname{sgn} F_n(x^*)$ form a set with cardinality $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\}$. Due to relation (8) for any $x^* \in \operatorname{int} C_0$ we obtain that $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\} = 1$. On the other hand, due to relation (7), for any $x^* \in \operatorname{int} C_j$, $j \in N^n_{\neg 0}$, we have $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\} = 2^{n-1}$. Therefore, for any $x^* \in \bigcup_{j \in N^n_{\neg 0}} \operatorname{int} C_j$ we obtain $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\} = \sum_{\ell=1}^{n} 2^{n-\ell}$ or, equivalently, $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\} = 2^n - 1$. Thus, due to relations (9) and (10) we have that for any $x^* \in \bigcup_{i \in N^n} \operatorname{int} C_i$, it holds that $\operatorname{card}\{\operatorname{sgn} F_n(x^*)\} = 2^n$. Therefore, we conclude that the following is valid:

$$\sigma^n = \bigcup_{i \in N^n} C_i\,. \tag{12}$$

Using relations (11) and (12) it is evident that the sets C_i, $i \in N^n$ are well defined according to the hypotheses of Lemma 2.2. Thus, by virtue of Lemma 2.2 we obtain $\bigcap_{i \in N^n} C_i \neq \emptyset$ and consequently $\bigcap_{i \in N^n} \vartheta C_i \neq \emptyset$. Therefore, due to the continuity of F_n, for the following solution set S_{F_n} it holds that $S_{F_n} = \bigl\{x \in \operatorname{int}\sigma^n \;\big|\; f_i(x) = 0,\ \forall\, i \in N^n_{\neg 0}\bigr\} \neq \emptyset$. Thus, the theorem is proved. $\square$

Proof based on Lemma 2.3. In order to avoid repetitions, let us assume that the relations (3) to (6) of the previous proof are fulfilled.

For each one of the vertices υ^j, $j \in N^n_{\neg 0}$ we correspondingly consider the following closed sets:

$$C'_j = \mathrm{cl}\left\{x \in \sigma^n \;\middle|\; \mathrm{sgn} f_j(x) = \mathrm{sgn} f_j(\upsilon^j)\right\}, \quad j \in N^n_{\neg 0}, \tag{13}$$

and

$$C_j = \mathrm{cl}\left\{x \in \sigma^n \;\middle|\; x \notin C'_j\right\}, \quad j \in N^n_{\neg 0}, \tag{14}$$

while for the vertex υ^0 we consider the following closed sets:

$$C'_0 = \mathrm{cl}\left\{x \in \sigma^n \;\middle|\; \mathrm{sgn} F_n(x) = \mathrm{sgn} F_n(\upsilon^0)\right\}, \tag{15}$$

and

$$C_0 = \mathrm{cl}\left\{x \in \sigma^n \;\middle|\; x \notin C'_0\right\}. \tag{16}$$

Since by hypothesis we have $f_j(\upsilon^i) \neq 0$, $\forall\, j \in N^n_{\neg 0}$, $i \in N^n$, it is obvious that the following holds:

$$\mathrm{int}\, C'_i \neq \emptyset, \quad \forall\, i \in N^n, \tag{17}$$

while by virtue of hypotheses (1) and (2) it is obvious that:

$$\mathrm{int}\, C_i \neq \emptyset, \quad \forall\, i \in N^n. \tag{18}$$

Due to hypotheses (1) and (2) and relations (14) and (16) it is obvious that the faces $\sigma^n_{\neg i}$, $i \in N^n$ are covered by the corresponding closed sets C_i. Thus,

$$\sigma^n_{\neg i} \subset C_i, \quad \forall\, i \in N^n. \tag{19}$$

By virtue of relations (14) and (15) and due to relations (4) it is evident that $C'_0 = \bigcap_{i \in N^n_{\neg 0}} C_i$. Thus, since $\sigma^n = C'_0 \cup C_0$ it is obvious that:

$$\sigma^n = \bigcup_{i \in N^n} C_i. \tag{20}$$

Using relations (19) and (20) it is evident that the sets C_i, $i \in N^n$ are well defined according to the hypotheses of Lemma 2.3. Thus, by virtue of Lemma 2.3 we obtain $\bigcap_{i \in N^n} C_i \neq \emptyset$ and consequently $\bigcap_{i \in N^n} \vartheta C_i \neq \emptyset$. Therefore, due to the continuity of F_n, for the following solution set S_{F_n} it holds that $S_{F_n} = \left\{x \in \mathrm{int}\,\sigma^n \;\middle|\; f_i(x) = 0, \ \forall\, i \in N^n_{\neg 0}\right\} \neq \emptyset$. Thus, the theorem is proved. $\square$

Remark 2.3. For $n = 1$, Theorem 2.1 clearly reduces to the Bolzano intermediate value theorem. For this reason, Theorem 2.1 was named *"intermediate value theorem for simplices"*.

