International Journal of Bifurcation and Chaos, Vol. 6, No. 8 (1996) 1425–1437

PERIODIC ORBITS AND INVARIANT SURFACES OF 4D NONLINEAR MAPPINGS

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Received April 25, 1995; Revised November 22, 1996

The accurate computation of periodic orbits and the knowledge of their stability properties are very important for studying the behavior of many physically interesting dynamical systems. In this paper, we describe first an efficient numerical method for computing periodic orbits of 4D mappings of any period and to any desired accuracy. This method always converges rapidly to a periodic orbit independently of the initial guess, which is very useful when the mapping has many periodic orbits close to each other, as in the case of conservative maps. We illustrate this method on a 4D symplectic mapping, by computing some of its periodic orbits and determining their particular arrangement in the 4D space, according to their stability characteristics. We then obtain periodic orbits associated with sequences of (rational) winding numbers converging on a pair of irrationals and discuss the possible existence of an analogue of Greene's criterion for 4D symplectic mappings.

1. Introduction

The four-dimensional (4D) symplectic mapping

$$T: \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} R(\omega_1) & \mathcal{O} \\ \mathcal{O} & R(\omega_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}, \tag{1}$$

where $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$, and

$$R(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}, \quad a \in [0, \pi], \tag{2}$$

is an interesting model, which is of direct relevance to the dynamics of particle beams passing repeatedly through FODO cells of magnetic focusing elements [Bountis & Tompaidis, 1991; Bountis & Kollmann, 1994]. In a recent study [Vrahatis & Bountis, 1994; Vrahatis, 1995] we demonstrated how one may obtain high period 4D orbits of a conservative (but not symplectic) mapping of the type

$$T: \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} R(\omega) & \mathcal{O} \\ \mathcal{O} & R(\omega) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - x_1^2 + x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}, \tag{3}$$

containing only one frequency ω. This mapping represents a complex extension (in 4D) of the well known Hénon's 2D mapping [Hénon, 1969]:

$$T: \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - x_1^2 \end{pmatrix}, \tag{4}$$

whose periodic orbits have also been studied extensively by our methods [Vrahatis *et al.*, 1993; Vrahatis & Bountis, 1994; Vrahatis, 1995].

As is well known, N-dimensional mappings of the form:

$$X' = T(X), \quad X = (x_1, x_2, \ldots, x_N), \qquad (5)$$

can model conservative dynamical systems, if the determinant of the Jacobian of the map is unity, i.e., $|\det J_T| = 1$, or dissipative ones if $|\det J_T| < 1$ [Birkhoff, 1917; Bountis & Helleman, 1981]. We say that $X = (x_1, x_2, \ldots, x_N)$ is a fixed point of T of *order* p (or a periodic orbit of *period* p), if:

$$X = T^p(X) \equiv \underbrace{T(T(\cdots(T(X))\cdots))}_{p \text{ times}},$$

$$p = 1, 2, 3, \ldots. \qquad (6)$$

In general, it is difficult to find in the literature efficient methods for computing periodic orbits of high period for $N(> 2)$-dimensional maps. Also, traditional iterative schemes, such as Newton's method and related classes of algorithms [Ortega & Rheinboldt, 1970], often fail since they converge to the same fixed point almost independently of the initial guess, while there may exist several other fixed points nearby, which are all desirable for the applications. Moreover, these methods are generally affected by the mapping evaluations taking large values, or may break down due to the nonexistence of derivatives or poorly behaved partial derivatives near fixed points [Ortega & Rheinboldt, 1970].

In this paper, we describe and apply an efficient numerical method for rapidly computing periodic orbits of N-dimensional maps to any desired accuracy. This method exploits topological degree theory to provide a criterion for the existence of a periodic orbit of an iterate of the mapping within a given region.

Our method, referred to here as the CP (characteristic polyhedron)-criterion, is especially useful for the computation of high period orbits and is quite efficient, since the only computable information required is the algebraic sign of the components of the mapping. Thus, it is not affected by the mapping evaluations taking large values in the neighborhood of unstable periodic orbits. It is important to note, however, that the computational effort of this method grows exponentially with the dimension of the map, N.

In Sec. 2, we start by describing how our criterion is used to locate periodic orbits inside a converging sequence of smaller and smaller characteristic polyhedra. Also, in that section, we describe a generalized bisection method used in combination with the CP-criterion to compute the desired periodic orbit to any accuracy.

In Sec. 3, this procedure is applied to the calculation of periodic orbits of the 4D symplectic mapping (1) for periods which reach up to the thousands. More specifically, we have been able to apply our procedure to compute periodic orbits which for large period approach invariant surfaces of dimensions 1 and 2, around the origin of 4D space.

