Solving Systems of Nonlinear Equations Using the Nonzero Value of the Topological Degree

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Two algorithms are described here for the numerical solution of a system of nonlinear equations $F(X) = \Theta$, where $\Theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$, and F is a given continuous mapping of a region $\mathscr{D}$ in $\mathbb{R}^n$ into $\mathbb{R}^n$. The first algorithm locates at least one root of the system within an n-dimensional polyhedron, using the nonzero value of the topological degree of F at Θ relative to the polyhedron; the second algorithm applies a new generalized bisection method in order to compute an approximate solution of the system. The size of the original n-dimensional polyhedron is arbitrary, and the method is globally convergent in a residual sense.

These algorithms, in the various function evaluations, only make use of the algebraic sign of F and do not require computations of the topological degree. Moreover, they can be applied to nondifferentiable continuous functions F and do not involve derivatives of F or approximations of such derivatives.

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1. INTRODUCTION

Several methods for the numerical solution of systems of nonlinear equations, based on the topological degree theory, have been proposed in the past few years [3, 6–8, 15, 17]. According to these methods, one establishes the existence of at least one root of a system of nonlinear equations,

$$F(X) = \Theta = (0, 0, \ldots, 0), \tag{1.1}$$

where $F = (f_1, f_2, \ldots, f_n)$ is a continuous mapping of a bounded region $\mathscr{D}$ in $\mathbb{R}^n$ into $\mathbb{R}^n$, in a polyhedron P in $\mathscr{D}$ [1, 2, 17–21], by computing the topological degree of F at Θ relative to P, $\deg(F, P, \Theta)$ [1, 2, 7, 8, 17–25, 27]; if a nonzero value of $\deg(F, P, \Theta)$ is obtained, then by Kronecker's theorem [1, 2, 14] at least one solution of system (1.1) is within P (provided that $\Theta \notin F(b(P))$, where $b(P)$ indicates the oriented boundary of P [2, 17–19]). On the other hand, if deg(F, P,

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$\Theta) = 0$, no conclusions can be drawn because more information about F is needed [1, 2, 14, 21].

However, although the nonzero value of deg(F, P, Θ) plays an important role in the existence of a root, its exact value is useless, since it does not give any additional information about the existence of the roots of system (1.1). Moreover, the computation of deg(F, P, Θ) is a time-consuming procedure [8] and cannot be accurately achieved unless the modulus of continuity of F on P is known [18, 21].

In this paper we implement the concept of the characteristic n-polyhedron, CP [22–24] (or admissible n-polygon [27]), by which we avoid all calculations concerning the topological degree. This is because, under suitable assumptions on its boundary, a characteristic n-polyhedron always contains at least one solution of system (1.1) since the absolute value of the topological degree of F at Θ relative to the CP is equal to 1 [22, 27].

In the literature several bisection methods exist [3, 6–8, 15, 17, 25] that require the computation of the topological degree in order to secure its nonzero value. In the bisection method described here, however, the computation of the topological degree is avoided by making sure that it retains a nonzero value at every iteration. More specifically, the method is applied to a characteristic n-polyhedron in such a way that the new refined n-polyhedron is also a characteristic one [24, 27].

A general description of how a root is isolated and evaluated is presented in Sections 2 and 3. In Section 4 a detailed step-by-step description of the algorithms is given. Finally, in Section 5, the method is illustrated on some model problems.

2. LOCALIZATION OF A ROOT

Let B_i^n be the n-digit binary representation of the integer number $(i - 1)$, counting the leftmost digit first. Then, the *n-binary* matrix

$$M_n^* = [C_{ij}^*] \tag{2.1}$$

is the $2^n \times n$ matrix whose entry in the ith row and jth column is the jth digit of B_i^n. Now, if we replace each zero element in the matrix M_n^* by -1, we shall come up with a new $2^n \times n$ matrix

$$M_n = [C_{ij}], \tag{2.2}$$

which we call an *n-complete* matrix. For example, when $n = 2$, 3 we have, respectively,

$$n = 2 \quad \begin{matrix} B_1^2 = 00 \\ B_2^2 = 01 \\ B_3^2 = 10 \\ B_4^2 = 11 \end{matrix} \rightarrow M_2^* = \begin{bmatrix} B_1^2 \\ B_2^2 \\ B_3^2 \\ B_4^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow M_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix};$$

$$n = 3 \quad \begin{matrix} B_1^3 = 000 \\ B_2^3 = 001 \\ B_3^3 = 010 \\ B_4^3 = 011 \\ B_5^3 = 100 \\ B_6^3 = 101 \\ B_7^3 = 110 \\ B_8^3 = 111 \end{matrix} \rightarrow M_3^* = \begin{bmatrix} B_1^3 \\ B_2^3 \\ B_3^3 \\ B_4^3 \\ B_5^3 \\ B_6^3 \\ B_7^3 \\ B_8^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow M_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now suppose that P is an n-dimensional polyhedron (n-polyhedron) with 2^n vertices. Suppose further that $F = (f_1, f_2, \ldots, f_n)\colon P \subset \mathbb{R}^n \to \mathbb{R}^n$. Then we define the *vector of signs* of F, relative to a vertex X_k of P, by

$$\operatorname{sgn} F(X_k) = (\operatorname{sgn} f_1(X_k), \operatorname{sgn} f_2(X_k), \ldots, \operatorname{sgn} f_n(X_k)), \tag{2.3}$$

where, for any real number a,

$$\operatorname{sgn} a = \begin{cases} -1 & \text{if} \quad a < 0 \\ 0 & \text{if} \quad a = 0. \\ 1 & \text{if} \quad a > 0 \end{cases} \tag{2.4}$$