Remark 2.4. The only computable information required by hypotheses (1) and (2) of Theorem 2.1 is the algebraic sign of the function values on the boundary of the n-simplex σ^n. Thus, Theorem 2.1 is applicable whenever the signs of the function values are computed correctly. The algebraic sign is the smallest amount of information (one bit of information) necessary for the purpose needed. Thus, the methods that require only algebraic signs are of major importance for tackling problems with imprecise (not exactly known) information. This kind of problems occurs in various scientific fields including mathematics, economics, engineering, computer science, biomedical informatics, medicine and bioengineering, among others. This is so, because, in a large variety of applications, precise function values are either impossible or time consuming and computationally expensive to obtain. One such application is provided in [57]. This application concerns the computation of all the periodic orbits (stable and unstable) of any period and accuracy which occur, among others, in the study of beam dynamics in circular particle accelerators like the Large Hadron Collider (LHC) machine at the European Organization for Nuclear Research (CERN). In this application, the method which is presented in [53] and is implemented in [54] is used.

3. Simplicial approximations of zeros of continuous mappings

Theorem 2.1 can be used for approximating fixed points and solutions to systems of nonlinear equations within a given simplex. To this end, we describe a well known and widely used approach that is based on the theory of simplicial approximation of continuous mappings. In general, *simplicial methods* are named the procedures which provide approximations to solutions by means of simplices [5]. In 1912 Brouwer [10] for the proof of his fixed point theorem developed the theory of simplicial approximation of continuous mappings and applied it in order to compute the well-known Brouwer's degree. The Brouwer's fixed point theorem and its generalizations were very useful in providing the existence of solutions to a lot of problems in mathematics. On the other hand, from the computational point of view, their usefulness was restricted. This is so, because all computational methods that used for the computation of an approximate fixed point of a given function were based on iterative procedures that required additional restrictions on the function in order to guarantee convergence.

In 1967 Scarf [41] developed a method for approximating a fixed point of a continuous function from a unit simplex into itself (cf. Remark 3.1 below). Scarf's method provided the first constructive proof to Brouwer's fixed point theorem. This approach is considered very important and various extensions of this have been proposed. Scarf's simplicial method is based on a simplicial subdivision (triangulation) of the given simplex and it uses a labeling of the vertices of the simplicial subdivision.

Definition 3.1. ([4, p. 153]). A *simplicial subdivision* or *triangulation* of an n-simplex σ^n is a partition of σ^n into n-simplices σ_i^n such that the intersection of two n-simplices σ_i^n and σ_j^n is a face (of any dimension) of each of them, or the empty set. The vertices (0-faces) of the n-simplices σ_i^n are called *vertices of the simplicial subdivision*.

Proposition 3.1. ([3, p. 607], [60, p. 812]). *The* diameter *of an m-simplex σ^m in $\mathbb{R}^n$, $m \leqslant n$, denoted by* $\operatorname{diam}(\sigma^m)$*, is the length of the longest edge (1-face) of σ^m.*

Definition 3.2. The *mesh* of a simplicial subdivision T of an n-simplex σ^n in $\mathbb{R}^n$ denoted by $\operatorname{mesh}(\sigma^n)$, is given by the $\sup_{\tau^n \in T} \operatorname{diam}(\tau^n)$.

Definition 3.3. Let T be a simplicial subdivision of an n-simplex σ^n in $\mathbb{R}^n$ and let T^0 be the set of the vertices of T. A *labeling* function of T is a function $\lambda : T^0 \to \mathbb{N}_0$.