As long as there still exist stable periodic orbits, by slightly varying the initial conditions of these orbits, we can find better and better approximations of invariant surfaces. On the other hand, by slightly changing the coefficient of one of the nonlinear terms of the mapping, it is possible to move collectively a sequence of periodic orbits closer to, or away from, destabilization.

Thus, we conclude that, if an analogue of Greene's criterion for 2D maps [Greene, 1979; MacKay, 1982; Falcolini & de la Llave, 1992; MacKay, 1992] exists for 4D maps of the type (1) it should be considerably more complicated than what has been recently found for higher-dimensional torus maps, satisfying the twist condition [Bollt & Meiss, 1992; Tompaidis, 1994].

Finally we end, in Sec. 4, with some concluding remarks and a discussion of ongoing work on the application of these methods, to gain a better understanding of the structure of the dynamics and global stability of orbits around the origin of higher-dimensional maps.

2. The CP-Criterion

In this section, we implement topological degree theory to give a criterion for the existence of a periodic orbit within a given region of the phase space of the system. This criterion is based on the construction of a sequence of "characteristic polyhedra" within a scaled translation of the unit cube. The concept of a characteristic polyhedron will be reviewed and a procedure for its construction will be presented. The theoretical development of the concepts employed here can be found in [Vrahatis & Iordanidis, 1986; Vrahatis, 1988a; 1988b].

As was said previously, the problem of finding periodic orbits of nonlinear mappings $T : \mathcal{D} \subset$

$\mathbf{R}^n \to \mathbf{R}^n$ of period p amounts to finding points $X^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star) \in \mathcal{D}$ which satisfy the following equation:

$$T^p(X^\star) = X^\star . \tag{7}$$

Obviously, finding such a periodic orbit is equivalent to solving the following system:

$$F(X) = \mathcal{O} , \tag{8}$$

with $F = (f_1, f_2, \ldots, f_n) = T^p - I_n$, where I_n is the $n \times n$ identity matrix and $\mathcal{O} = (0, 0, \ldots, 0)$ is the origin of $\mathbf{R}^n$.

Many problems require the solution of systems of nonlinear equations for which Newton's method and related classes of algorithms [Ortega & Rheinboldt, 1970] fail due to the nonexistence of derivatives or poorly behaved partial derivatives. Also, Newton's method (as well as Newton-like methods), often converge to the same solution $X^\star$ of $F(X) = \mathcal{O}$ almost independently of the initial guess, while there may exist several solutions nearby all of which are desired for the application. Because of these reasons, various approaches based upon topological degree theory and generalized bisection methods have been investigated in recent years (see, for example, [Greene, 1992; Vrahatis, 1986; 1988a]).

Bisection methods for finding solutions of systems of equations depend on a criterion, which guarantees that a solution lies within a given region. Then this region is subdivided in such a way that the criterion can be applied again to the refined region, etc.

In one dimension, this criterion consists of the product of the signs of the function evaluations at the endpoints of a given interval. Specifically, if one wishes to locate a solution of an equation $f(x) = 0$ in the interval (a, b) where $f : (a, b) \subset \mathbf{R} \to \mathbf{R}$ one examines whether the following relation is fulfilled:

$$\operatorname{sgn} f(a) \operatorname{sgn} f(b) = -1 , \tag{9}$$

where sgn is the sign function with values:

$$\operatorname{sgn} \psi = \begin{cases} -1, & \text{if} \quad \psi < 0; \\ 0, & \text{if} \quad \psi = 0; \\ 1, & \text{if} \quad \psi > 0. \end{cases} \tag{10}$$

If (9) holds, then we know that there is at least one solution within (a, b). This is known as *Bolzano's existence criterion* and can be generalized to higher dimensions [Miranda, 1940; Vrahatis, 1989].

Instead of Bolzano's criterion, however, one may alternatively consider

$$\deg[f,\ (a, b),\ 0] = \frac{1}{2}\{\operatorname{sgn} f(b) - \operatorname{sgn} f(a)\} , \tag{11}$$

where $\deg[f,\ (a, b),\ 0]$ is the *topological degree of f at the origin relative to* (a, b). Now, if the value of $\deg[f,\ (a, b),\ 0]$ is not zero, we know that there is at least one solution in (a, b), since, in that case, Bolzano's criterion is fulfilled. The value of $\deg[f,\ (a, b),\ 0]$ gives additional information concerning the behavior of the solutions of $f(x) = 0$ in (a, b) relative to the slopes of f [Greene, 1992]. For example, if $\deg[f,\ (a, b), 0] = 1$ (which means that $f(b) > 0$ and $f(a) < 0$), then the number of solutions at points where $f(x)$ has a positive slope exceeds by one the number of solutions at points at which $f(x)$ has a negative slope.