Moreover, we define the *matrix of signs* of F, relative to the vertices of P, by

$$S(F, P) = [\sigma_{ij}], \tag{2.5}$$

where

$$\sigma_{ij} = \operatorname{sgn} f_j(X_i). \tag{2.6}$$

Of course, the matrix of signs $S(F, P)$ is a $2^n \times n$ matrix such that the entries in its kth row are the corresponding coordinates of the vector $\operatorname{sgn} F(X_k)$. Now, an n-polyhedron CP in $\mathbb{R}^n$ is called a *characteristic n-polyhedron* relative to $F = (f_1, f_2, \ldots, f_n)\colon \mathrm{CP} \to \mathbb{R}^n$, iff the following relationship exists:

$$S(F, \mathrm{CP}) \equiv M_n. \tag{2.7}$$

The *ith vertex* of CP is the vertex of CP for which the coordinates of the vector $\operatorname{sgn} F(X_i)$ are identical to the corresponding entries of the ith row of M_n. Of course, each characteristic n-polyhedron has exactly 2^n vertices. Now, a *proper 1-simplex* of CP is an oriented 1-simplex whose extreme points are vertices of CP, for example, $\langle X_p, X_q \rangle$, for which the corresponding coordinates of the vectors $\operatorname{sgn} F(X_p)$ and $\operatorname{sgn} F(X_q)$ differ from each other only in one case. So, for each vertex of CP, for example, X_i, there are exactly n other vertices X_k of CP such that the 1-simplexes $\langle X_i, X_k \rangle$ are proper 1-simplexes of CP. The subindexes k are given by

$$k = i - 2^{n-j} C_{ij}, \qquad j = 1, 2, \ldots, n, \tag{2.8}$$

where C_{ij} are the corresponding entries of M_n (see [27] for a proof). Finally, each characteristic n-polyhedron has exactly $n2^{n-1}$ proper 1-simplexes. Now suppose that $F = (f_1, f_2, \ldots, f_n)\colon \mathrm{CP} \to \mathbb{R}^n$ is continuous; then, under suitable assumptions on the boundary of CP, the topological degree of F at Θ relative to CP is given by

$$\deg(F, \mathrm{CP}, \Theta) = \pm 1 \tag{2.9}$$

(see [27] for a proof).

Next, we give a procedure for the construction of CP. This procedure starts with an arbitrary point X^o in $\mathbb{R}^n$,

$$X^o = (x_1^o, x_2^o, \ldots, x_n^o), \tag{2.10}$$

and with arbitrary stepsizes in each coordinate direction,

$$H = (h_1, h_2, \ldots, h_n). \tag{2.11}$$

On the basis of X^o and H, we construct an initial n-polyhedron P_o that is a scaled translation of the unit n-cube. More specifically, let G be the rank-1, $n \times n$ matrix, all of whose rows are equal to the row vector X^o; thus,

$$G = [g_{ij}], \tag{2.12}$$

where

$$g_{ij} = x_j^o, \tag{2.13}$$

and let B be the $n \times n$ diagonal matrix with elements $h_1, h_2, \ldots, h_n$. Thus,

$$B = [b_{ij}], \tag{2.14}$$

where

$$b_{ij} = \delta_i^j h_j \tag{2.15}$$

and where δ_i^j is the well-known Kronecker's delta. Then, the coordinates of the vertices of the initial n-polyhedron P_o are given by the corresponding entries of the $2^n \times n$ matrix R, where

$$R = G + M_n^* \cdot B. \tag{2.16}$$

The *ith vertex* of P_o is the vertex of P_o whose coordinates are identical to the corresponding entries of the ith row of the matrix R. Of course, P_o has exactly 2^n vertices. An *edge* of P_o is an oriented 1-simplex whose extreme points are vertices of P_o, for example, $\langle V_p, V_q \rangle$, for which the corresponding coordinates of V_p and V_q differ from each other only in one case. Of course, for each vertex of P_o, for example, V_i, there are exactly n other vertices V_k of P_o such that the 1-simplexes $\langle V_i, V_k \rangle$ are edges of P_o. Moreover, on the basis of the correlation we have determined among the elements of M_n and M_n^*, we easily observe that the subindexes k are also given by the relationship (2.8) and that each P_o has exactly $n2^{n-1}$ edges.

Now, to construct a characteristic n-polyhedron, we compare the matrix $S(F, P_o)$ with the matrix M_n. If they are identical, then P_o is a characteristic n-polyhedron; otherwise, the procedure creates "suitable" points X^* in $\mathbb{R}^n$, such that their vectors of signs of F relative to these points produce the rows of M_n that are missing in $S(F, P_o)$. The points X^* lie in neighborhoods of the roots of the components of F lying on the edges of P_o. More specifically, suppose $A = \{V_1, V_2, \ldots, V_{2^n}\}$ is the ordered set of the vertices of P_o, and let I: $A \to \{1, 2, \ldots, 2^n\}$ be the one-to-one function such that $I(V_i) = i$ for all $i \in \{1, 2, \ldots, 2^n\}$; then, for the ith vertex of P_o, using (2.8) we can find n other subindexes k such that the 1-simplexes $\langle I^{-1}(i), I^{-1}(k) \rangle$ are edges of P_o. Now, for each one of the above pair, for example, (i, l), we assume the corresponding vertices

$$V_i = (v_1, v_2, \ldots, v_m, \ldots, v_n) \quad \text{and} \quad V_l = (v_1, v_2, \ldots, v_m', \ldots, v_n) \tag{2.17}$$

of P_o, which by construction of P_o are determined to have corresponding coordinates that differ from each other only in one case, for example, in the mth. Next, by holding the $v_1, v_2, \ldots, v_{m-1}, v_{m+1}, \ldots, v_n$ fixed, we solve the equations

$$f_s(v_1, v_2, \ldots, v_{m-1}, r_s, v_{m+1}, \ldots, v_n) = 0, \qquad s = 1, 2, \ldots, n \tag{2.18}$$

for r_s in the interval $(\alpha, \alpha + \beta)$, where $\alpha = \min\{|v_m|, |v'_m|\}$ and $\beta = |v_m - v'_m|$, with an accuracy DELTA. Now, suppose that for some s there is a solution r_s of equations (2.18) in $(\alpha, \alpha + \beta)$; then, the following point