Definition 3.4. A labeling function $\lambda(\upsilon)$ which is defined on an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is called a *proper labeling function* if it satisfies the following conditions:

(a) $\lambda\ (\upsilon) \in N^n = \{0, 1, \ldots, n\}$,

(b) $\{\lambda(\upsilon^0), \lambda(\upsilon^1), \ldots, \lambda(\upsilon^n)\} = N^n = \{0, 1, \ldots, n\}$,

(c) If the i-face determined by the set of vertices $\{\upsilon^{k_0}, \upsilon^{k_1}, \dots, \upsilon^{k_i}\}$ is the carrier of a point υ then $\lambda(\upsilon) \in \{\lambda(\upsilon^{k_0}), \lambda(\upsilon^{k_1}), \dots, \lambda(\upsilon^{k_i})\}$.

Similarly, we give the following definition.

Definition 3.5. A labeling is called *admissible* if for a simplicial subdivision T of an n-simplex σ^n in $\mathbb{R}^n$ each vertex of T is labeled with an integer in $\mathbb{N}_0$ such that no vertex in the i-face $\sigma^n_{\neg i}$ is labeled with i.

Definition 3.6. Assume that $\sigma^k = [\omega^0, \omega^1, \dots, \omega^k]$ is a k-simplex, $k \leqslant n$, of a simplicial subdivision of σ^n, then σ^k is said to have a *complete set of labels* if it holds that $\{\lambda(\omega^0), \lambda(\omega^1), \dots, \lambda(\omega^k)\} = \{0, 1, \dots, k\}$. This simplex is also called *completely-labeled simplex*.

Definition 3.7. A completely-labeled n-simplex (with a complete set of labels) is also called *Sperner* or *panchromatic* n-simplex.

The very important and pioneering Sperner's lemma [12,58] states that: *"For any simplicial subdivision and proper labeling function of an n-simplex σ^n in $\mathbb{R}^n$ there is at least one n-simplex of the subdivision with a complete set of labels"*. This lemma can also be formulated as follows:

Lemma 3.1 *(Sperner [48]). Let T be a simplicial subdivision of an n-simplex σ^n in $\mathbb{R}^n$. Assume that the vertices of T are labeled using an admissible labeling. Then there is at least one completely-labeled n-simplex τ^n in T.*

Remark 3.1. It is well-known that for continuous mappings and a fine enough simplicial subdivision the vertices of a completely-labeled n-simplex τ^n of the simplicial subdivision approximate a fixed point of the mapping. To this end, there are various methods including the fundamental Scarf's method [41,61]. Scarf's method approximates a fixed point of a continuous mapping $F_n = (f_1, f_2, \dots, f_n) : u^n \to u^n$, where u^n is the unit simplex $u^n = \{x \geqslant 0 \mid \sum_i x_i = 1\}$. The method considers a simplicial subdivision T of u^n into simplices of sufficiently small mesh mesh(u^n), so that $\|x - y\| \leqslant \text{mesh}(u^n)$ implies $\|F_n(x) - F_n(y)\| \leqslant \varepsilon/n$, for $\varepsilon > 0$. Then, it labels each vertex w of the subdivision using an index $i = 1, 2, \dots, n$ such that $w_i > f_i(w)$. In the case where w is not a fixed point there is at least one such index, while, if w is a fixed point then it labels w with, i.e., $\arg_i \max(w_i)$. The unit vectors e_i at the n corners of the simplex u^n are labeled with i, and all the vertices on the face $x_j = 0$ are labeled with an index different to j. Then, Sperner's lemma implies that the subdivision has at least one completely-labeled simplex (panchromatic simplex) τ^n (a small simplex τ^n whose vertices have distinct labels). From the choices of the labels and the value of mesh(u^n) it follows that any point $x \in \tau^n$ satisfies $\|F(x) - x\| \leqslant \varepsilon$.

Remark 3.2. It is worth mentioning that, since the condition that no vertex in the i-face $\tau^n_{\neg i}$ is labeled with i may not be fulfilled, then τ^n may not enclose a fixed point. Thus, in general, the term "approximate fixed point" is used in the sense that the point is close to its image while it is not necessary close to a fixed point. On the other hand, by considering finer subdivisions such as the mesh of the subdivision to tend to zero then the resulting sequence of approximate fixed points must contain (by compactness) a subsequence that converges to a point, which must be a fixed point [61].

Next we present a theorem for simplicial approximations of zeros of continuous mappings which is based on Theorem 2.1 and Sperner's Lemma 3.1.