The topological degree, as well as Bolzano's criterion, transfer all information regarding the roots to the boundary of the given region. Now, using the value of the topological degree (or Bolzano's criterion), one can calculate a solution of $f(x) = 0$ by bisecting the interval (a, b). So we subdivide (a, b) into two subintervals $(a, c]$, $[c, b)$, where $c = (a + b)/2$ is the midpoint of (a, b), and keep the subinterval for which the value of the topological degree is not zero (relative to itself), by checking the information on the boundaries. In this way, we keep at least one solution within a smaller interval and continue this procedure until the endpoints of the final subinterval differ from each other by less than a fixed amount. This method is called *bisection method* and can be expressed as follows [Vrahatis & Iordanidis, 1986; Vrahatis, 1988a; Vrahatis *et al.*, 1990]:

$$x_{n+1} = x_n + \operatorname{sgn} f(a) \operatorname{sgn} f(x_n)(b - a)/2^{n+1} , \qquad x_0 = a , \qquad n = 0, 1, \ldots . \tag{12}$$

It converges, of course, to a solution $x^\star$ in (a, b) if for some x_n, $n = 1, 2, \ldots$, we have

$$\operatorname{sgn} f(x_0) \operatorname{sgn} f(x_n) = -1 . \tag{13}$$

Moreover, the minimum number of iterations γ, which are required to obtain an approximate solution x' such that $|x' - x^\star| \le \varepsilon$ for some $\varepsilon \in (0, 1)$ is given by:

$$\gamma = \lceil \log_2((b - a)\varepsilon^{-1}) \rceil , \tag{14}$$

here the notation $\lceil \cdot \rceil$ refers to the smallest integer, hich is not less than the real number quoted.

It would be desirable, of course, to generalize ıe above bisection method to higher dimensions. o do this we extend Bolzano's criterion in the fol- wing way: Let us define a characteristic polyhe- ron by constructing the $2^n \times n$ matrices $\mathcal{M}_n$ whose ›ws are formed by all possible combinations of -1, . For example, for $n = 1, 2, 3$, we have:

$$\mathcal{M}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{M}_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{15}$$

Jow, for $n = 1$, we consider the segment $[x_1, x_2]$,nd evaluate the sign of $f(x)$ at the endpoints. Then, if the matrix

$$\mathcal{S}(f;\ [x_1, x_2]) = \begin{bmatrix} \operatorname{sgn} f(x_1) \\ \operatorname{sgn} f(x_2) \end{bmatrix} \tag{16}$$

ıgrees with $\mathcal{M}_1$, up to a permutation of the rows, ve say that $[x_1, x_2]$ is a *characteristic polyhedron*. Suppose now that $\Pi^n = \langle \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{2^n} \rangle$ is an ›riented n-dimensional polyhedron with 2^n vertices, $\Upsilon_k \in \mathbf{R}^n$ (i.e., an orientation has been assigned to ts vertices), and let $F = (f_1, f_2, \ldots, f_n) : \Pi^n \subset \mathbf{R}^n \to \mathbf{R}^n$ be a nonlinear mapping from Π^n into $\mathbf{R}^n$. Then we call a *matrix of signs associated with F and* Π^n, and denote it by $\mathcal{S}(F;\ \Pi^n)$, the $2^n \times n$ matrix vhose entries in the kth row are the corresponding :oordinates of the vector:

$$\operatorname{sgn} F(\Upsilon_k) = (\operatorname{sgn} f_1(\Upsilon_k),\ \operatorname{sgn} f_2(\Upsilon_k), \ldots,\ \operatorname{sgn} f_n(\Upsilon_k)). \tag{17}$$

The n-polyhedron $\Pi^n = \langle \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{2^n} \rangle$ in $\mathbf{R}^n$ is :alled a *characteristic polyhedron (CP) relative to* $F = (f_1, f_2, \ldots, f_n) : \Pi^n \subset \mathbf{R}^n \to \mathbf{R}^n$, if the matrix of signs associated with F and Π^n, $\mathcal{S}(F;\ \Pi^n)$, is identical with the n-complete matrix $\mathcal{M}_n$. In other words, the 2^n vertices of Π^n obtain every combination of ± 1 in (17).

Suppose now that Π^n is CP and that $F = (f_1, f_2, \ldots, f_n) : \Pi^n \subset \mathbf{R}^n \to \mathbf{R}^n$ is continuous. Then, under suitable assumptions on the boundary of Π^n, the value of the topological degree of F at $\mathcal{O}$ relative to Π^n is given by

$$\deg[F, \Pi^n, \mathcal{O}] = \sum_{X \in F^{-1}(\mathcal{O})} \operatorname{sgn} \det J_F(X) = \pm 1 \neq 0\,, \tag{18}$$

(see [Vrahatis & Iordanidis, 1986], Theorem 2.9), which implies the existence of a periodic orbit inside Π^n. Of course, there are certain subtleties of the method which concern the possibility of a root occurring (at some stage) outside the CP, or the diameter of the CP staying bounded away from zero, while its volume vanishes. These possibilities however are taken care of by some well-defined strategies, as described in detail in [Vrahatis, 1988a; 1988b; 1995].