$$X^* = (v_1, v_2, \ldots, v_{m-1}, r_s, v_{m+1}, \ldots, v_n) \tag{2.19}$$

lies on the edge $\langle V_i, V_l \rangle$ of P_o. Next, we create the points

$$X_1^* = (v_1, v_2, \ldots, v_{m-1}, r_s + \text{DSTAR}, v_{m+1}, \ldots, v_n)$$
and
$$X_2^* = (v_1, v_2, \ldots, v_{m-1}, r_s - \text{DSTAR}, v_{m+1}, \ldots, v_n), \tag{2.20}$$

where DSTAR is a small positive real number such that DELTA $\le$ DSTAR $< \min\{(r_s - \alpha), (\alpha + \beta - r_s)\}$. Finally, if the sgn $F(X_u^*)$ for some $u = 1, 2$ coincides with any row of M_n, for example, the λth one, which was not present before in $S(F, P_o)$, the vertex V_λ of P_o is replaced by the X_u^*. Now, since F and P_o are arbitrary, we do not know a priori for which edge of P_o and for which component of F we must search for the points X^*. So, we search along all the edges of P_o and all the components of F until a characteristic n-polyhedron emerges. An order of the edges of P_o, which are going to be searched using the above process, can be easily found (see Lemma 4.2).

The construction of a characteristic n-polyhedron fails if either the topological degree of F at Θ relative to the initial n-polyhedron is zero, or the roots of (2.18) cannot be obtained. Of course, we can use any one of the well-known one-dimensional methods [14] to solve equations (2.18). Here, we use the traditional one-dimensional bisection method (see [3] and [16] for a discussion of usefulness), since frequently the edges of P_o are very long and, also, since a few significant digits for the computation of the roots of equations (2.18) are required. So, to solve an equation of the form

$$\varphi(t) = 0, \tag{2.21}$$

where $\varphi: [\gamma_1, \gamma_2] \subset \mathbb{R} \to \mathbb{R}$ is continuous, we recommend the following iterative scheme [5, 24, 26, 27] (which is a simplified version of the bisection method):

$$t_{n+1} = t_n + \operatorname{sgn} \varphi(t_0) \cdot \operatorname{sgn} \varphi(t_n) \cdot h/2^{n+1}, \qquad n = 0, 1, \ldots, \tag{2.22}$$

with $t_0 = \gamma_1$ and $h = \gamma_2 - \gamma_1$. Of course, it converges to a root t^* in (γ_1, γ_2) if, for some t_n, $n = 1, 2, \ldots$ holds that

$$\operatorname{sgn} \varphi(t_0) \cdot \operatorname{sgn} \varphi(t_n) = -1. \tag{2.23}$$

Also, the minimum number of iterations μ, which are required in obtaining an approximate root t' such that $|t' - t^*| \le \varepsilon$, for some $\varepsilon \in (0, 1)$, is given by

$$\mu = \lceil \log_2(h \cdot \varepsilon^{-1}) \rceil, \tag{2.24}$$

where the notation $\lceil a \rceil$ refers to the smallest integer not less than the real number a.

3. EVALUATION OF A ROOT

The generalized method of bisection, which we describe in this section, bisects a characteristic n-polyhedron, CP, in such a way that the new refined n-polyhedron is also a characteristic one. To do this, we bisect a proper 1-simplex of CP in the following way: Let $\langle X_i, X_j \rangle$ be a proper 1-simplex of CP, and let

$$B = \frac{X_i + X_j}{2} \tag{3.1}$$

be its midpoint. We then distinguish the following three cases:

(1) If the vectors sign $F(B)$ and sign $F(X_i)$ are identical, then we replace X_i by B, and the process continues with the next proper 1-simplex.
(2) If the vectors sign $F(B)$ and sign $F(X_j)$ are identical, then we replace X_j by B, and the process continues with the next proper 1-simplex.
(3) Otherwise, the process continues with the next proper 1-simplex.

Where the above function sign a, $a \in \mathbb{R}$ is the known sign function with values

$$\text{sign } a = \begin{cases} -1 & \text{if} \quad a < 0 \\ 1 & \text{if} \quad a \geq 0. \end{cases} \tag{3.2}$$

The above bisection process will be called *characteristic bisection.* Now, it is not difficult to show that, when the characteristic bisection is applied to a proper 1-simplex of CP, the new refined n-polyhedron is also a characteristic one (see [27] for a proof). Of course, an order of the proper 1-simplexes of CP that are going to be bisected using the characteristic bisection can be easily found using relationship (2.8) (see Lemma 4.2). Now, if for the midpoint B of a proper 1-simplex of CP, for example, $\langle X_i, X_j \rangle$, sign $F(B) \neq$ sign $F(X_i)$ and sign $F(B) \neq$ sign $F(X_j)$ exist, then we apply the *relaxation process* [24, 27], which is briefly described below. So, since sign $F(B)$ does not coincide with sign $F(X_i)$ and sign $F(X_j)$, of course, it coincides with sign $F(X_u)$ for some vertex X_u of CP such that $X_u \neq X_i$ and $X_u \neq X_j$. Now, the relaxation process creates the point

$$X_u' = 2B - X_u \tag{3.3}$$

and applies the characteristic bisection using X_u' instead of B. This process is applied repeatedly until, for the point X_u', it obtains one of the following:

$$\begin{array}{ll} (1) & \text{sign } F(X_u') = \text{sign } F(X_i), \\ (2) & \text{sign } F(X_u') = \text{sign } F(X_j); \end{array} \tag{3.4}$$

otherwise, it terminates when the number of its iterations becomes 2; this number is heuristically chosen and very difficult to determine a priori in the general class of problems to which the method is applied.