Theorem 3.1 *(Simplicial approximation of zeros). Let $\sigma^n = [\upsilon^0, \upsilon^1, \dots, \upsilon^n]$ be an n-simplex in $\mathbb{R}^n$ and let $F_n = (f_1, f_2, \dots, f_n)\colon \sigma^n \to \mathbb{R}^n$ be a continuous function such that the hypotheses of Theorem 2.1 are* *fulfilled. Assume that T is a simplicial subdivision of σ^n and let T^0 be the set of the vertices of T. Consider the following labeling function $\lambda : T^0 \to \mathbb{N}_0$ of T:*

$$\lambda(x) = \begin{cases} i, & \textit{if} \quad \mathrm{sgn} F_n(x) = \mathrm{sgn} F_n(\upsilon^i), \quad \textit{for} \quad i \in N^n, \\ \min\{j \,|\, j \in N^n_{\neg 0 \ell}\}, & \textit{if} \quad \textit{for} \;\; \ell \in N^n_{\neg 0},\ \exists\, y \in \sigma^n_{\neg \ell} \text{ s.t. } \forall\, i \in N^n,\ \mathrm{sgn} F_n(x) = \mathrm{sgn} F_n(y) \neq \mathrm{sgn} F_n(\upsilon^i), \\ 1, & \textit{otherwise.} \end{cases}$$

Then there is at least one completely-labeled n-simplex τ^n in T.

Proof. Obviously each vertex of the simplicial subdivision T is labeled with an integer in $N^n = \{0, 1, \dots, n\}$. Due to hypotheses (1) and (2) of Theorem 2.1 it is evident that no vertex in the i-face $\sigma^n_{\neg i}$ is labeled with i. Thus, the vertices of T are labeled using an admissible labeling. By virtue of Sperner Lemma 3.1 there is at least one completely-labeled n-simplex τ^n in T. Thus, the theorem is proved. □

Remark 3.3. Similarly to the discussion of Remark 3.1 by considering finer subdivisions such as the mesh of the subdivision to tend to zero then the resulting sequence of approximate zeros must contain (by compactness) a subsequence that converges to a point, which must be a zero point. The approximate zero point can be obtained by an interior point in τ^n such as the barycenter of τ^n that can be computed by its vertices.

Definition 3.8. Let $\sigma^m = [\upsilon^0, \upsilon^1, \dots, \upsilon^m]$ be an m-simplex in $\mathbb{R}^n$, $m \leqslant n$. Then the *barycenter* of σ^m denoted by K is a point in $\mathbb{R}^n$ such that:

$$K = \frac{1}{m+1} \sum_{i=0}^{m} \upsilon^i.$$

Remark 3.4. By convexity it is obvious that the barycenter of any m-simplex σ^m in $\mathbb{R}^n$ is a point in the relative interior of σ^m.

Next we give estimates of the above approximation where the Euclidean norm is used to measure distances.

Theorem 3.2. ([52]). *Suppose that $\sigma^m = [\upsilon^0, \upsilon^1, \dots, \upsilon^m]$ is an m-simplex in $\mathbb{R}^n$, $m \leqslant n$. Let K be the barycenter of σ^m and let K_i be the barycenter of the i-th face $\sigma^m_{\neg i} = [\upsilon^0, \upsilon^1, \dots, \upsilon^{i-1}, \upsilon^{i+1}, \dots, \upsilon^m]$ of σ^m then the following relationships hold for all $0 \leqslant i \leqslant m$,*

(*a*) *The points υ^i, K and K_i are collinear points,*

(*b*) $$\|K - \upsilon^i\| = \frac{m}{m+1} \left(\frac{1}{m} \sum_{\substack{j=0 \\ j \neq i}}^{m} \|\upsilon^i - \upsilon^j\|^2 - \frac{1}{m^2} \sum_{\substack{p=0 \\ p \neq i}}^{m-1} \sum_{\substack{q=p+1 \\ q \neq i}}^{m} \|\upsilon^p - \upsilon^q\|^2 \right)^{1/2},$$

(*c*) $\|K - K_i\| = m^{-1} \|K - \upsilon^i\|$.

Proof. See [52]. □

Definition 3.9. ([55]). The *barycentric radius* $\beta(\sigma^m)$ of an m-simplex σ^m in $\mathbb{R}^n$ is the radius of the smallest ball centered at the barycenter of σ^m and containing the simplex. The barycentric radius $\beta(A)$ of a subset A of $\mathbb{R}^n$ is the supremum of the barycentric radii of simplices with vertices in A.