Now suppose that $\langle X_1, X_2, \ldots, X_{2^n} \rangle$ is the ordered set of vertices of a characteristic n-polyhedron CP; then a *diagonal* of CP is a 1-simplex, say, $\langle X_k, X_l \rangle$, such that all the corresponding components of the vectors sgn $F(X_k)$ and sgn $F(X_l)$ are different from each other [24, 27]. Now, for each vertex of CP, for example, X_k, there is exactly one other vertex X_l such that the 1-simplex $\langle X_k, X_l \rangle$ is a diagonal of CP, where the subindex 1 is given by

$$l = 2^n + 1 - k \tag{3.5}$$

(see [27] for a proof). Of course, each characteristic n-polyhedron has exactly 2^{n-1} diagonals. Moreover, we define the *diameter* of CP as the length of the longest proper 1-simplex of CP (where the distances are measured in Euclidean norms) and denote it by $\Delta(\mathrm{CP})$, while the length of the longest 1-simplex of CP is called the *mesh* of CP and is denoted by $m(\mathrm{CP})$. Finally, we define the midpoint of the longest diagonal of CP as an *estimate of the solution of the system* (1.1), and we denote it by AS.

Now, the number of characteristic bisections of the proper 1-simplexes of a CP that are required in obtaining a new refined characteristic n-polyhedron CP′, such that $\Delta(\mathrm{CP}') \leq \mathrm{EPSILO}$ for some EPSILO $\in (0, 1)$, is given by

$$v = \left\lceil \log_2\left(\frac{\Delta(\mathrm{CP})}{\mathrm{EPSILO}}\right)\right\rceil \tag{3.6}$$

(see [27] for a proof). Furthermore, if $\Delta(\mathrm{CP}) \leq \mathrm{EPSILO}$ exists for some CP, then the following relationships are true:

$$m(\mathrm{CP}) \leq n\ \mathrm{EPSILO}, \tag{3.7}$$

$$\| AS - S \|_2 \leq \frac{n\ \mathrm{EPSILO}}{2} \tag{3.8}$$

(see [24] and [27]), where S in CP is such that $F(S) = \Theta$.

The above generalized method of bisection can become more efficient when the characteristic bisection is repeatedly applied to the diagonals, starting with $\langle X_k, X_l\rangle$, for example, until sign $F((X_k + X_l)/2)$ becomes different from sign $F(X_k)$ and from sign $F(X_l)$. One such order, in which the diagonals can be considered, is easily derived from the result of Lemma 4.3.

4. THE ALGORITHMS AND THEIR DESCRIPTION

In this section we give a detailed step-by-step description of the algorithms. To do this we need to construct the n-complete matrix, as well as specify the order in which the edges of the initial n-polyhedron will be considered for the construction of a characteristic n-polyhedron. The following lemmas will facilitate this whole procedure:

LEMMA 4.1. *The entries C^*_{ij} of the n-binary matrix M^*_n are given by*

$$C^*_{ij} = \left\lfloor \frac{i-1}{2^{n-j}} \right\rfloor - 2\left\lfloor \frac{i-1}{2^{n-j+1}} \right\rfloor, \tag{4.1}$$

where the notation $\lfloor a \rfloor$ refers to the largest integer that is not greater than the real number a.

PROOF. It is easy to see that the above formula is a slight modification of the radix conversion scheme described by Knuth [11, p. 281]. Consequently, since C^*_{ij} is the jth digit of the n-digit binary representation of the number $(i - 1)$, counting the leftmost digit first, then the relationship (4.1) is true. □

LEMMA 4.2. *Let $A = \langle V_1, V_2, \ldots, V_{2^n}\rangle$ be the ordered set of the vertices of an initial n-polyhedron P_o, and let $I: A \rightarrow \{1, 2, \ldots, 2^n\}$ be the one-to-one function such that $I(V_p) = p$, $p = 1, 2, \ldots, 2^n$. Then the 1-simplexes $\langle I^{-1}(i), I^{-1}(k)\rangle$, where*

$$i = (t-1)2^{n-j} + m, \tag{4.2}$$
$$k = i + 2^{n-j},$$

for all $j = 1, 2, \ldots, n$, $t = 1, 3, \ldots, 2^j - 1$, *and* $m = 1, 2, \ldots, 2^{n-j}$, *determine all the* $n2^{n-1}$ *edges of* P_o.

PROOF. First, we observe that

$$\begin{aligned} 1 \le i < 2^n, \\ 1 < k \le 2^n. \end{aligned} \tag{4.3}$$

Next, by combining (4.1) and (4.2) it follows that

$$C_{ij}^* = \left\lfloor t - 1 + \frac{m-1}{2^{n-j}} \right\rfloor - 2\left\lfloor \frac{t-1}{2} + \frac{m-1}{2^{n-j+1}} \right\rfloor. \tag{4.4}$$

Now, since $0 \le (m-1)/2^{n-j} < 1$, we obtain

$$C_{ij}^* = (t-1) - 2\left\lfloor \frac{t-1}{2} \right\rfloor. \tag{4.5}$$

Finally, since $(t-1)$ is an even number, we have

$$C_{ij}^* = (t-1) - \frac{2(t-1)}{2} = 0. \tag{4.6}$$

Consequently, the corresponding entry C_{ij} of the n-complete matrix M_n, is given by

$$C_{ij} = -1. \tag{4.7}$$

From (2.8), using the above relationship and on the basis of the correlation we have determined among the elements of M_n^* and M_n, we observe that the corresponding coordinates of the vertices V_i and V_k of P_o (where the indexes i and k are given by (4.2)) differ from each other only in one case. Thus, the lemma is proven. □

Moreover, using (4.2), we can also find an order for the proper 1-simplexes of a characteristic n-polyhedron that are going to be bisected using the process described in Section 3.

Finally, we give an analogous result for the diagonals of a characteristic n-polyhedron.

LEMMA 4.3. *Let* $X = \langle X_1, X_2, \ldots, X_{2^n} \rangle$ *be the ordered set of the vertices of a characteristic n-polyhedron CP, and let* $T: X \to \{1, 2, \ldots, 2^n\}$ *be the one-to-one function such that* $T(X_p) = p$, $p = 1, 2, \ldots, 2^n$. *Then, the 1-simplexes* $\langle T^{-1}(k), T^{-1}(l) \rangle$, *where*

$$l = 2^n + 1 - k, \qquad \textit{for all} \quad k = 1, 2, \ldots, 2^{n-1}, \tag{4.8}$$

determine all the 2^{n-1} *diagonals of CP.*

PROOF. It follows directly from relationship (3.5). □

Now we can proceed with a detailed step-by-step description of the algorithms:

Algorithm 4.1. Construction of a characteristic n-polyhedron.

(1) Input $\{n, F, X^o, H, \text{DELTA}, \text{EPSILO}\}$.

(2) Compute the machine precision EPSMCH [4, 12].
- (2.1) Check whether DELTA < EPSMCH.
  (Yes): Set DELTA = $\frac{1}{16}$.
  (No): Go to (2.2).
- (2.2) Check whether EPSILO < EPSMCH.
  (Yes): Set EPSILO = EPSMCH.
  (No): Go to (2.3).
- (2.3) Set DSTAR = DELTA + 2EPSMCH.

(3) Construct the n-binary matrix $M_n^* = [C_{ij}^*]$ and the n-complete matrix $M_n = [C_{ij}]$. To do this, execute Steps (3.1)–(3.3).
- (3.1) Repeat Steps (3.2)–(3.3) in sequence for $i = 1$ to 2^n.
- (3.2) Repeat Step (3.3) in sequence for $j = 1$ to n.
- (3.3) Set $C_{ij}^* = \lfloor (i-1)/2^{n-j} \rfloor - 2\lfloor (i-1)/2^{n-j+1} \rfloor$,
  $C_{ij} = 2C_{ij}^* - 1$.

(4) Construct the initial n-polyhedron P_o. Thus, on the basis of X^o and H, using relationships (2.12)–(2.16), construct P_o and store the coordinates of its vertices in the corresponding entries of a $2^n \times n$ matrix $R = [v_{ij}]$.

(5) Set all the entries of a $2^n \times n$ matrix $X = [x_{ij}]$ equal to the corresponding entries of the matrix R.

(6) Construct the indexing array $W = [w_i]$ with length 2^n and entries $w_i = i$. Also, set all the entries of an array AS with length n, equal to zero.

(7) Check whether the initial n-polyhedron is a characteristic one. To do this, execute Steps (7.1)–(7.8).
- (7.1) Set $i = 0$,
  $m = 0$.
- (7.2) If $i = 2^n$ go to (8); otherwise, replace i by $i + 1$ and continue.
- (7.3) Compute the function value at the entries of the ith row, R_i, of R.
- (7.4) Check whether $\| F(R_i) \|_\infty \le$ EPSILO.
  (Yes): AS $\leftarrow R_i$ (the symbol "$\leftarrow$" reads "is replaced by").
  Go to (21).
  (No): Go to (7.5).
- (7.5) If $m = 2^n$ go to (7.2); otherwise, replace m by $m + 1$ and continue.
- (7.6) Test whether sgn $F(R_i)$ coincides with the mth row of M_n.
  (Yes): Go to (7.7).
  (No): Return to (7.5).
- (7.7) Check whether $w_m = 0$.
  (Yes): Go to (7.2).
  (No): Store R_i in the mth row of X,
  Set $w_m = 0$,
  Go to (7.2).
- (7.8) Check whether there is any entry of W other than zero.
  (Yes): Go to (8).
  (No): Go to (22).

(8) Find the order in which the extreme points of the edges of P_o occur. To do this, set all the entires of an $n2^{n-1} \times 2$ matrix $O = [o_{ij}]$ equal to zero, and execute Steps (8.1)–(8.6).

(8.1) Set $i = 0$.

(8.2) Repeat Steps (8.3)–(8.6) in sequence for $j = 1$ to n.

(8.3) Repeat Steps (8.4)–(8.6) in sequence for $t = 1, 3, 5, \ldots, 2^j - 1$.

(8.4) Repeat Steps (8.5)–(8.6) in sequence for $m = 1$ to 2^{n-j}.

(8.5) $i \leftarrow i + 1$

(8.6) Set $o_{i1} = (t - 1)2^{n-j} + m$,
$o_{i2} = o_{i1} + 2^{n-j}$.

(9) Set $i = 0$.

(10) If $i = n2^{n-1}$ go to (19); otherwise, replace i by $i + 1$ and continue.

(11) Set $p = o_{i1}$,
$q = o_{i2}$,
$j = 0$.

(12) If $j = n$ go to (10); otherwise, replace j by $j + 1$ and continue.

(13) Check whether $v_{pj} \neq v_{qj}$ and $v_{pk} = v_{qk}$ for all $k = 1, 2, \ldots, j - 1, j + 1, \ldots, n$.
(Yes): Go to (14).
(No): Return to (12).

(14) Set all the entries of an array $Y = [y_l]$ with length n equal to zero.

(15) Set $\beta = | v_{pj} - v_{qj} |$,
$\alpha = \min\{| v_{pj} |, | v_{qj} |\}$,
$y_k = v_{pk}$, for all $k = 1, 2, \ldots, j - 1, j + 1, \ldots, n$,
$y_j = \alpha$.

(16) Set all the entries of an $n \times 2$ matrix $Z = [z_{lm}]$ equal to zero. Also,
set $s = 0$,
$k = 0$.

(17) If $s = n$ go to (18); otherwise, replace s by $s + 1$ and continue.

(17.1) Solve the equation $f_s(y_1, y_2, \ldots, y_{j-1}, r_s, y_{j+1}, \ldots, y_n) = 0$ for r_s, within the interval $(\alpha, \alpha + \beta)$, using formula (2.22) with accuracy DELTA.

(17.2) Check whether $\alpha + \text{DSTAR} \leq r_s \leq \alpha + \beta - \text{DSTAR}$.
(Yes): Set $z_{s1} = r_s + \text{DSTAR}$,
$z_{s2} = r_s - \text{DSTAR}$,
Return to (17).
(No): Return to (17).

(18) If $k = 2$ go to (19); otherwise, replace k by $k + 1$ and continue.

(18.1) Set $s = 0$,
$m = 0$.

(18.2) If $s = n$ go to (18); otherwise, replace s by $s + 1$ and continue.

(18.3) Check whether $z_{sk} = 0$.
(Yes): Return to (18.2).
(No): Set $y_j = z_{sk}$,
Go to (18.4).