Remark 3.5. The length of the barycentric radius $\beta(\sigma^m)$ of an m-simplex σ^m in $\mathbb{R}^n$, $m \leqslant n$, is given by $\max\limits_{0 \leqslant i \leqslant m} \|K - \upsilon^i\|$.

Theorem 3.3. ([55]). *Any m-simplex $\sigma^m = [\upsilon^0, \upsilon^1, \dots, \upsilon^m]$ in $\mathbb{R}^n$, $m \leqslant n$ is enclosable by the spherical surface S_β^{m-1} centered at the barycenter of σ^m and with radius the barycentric radius $\beta(\sigma^m)$ given by:*

$$\beta(\sigma^m) = \frac{1}{m+1} \max_{0 \leqslant i \leqslant m} \left\{ m \sum_{\substack{j=0 \\ j \neq i}}^{m} \|\upsilon^i - \upsilon^j\|^2 - \sum_{\substack{p=0 \\ p \neq i}}^{m-1} \sum_{\substack{q=p+1 \\ q \neq i}}^{m} \|\upsilon^p - \upsilon^q\|^2 \right\}^{1/2}.$$

Proof. See [55]. □

Remark 3.6. The barycentric radius $\beta(\sigma^n)$ of an n-simplex σ^n in $\mathbb{R}^n$ can be used to estimate error bounds for approximate fixed points or approximate roots of mappings in $\mathbb{R}^n$, by approximating a fixed point or a root by the barycenter of σ^n. Note that the computation of $\beta(\sigma^n)$ requires only the lengths of the edges of σ^n, which are also required in order to compute the diameter $\mathrm{diam}(\sigma^n)$ of σ^n. Furthermore, since the distance of the barycenter K of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \dots, \upsilon^n]$ in $\mathbb{R}^n$ from the barycenter K_i of the i-th face $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \dots, \upsilon^{i-1}, \upsilon^{i+1}, \dots, \upsilon^n]$ of σ^n is equal to $\|K - \upsilon^i\|/n$ [52,55], then using Theorem 3.3 we can easily compute the value of

$$\gamma(\sigma^n) = \min_{0 \leqslant i \leqslant n} \|K - K_i\| / \mathrm{diam}(\sigma^n).$$

The value $\gamma(\sigma^n)$ can be used to estimate the thickness $\theta(\sigma^n)$ of σ^n, that is given by [5,27,40,60]:

$$\theta(\sigma^n) = \min_{0 \leqslant i \leqslant n} \left\{ \min_{x \in \sigma^n_{\neg i}} \big\|K - x\big\| \right\} / \mathrm{diam}(\sigma^n).$$

The thickness $\theta(\sigma^n)$ is important to piecewise linear approximations of smooth mappings and, in general, to simplicial and continuation methods for approximating fixed points or solutions of systems of nonlinear equations.

4. Synopsis

Two short proofs of a intermediate value theorem for simplices [59] are given. The proofs are stemmed from Sperner covering principles of a simplex. Also, this intermediate value theorem is applied for the localization and approximation of fixed points and zeros of continuous mappings using a simplicial subdivision of a simplex. Furthermore, a theorem for the existence of a Sperner simplex (panchromatic simplex) in the considered simplicial subdivision is proved. In addition, an error estimate is presented.

Acknowledgement

The author would like to thank the anonymous reviewer for his very helpful comments.

References

[1] G. Alefeld, A. Frommer, G. Heindl, J. Mayer, On the existence theorems of Kantorovich, Miranda and Borsuk, Electron. Trans. Numer. Anal. 17 (2004) 102–111.

[2] P.S. Aleksandrov (Alexandroff), Combinatorial Topology, Vol. 1, Translated from the first (1947) Russian edition by Horace Komm, Graylock Press, Rochester, New York, 1956.