(18.4) Check whether $\| F(Y) \|_\infty \leq$ EPSILO.
(Yes): AS $\leftarrow Y$,
Go to (21).
(No): Go to (18.5).

(18.5) If $m = 2^n$ go to (18); otherwise, replace m by $m + 1$ and continue.

(18.6) Test whether sgn $F(Y)$ is identical to the mth row of M_n.
(Yes): Go to (18.7).
(No): Return to (18.5).

(18.7) Check whether $w_m = 0$.
(Yes): Return to (18.2).
(No): Store Y in the mth row of X.
Set $w_m = 0$.
Check whether there is any entry of W other than zero.
(Yes): Return to (18).
(No): Go to (22).

(Steps (19) and (20), which follow, are based on numerical experience gathered in the process of testing this algorithm in accordance with the bisection portion of the method. The idea behind these steps is to construct a new characteristic n-polyhedron that is a scaled translation of the unit n-cube in such a way that not more than two of its vertices lie on the same edge of the initial n-polyhedron).

(19) Reconstruct a characteristic n-polyhedron in such a way that its vertices are of a scaled translation of the unit n-cube. To do this, execute Steps (19.1)–(19.11).

(19.1) Set all the entries of the arrays $E = [e_j]$, $U = [u_j]$ of length n equal to zero.

(19.2) Set $i = 0$,
$m = 0$.

(19.3) Repeat Steps (19.4)–(19.5) in sequence for $j = 1$ to n.

(19.4) $e_j = \min_j \{x_{ij}, \text{ for all } i = 1, 2, \ldots, 2^n\}$.

(19.5) $u_j = \max_j \{x_{ij}, \text{ for all } i = 1, 2, \ldots, 2^n\} - e_j$.

(19.6) Construct the matrix R using relationships (2.12)–(2.16) and vectors E and U, instead of X^o and H, respectively.

(19.7) If $i = 2^n$ go to (20); otherwise, replace i by $i + 1$ and continue.

(19.8) Compute $F(R_i)$ for the ith row R_i of R.

(19.9) Check whether $\| F(R_i) \|_\infty \leq$ EPSILO.
(Yes): AS $\leftarrow R_i$,
Go to (21).
(No): Go to (19.10).

(19.10) If $m = 2^n$ go to (19.7); otherwise, replace m by $m + 1$ and continue.

(19.11) Test whether sgn $F(R_i)$ coincides with the mth row of M_n.
(Yes): Store R_i in the mth row of X.
Go to (19.7).
(No): Return to (19.10).

(20) Reconstruct a characteristic n-polyhedron in such a way that not more than two of its vertices lie on the same edge of the initial n-polyhedron. To do this, execute Steps (20.1)–(20.11).

(20.1) Set $i = 0$,
$j = 0$,
$m = 0$.

(20.2) If $i = 2^{n-1}$ go to (22); otherwise, replace i by $i + 1$ and continue.

(20.3) Set $l = 1 - i + 2^n$.

(20.4) If $j = n$ go to (20.2); otherwise, replace j by $j + 1$ and continue.

(20.5) Check whether $x_{ij} = x_{lj}$.
(Yes): Go to (20.6).
(No): Return to (20.4).

(20.6) Execute Steps (20.7)–(20.10) in sequence for $k = i$ and $k = l$.

(20.7) Check whether $x_{kj} = x_j^o$.
(Yes): Set $x_{kj} = x_j^o + h_j$.
Go to (20.8).
(No): $x_{kj} \leftarrow x_{kj} - h_j$.
Go to (20.8).

(20.8) Check whether $\| F(X_k) \|_\infty \leq$ EPSILO, where $X_k = (x_{k1}, x_{k2}, \ldots, x_{kn})$.
(Yes): AS $\leftarrow X_k$.
Go to (21).
(No): Go to (20.9).

(20.9) If $m = 2^n$ go to (20.2); otherwise, replace m by $m + 1$ and continue.

(20.10) Check whether sgn $F(X_k)$ is identical to the mth row of M_n.
(Yes): Check whether $k = m$.
(Yes): Store X_k in the mth row of X.
Return to (20.2).
(No): Return to (20.6).
(No): Return to (20.9).

(20.11) Go to (22).

(21) Take AS as an approximate solution of system (1.1) and terminate.

(22) Take the characteristic n-polyhedron CP so that the coordinates of its vertices are the corresponding entries of the matrix X, and terminate.

Algorithm 4.2 A generalized method of bisection.

(1) Input $\{n, F, \text{EPSILO}, M_n, O, \text{and } X\}$.

(1.1) Set ZETA = $2n$ EPSILO

(2) Compute the number of iterations. To do this, execute Steps (2.1)–(2.5).

(2.1) Repeat Steps (2.2)–(2.3) in sequence for $i = 1$ to $n2^{n-1}$.

(2.2) Set $p = o_{i1}$,
$q = o_{i2}$.

(2.3) Compute $D_i = \| X_p - X_q \|_2$.

(2.4) Compute $D = \max_i \{D_i\}$.

(2.5) Compute $\nu = \lceil \log_2(D(n \text{ EPSILO}/2)^{-1}) \rceil$.

(3) Set all the entries of an array AS, with length n, equal to zero. Also, set $t = 0$.

(4) If $t = v$ go to (11); otherwise, replace t by $t + 1$ and continue.

(5) Bisect the diagonals. To do this, execute Steps (5.1)–(5.6).
(5.1) Set $i = 0$,
$m = 0$.
(5.2) If $i = 2^{n-1}$ go to (6); otherwise, replace i by $i + 1$ and continue.
(5.3) Set $j = 1 - i + 2^n$,
$B = (X_i + X_j)/2$.
(5.4) Check whether $\| F(B) \|_\infty \leq$ EPSILO.
(Yes): AS $\leftarrow B$.
Go to (12).
(No): Go to (5.5).
(5.5) If $m = 2^n$ go to (5.2); otherwise, replace m by $m + 1$ and continue.
(5.6) Check whether sign $F(B)$ is identical to the mth row of M_n.
(Yes): $X_m \leftarrow B$,
Check whether ($m = i$ or $m = j$).
(Yes): Go to (5.3).
(No): Go to (5.2).
(No): Return to (5.5).

(6) Check whether the length of the longest diagonal of the characteristic n-polyhedron is smaller than ZETA.
(Yes): Go to (11).
(No): Go to (7).

(7) Bisect the proper 1-simplexes. To do this, execute Steps (7.1)–(7.8).
(7.1) Construct the indexing array $W = [w_i]$ with length 2^n and entries $w_i = i$.
(7.2) Set $i = 0$,
$m = 0$,
$r' = 0$.
(7.3) If $i = n2^{n-1}$ go to (8); otherwise, replace i by $i + 1$ and continue.
(7.4) Set $p = o_{i1}$,
$q = o_{i2}$,
$r = 0$,
$B = (X_p + X_q)/2$.
(7.5) Check whether $\| F(B) \|_\infty \leq$ EPSILO.
(Yes): AS $\leftarrow B$.
Go to (12).
(No): Go to (7.6).
(7.6) If $m = 2^n$ go to (7.3); otherwise, replace m by $m + 1$ and continue.
(7.7) Check whether sign $F(B)$ is identical to the mth row of M_n.
(Yes): $G \leftarrow X_m$,
$X_m \leftarrow B$,
$w_m = 0$,
Check whether ($m \neq p$ and $m \neq q$).

(Yes): $r \leftarrow r + 1$,
$r' = 1$.
Go to (7.8).
(No): Go to (7.3).
(No): Return to (7.6).

(7.8) Test whether $r > 2$.
(Yes): Return to (7.3).
(No): $B \leftarrow 2X_m - G$.
Go to (7.5).

(8) Check whether there is any entry of W other than zero.
(Yes): Check whether $r' = 0$.
(Yes): Go to (4).
(No): Go to (9).
(No): Go to (4).

(9) Execute Step (19) of Algorithm 4.1. If a point AS is obtained so that $\| F(\text{AS}) \|_\infty \leq$ EPSILO, then go to (12).

(10) Go to (4).

(11) Find the longest diagonal of the characteristic n-polyhedron; take its mid-point as the final approximate solution and terminate.

(12) Take AS as the final approximate solution and terminate.

5. EXAMPLES

The algorithms described in Section 4 have been implemented using the new FORTRAN program CHABIS (CHAracteristic BISection). CHABIS was tested on the University of Patras UNIVAC 1100/60 system, and on the Cornell University IBM 3090-600E (supercomputer mainframe), IBM 4381, VAX 8530, as well as on a SPERRY IT IBM PC compatible, with random problems of varying dimensions. Our experience is that the procedures behaved predictably and reliably, and the results were quite satisfactory. Some typical computational results are given below. For the following problems, the reported parameters are

—n dimension,
—$X^o = (x_1^o, x_2^o, \ldots, x_n^o)$ starting point,
—$H = (h_1, h_2, \ldots, h_n)$ stepsizes in each coordinate direction,
—CP characteristic n-polyhedron obtained using DELTA $= \frac{1}{16}$,
—AS approximate solution computed so that $\| F(\text{AS}) \|_\infty \leq$ EPSILO $= 10^{-8}$, and
—NFCALL number of function evaluations on an IBM 4381.

Problem 5.1. *STENGER function* (1975) [3, 6, 10, 17]. This example gives the solution of system (1.1) for

$$F(x_1, x_2) = (x_1^2 - 4x_2, x_2^2 - 2x_1 + 4x_2).$$

As our starting values, we utilized $X^o = (0.1, 0.1)$ and $H = (4000, 4000)$. We obtained a CP and AS = (1.6954152, 0.71860817), after NFCALL = 107. By changing X^o to $X^o = (-2000, -2000)$ and H to $H = (2000\frac{1}{3}, 4000)$, we obtained a

CP and AS = (0, 0), after NFCALL = 94. Finally, starting with $X^o = (-1, -0.4)$ and $H = (2, 0.8)$, we obtained a CP and AS = (0, 0), after NFCALL = 5.

Problem 5.2. *ROSENBROCK function* (1960) [3, 10, 13]. In this case, F is given by

$$F(x_1, x_2) = (1 - x_1, 10(x_2 - x_1^2)).$$

Starting with the point $X^o = (-2000, -2000)$ and taking $H = (4000, 4000)$ after NFCALL = 113, we obtained a CP and AS = (1, 1). Then, by changing X^o to $X^o = (-2, -10)$ and H to $H = (4, 16)$, we obtained a CP and AS = (1, 1), after NFCALL = 24.

Problem 5.3. Here we have the solution of system (1.1) for

$$f_1(x_1, x_2) = \begin{cases} \dfrac{x_1^3 - x_2^3}{x_1^2 + x_2^2} & \text{if} \quad (x_1, x_2) \neq (0, 0), \\ 0 & \text{if} \quad (x_1, x_2) = (0, 0), \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} \dfrac{x_1^3 + x_2^3}{x_1^2 + x_2^2} & \text{if} \quad (x_1, x_2) \neq (0, 0), \\ 0 & \text{if} \quad (x_1, x_2) = (0, 0). \end{cases}$$

Starting with the point $X^o = (-100, -1000)$ and stepsizes $H = (120, 1020)$, we obtained the corresponding CP and AS = (0, 0), utilizing NFCALL = 115. Then for $X^o = (-100, -100)$ and $H = (200, 200)$, we obtained the corresponding CP and AS = (0, 0), utilizing NFCALL = 38.

Problem 5.4. *IDENTITY function* [3, 10]. In this case, F is defined by

$$F(X) = X, \qquad \text{where} \quad X \in \mathbb{R}^3.$$

Starting with $X^o = (-2000, -2000, -2000)$ and stepsizes $H = (3000, 3000, 3000)$, we obtained the corresponding CP and AS = (0, 0, 0), utilizing NFCALL = 45.