[3] P.S. Alexandroff, H. Hopf, Topologie, Erster Band, Berichtigter Reprint, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
[4] A.D. Alexandrov, Convex Polyhedra, English translation by N.S. Dairbekov, S.S. Kutateladze, A.B. Sossinsky, Comments and bibliography by V.A. Zalgaller, Appendices by L.A. Shor, Yu. A. Volkov, Springer-Verlag, Berlin, Heidelberg, 2005, The Russian edition was published by Gosudarstv. Izdat. Tekhn.-Teor. Lit., Moscow-Leningrad, 1950.
[5] E.L. Allgower, K. Georg, Simplicial and continuation methods for approximating fixed points and solutions to systems of equations, SIAM Rev. 22 (1) (1980) 28–85.
[6] D. Ariza-Ruiz, J. Garcia-Falset, S. Reich, The Bolzano-Poincaré-Miranda theorem in infinite-dimensional Banach spaces, J. Fixed Point Theory Appl. 21 (59) (2019) 1–12.
[7] K.J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (3) (1954) 265–290.
[8] B. Bolzano, Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werten, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege, Prague, 1817.
[9] K. Borsuk, Drei Sätze über die n-dimensionale Euklidische Sphäre, Fundam. Math. 20 (1933) 177–190.
[10] L.E.J. Brouwer, Über Abbildungen von Mannigfaltigkeiten, Math. Ann. 71 (1912) 97–115.
[11] A.-L. Cauchy, Cours d'Analyse L'École Royale Polytechnique, Paris, 1821, reprinted in Oeuvres Completes, Ser. 2., vol. 3.
[12] J.A. De Loera, X. Goaoc, F. Meunier, N.H. Mustafa, The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg, Bull. Am. Math. Soc. (N.S.) 56 (2019) 415–511.
[13] J. Freidenfelds, A set intersection theorem and applications, Math. Program. 7 (1) (1974) 199–211.
[14] J. Geanakoplos, Nash and Walras equilibrium via Brouwer, Econ. Theory 21 (2–3) (2003) 585–603.
[15] G. Heindl, Generalizations of theorems of Rohn and Vrahatis, Reliab. Comput. 21 (2016) 109–116.
[16] J.-J.P. Herings, An extremely simple proof of the KKMS theorem, Econ. Theory 10 (1997) 361–367.
[17] T. Ichiishi, On the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem, J. Math. Anal. Appl. 81 (1981) 297–299.
[18] T. Ichiishi, Game Theory for Economic Analysis, Academic Press, New York, NY, USA, 1983.
[19] T. Ichiishi, Alternative version of Shapley's theorem on closed coverings of a simplex, Proc. Am. Math. Soc. 104 (1988) 759–763.
[20] T. Ichiishi, A. Idzik, On a covering theorem, Econ. Theory 19 (2002) 833–838.
[21] V. Jarník, Bernard Bolzano and the foundations of mathematical analysis, in: Bolzano and the Foundations of Mathematical Analysis, Society of Czechoslovak Mathematicians and Physicists, Prague, 1981, pp. 33–42.
[22] S. Kakutani, A generalization of Brouwer's fixed point theorem, Duke Math. J. 8 (3) (1941) 457–459.
[23] Y. Kannai, M.H. Wooders, A further extension of the KKMS theorem, Math. Oper. Res. 25 (2000) 539–551.
[24] L. Kantorovich, On Newton's method for functional equations, Dokl. Akad. Nauk SSSR 59 (1948) 1237–1240.
[25] R.B. Kearfott, J. Dian, A. Neumaier, Existence verification for singular zeros of complex nonlinear systems, SIAM J. Numer. Anal. 38 (2000) 360–379.
[26] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunkt-satzes für n-dimensionale Simplexe, Fundam. Math. 14 (1929) 132–137.
[27] M. Kojima, Studies on piecewise-linear approximations of piecewise-C^1 mappings in fixed points and complementarity theory, Math. Oper. Res. 3 (1) (1978) 17–36.
[28] H. Komiya, A simple proof of K-K-M-S theorem, Econ. Theory 4 (1997) 463–466.
[29] S. Krasa, N.C. Yannelis, An elementary proof of the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem, Econ. Theory 4 (1994) 467–471.
[30] W. Kulpa, The Poincaré-Miranda theorem, Am. Math. Mon. 104 (6) (1997) 545–550.
[31] J. Liu, H.-Y. Tian, Existence of fuzzy cores and generalizations of the K-K-M-S theorem, J. Math. Econ. 52 (2014) 148–152.
[32] P. Milgrom, J. Mollner, Equilibrium selection in auctions and high stakes games, Econometrica 86 (1) (2018) 219–261.
[33] C. Miranda, Un' osservatione su un theorema di Brouwer, Boll. Unione Mat. Ital. 3 (1940) 5–7.
[34] R. Moore, J. Kioustelidis, A simple test for accuracy of approximate solutions to nonlinear (or linear) systems, SIAM J. Numer. Anal. 17 (1980) 521–529.
[35] J. Nash, Non-cooperative games, Ann. Math. 54 (2) (1951) 286–295.
[36] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, 1990.
[37] H. Poincaré, Sur certaines solutions particulières du problème des trois corps, C. R. Acad. Sci. Paris 91 (1883) 251–252.
[38] H. Poincaré, Sur certaines solutions particulières du problème des trois corps, Bull. Astron. 1 (1884) 63–74.
[39] P.J. Reny, M.H. Wooders, An extension of the KKMS theorem, J. Math. Econ. 29 (1998) 125–134.
[40] R. Saigal, On piecewise linear approximations to smooth mappings, Math. Oper. Res. 4 (2) (1979) 153–161.
[41] H. Scarf, The approximation of fixed points of a continuous mapping, SIAM J. Appl. Math. 15 (5) (1967) 1328–1343.
[42] H. Scarf (with the collaboration of T. Hansen), The Computation of Economic Equilibria, Cowles Foundation Monograph Series, vol. 24, Yale University Press, New Haven, London, 1973.
[43] U. Schäfer, A fixed point theorem based on Miranda, Fixed Point Theory Appl. 2007 (2007) 78706.
[44] U. Schäfer, From Sperner's Lemma to Differential Equations in Banach Spaces: An Introduction to Fixed Point Theorems and Their Applications, Karlsruhe Institute of Technology (KIT), KIT Scientific Publishing, Karlsruhe, Germany, 2014.
[45] L.S. Shapley, On balanced games without side payments, in: T.C. Hu, S.M. Robinson (Eds.), Mathematical Programming, Academic Press, New York, NY, USA, 1973, pp. 261–290.
[46] L.S. Shapley, R. Vohra, On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game, Econ. Theory 1 (1991) 108–116.
[47] S. Smale, Algorithms for solving equations, in: A. Gleason (Ed.), Proceedings of the International Congress of Mathematicians (ICM 1986), August 3–11, 1986, Berkeley CA, USA, Amer. Math. Soc., Providence, RI, USA, 1987, pp. 172–195.
[48] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Semin. Univ. Hamb. 6 (1928) 265–272.
[49] F. Stenger, Computing the topological degree of a mapping in $\mathbb{R}^n$, Numer. Math. 25 (1975) 23–38.

[50] M.J. Todd, The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems, vol. 124, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
[51] M. Turzański, The Bolzano-Poincaré-Miranda theorem–discrete version, Topol. Appl. 159 (2012) 3130–3135.
[52] M.N. Vrahatis, An error estimation for the method of bisection in $\mathbb{R}^n$, Bull. Greek Math. Soc. 27 (1986) 161–174.
[53] M.N. Vrahatis, Solving systems of nonlinear equations using the nonzero value of the topological degree, ACM Trans. Math. Softw. 14 (4) (1988) 312–329.
[54] M.N. Vrahatis, CHABIS: a mathematical software package for locating and evaluating roots of systems of nonlinear equations, ACM Trans. Math. Softw. 14 (4) (1988) 330–336.
[55] M.N. Vrahatis, A variant of Jung's theorem, Bull. Greek Math. Soc. 29 (1988) 1–6.
[56] M.N. Vrahatis, A short proof and a generalization of Miranda's existence theorem, Proc. Am. Math. Soc. 107 (1989) 701–703.
[57] M.N. Vrahatis, An efficient method for locating and computing periodic orbits of nonlinear mappings, J. Comput. Phys. 119 (1995) 105–119.
[58] M.N. Vrahatis, Simplex bisection and Sperner simplices, Bull. Greek Math. Soc. 44 (2000) 171–180.
[59] M.N. Vrahatis, Generalization of the Bolzano theorem for simplices, Topol. Appl. 202 (2016) 40–46.
[60] J.H.C. Whitehead, On C^1-complexes, Ann. Math. 41 (4) (1940) 809–824.
[61] M. Yannakakis, Equilibria, fixed points, and complexity classes, Comput. Sci. Rev. 3 (2) (2009) 71–85.