Problem 5.5. *Extended EIGER–SIKORSKI–STENGER function* (1984) [3, 10]. In this case, F is defined by

$$f_i(x_1, x_2, \ldots, x_n) = (x_i - 0.1)^2 + x_{i+1} - 0.1, \qquad \text{for all} \quad i = 1, 2, \ldots, n-1,$$
$$f_n(x_1, x_2, \ldots, x_n) = (x_n - 0.1)^2 + x_1 - 0.1.$$

We utilized $n = 2, 3, \ldots, 9$, the corresponding starting points $X^o = (-2000, -2000, \ldots, -2000)$, and the corresponding stepsizes $H = (2000, 2000, \ldots, 2000)$. Then we obtained the corresponding CP and AS = $(-0.9, -0.9, \ldots, -0.9)$, utilizing NFCALL = 41, 45, 53, 69, 101, 165, 293, and 549, respectively.

Problem 5.6. *Extended KEARFOTT function* (1977) [7, 8]. Here we have the solution of system (1.1) for

$$f_i(x_1, x_2, \ldots, x_n) = x_i^2 - x_{i+1}, \qquad \text{for all} \quad i = 1, 2, \ldots, n-1,$$
$$f_n(x_1, x_2, \ldots, x_n) = x_n^2 - x_1.$$

Table I. Comparison of ESS Bisection Method with Our New Generalized Bisection Method

		ESS bisection method			Characteristic bisection method		
Problem	n	$\Delta(SS)$	min NFCALL	max NFCALL	$\Delta(P_o)$	$m(P_o)$	NFCALL
5.1	2	0.75	45	65	4,000	5,656.85	107
					4,000	4,472.28	94
					2	2.15	5
5.2	2	7.50	58	70	4,000	5,656.85	113
					16	16.49	24
5.4	3	0.61	66	106	3,000	5,196.15	45
5.5	4	0.53	106	132	2,000	4,000.00	53

We took $n = 2, 3, \ldots, 9$, the corresponding starting points $X^o = (0.1, 0.1, \ldots, 0.1)$, and the corresponding stepsizes $H = (2000, 2000, \ldots, 2000)$. Then we obtained the corresponding CP and AS = (1, 1, . . . , 1), after NFCALL = 41, 45, 53, 69, 101, 165, 293, and 549, respectively.

In Table I we compare the numerical results obtained by the Eiger–Sikorski–Stenger (ESS) bisection method [3] with the corresponding numerical results of the method presented in this paper, obtained on the Cornell University IBM 4381. The reported parameters of Table I indicate:

n: dimension;
Δ(SS): the length of the longest edge of the starting n-simplex, SS, of the ESS method;
$\Delta(P_o)$: the length of the longest edge of our initial n-polyhedron P_o;
$m(P_o)$: the length of the longest 1-simplex of P_o; and
NFCALL: the total number of evaluations of F that are required in localizing and evaluating a root with accuracy 10^{-8}.

We now compare the numerical results obtained by the Kearfott nontopological degree-based bisection method [9, 10] with the corresponding numerical results obtained by the method used in this paper. The goal of Kearfott's method is to find all roots within a specified region reliably, though this requires Jacobian matrix evaluations. So, for comparison purposes, we use Kearfott's "equivalent function evaluations" [10], defined as $5 \times (\text{NFCALL} + n \times \text{NJCALL})$, where NFCALL and NJCALL are the number of function evaluations and the number of Jacobian matrix evaluations, respectively. Computational results of Problems 5.1, 5.2, 5.4, and 5.5 are reported in [10], where a domain-stopping tolerance of $\varepsilon = 10^{-5}$ and a range-stopping tolerance $\varepsilon_F = 10^{-10}$ are used. In this comparison, we utilized initial n-polyhedrons with the same size of initial boxes as in [10]. In Table II we present Kearfott's equivalent NFCALL and our number of function evaluations NFCALL required for the construction of a characteristic n-polyhedron, and the evaluation of an approximate solution with accuracy EPSILO = 10^{-10}. In Problem 5.1 the value of the topological degree relative to the initial 2-polyhedron $[-4, 4]^2$ is zero, causing the nonconstruction of a

Table II. Comparison of Kearfott's Bisection Method with Our New Generalized Bisection Method

		Kearfott's bisection method		Characteristic bisection method	
Problem	n	Initial box	Equivalent NFCALL	Initial n-polyhedron	NFCALL
5.1	2	$[-4, 4]^2$	695	$[-4, 4]^2$	21
5.2	2	$[-4, 4]^2$	30	$[-4, 4]^2$	19
5.4	3	$[-.25, .25]^3$	20	$[-.25, .25]^3$	9
5.5	4	$[-.2, .2]^4$	100	$[-.2, .2]^4$	18

characteristic 2-polyhedron. Nevertheless, when using this nonconstructed polyhedron, the bisection portion of the algorithm converges to an approximate solution within the required accuracy.

6. CONCLUDING REMARKS

This paper describes two algorithms for the numerical solution of a system of n nonlinear equations in n variables ($n \geq 2$). The first algorithm locates at least one root of system (1.1) within the n-polyhedron; the second applies a generalized bisection method in order to compute an approximate solution of the system. These algorithms are primarily useful for small dimensions ($n \leq 10$), since they cannot avoid a generic attribute of all n-dimensional bisection methods, that is, the significant amount of computational work needed for large dimensions (cf. [3, 7, 8]).

However, the algorithms in this paper avoid all computations concerning topological degree. They require only that the algebraic signs of the function evaluations be correct, so that they can be applied to problems with imprecise function values. Moreover, they can be applied to nondifferentiable continuous functions and do not involve calculations of derivatives or approximations of such derivatives. Furthermore, their analysis is such that its generalization to a higher dimension is quite straightforward, since they are fully automated and handle the dimensionality of the problem just as a parameter (cf. [6, 15]).

In addition, the bisection algorithm has the advantages of the traditional one-dimensional bisection method; that is, we can find out the number of iterations that are needed for the attainment of an approximate root to a predetermined accuracy beforehand. Also, the starting estimate of the root does not have to be near the root.

Furthermore, the algorithm for the localization of the root can be used as a starting procedure for obtaining good initial estimations of the root for other methods, for which good initial approximations are a condition sine qua non.

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