Next, we turn to a generalized bisection method, used in combination with the CP-criterion outlined above, for computing periodic orbits of any period and to any desired accuracy. This method has all the advantages of one-dimensional bisection and is particularly useful in cases where the period of the orbit is high since it always converges within the initial specified region. Moreover, the only information it requires is the algebraic signs of the components of the mapping.

This method is based on the refinement of a CP and may be called *characteristic bisection*. Several bisection methods are available in the literature [Eiger *et al.*, 1984; Greene, 1992; Vrahatis, 1986] which require the computation of the topological degree to secure its nonzero value. In the bisection method outlined here, however, the computation of the topological degree is avoided by making sure that it remains nonzero at every iteration.

The method bisects a characteristic polyhedron Π^n in such a way that the new refined n-polyhedron is also a characteristic one. To do this, one computes the midpoint of a proper one-simplex (edge) of Π^n and uses it to replace that vertex of Π^n for which the vectors of their signs are identical (see [Vrahatis & Iordanidis, 1986; Vrahatis, 1988a; 1995] for details). Finally, in analogy with the $n = 1$ case, (14), the number B of characteristic bisections of the proper one-simplexes of a Π^n required to obtain a new refined characteristic polyhedron $\Pi^n_\star$ whose longest edge length, $\Delta(\Pi^n_\star)$, satisfies $\Delta(\Pi^n_\star) \le \varepsilon$, for some $\varepsilon \in (0, 1)$, is given by [Vrahatis & Iordanidis, 1986]:

$$B = \lceil \log_2(\Delta(\Pi^n)\varepsilon^{-1}) \rceil . \qquad (19)$$

3. Periodic Orbits and Invariant Surfaces

Let us now proceed to illustrate the method of Sec. 2 on the 4D mapping

$$T: \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}, \qquad (20)$$

for two generally different values of the "linear" frequencies ω_1, ω_2.

As was mentioned previously, in order to compute a periodic orbit of (1), with period p, we have to find a point $X^\star$ such that $T^p(X^\star) = X^\star$. So we consider

$$F_4 = (f_1, f_2, f_3, f_4) = T^p - I_4 , \qquad (21)$$

(where I_4 is the identity matrix) and solve, for any given p, the following system of four equations in four unknowns:

$$F_4(X) = \mathcal{O} = (0, 0, 0, 0) . \qquad (22)$$

To this end, we choose an appropriate starting point

$$X^0 = (x_1^0, x_2^0, x_3^0, x_4^0) , \qquad (23)$$

and (generally different) stepsizes in each coordinate direction,

$$H = (h_1, h_2, h_3, h_4) , \qquad (24)$$

in such a way that the box thus constructed forms a domain, within which we shall attempt to locate and compute a solution of (22) which is a periodic orbit of the mapping T^p.

Suppose now that a periodic point $X_1^\star$ has been computed within a predetermined accuracy ε such that

$$\|T^p(X_1^\star) - X_1^\star\| \le \varepsilon . \qquad (25)$$

Then, in order to compute all the other points $X_i^\star$, $i = 2, \ldots, p$, with the same accuracy ε, we iterate the mapping T as follows: First we obtain an approximation $\hat{X}_2$ of the next point of the orbit $X_2^\star$ by one iteration of T:

$$\hat{X}_2 = T(X_1^\star) , \qquad (26)$$

and check if the following relation is fulfilled:

$$\|T^p(\hat{X}_2) - \hat{X}_2\| \le \varepsilon . \qquad (27)$$

If (27) holds, we set $X_2^\star = \hat{X}_2$ and proceed to determine $X_3^\star, \ldots$, etc., in the same way. Let us apply the above procedure on (20). In general, a visualization of the orbits of the mapping is very helpful for choosing the starting point X^0 and the stepsizes (24). In any case, if such a visualization is not available, one can search within various boxes taking a suitable grid for the domain of interest.

The stability of these orbits is determined by the eigenvalues of the return matrix [Greene, 1979; Bountis & Helleman, 1981]:

$$J_T^{(p)} = \prod_{i=1}^{p} J_T(X_i^\star) = J_T(X_p^\star) J_T(X_{p-1}^\star) \cdots J_T(X_1^\star) . \qquad (28)$$

In the case of our mapping these Jacobian matrices are of the form:

$$J_T(X) = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2x_1 & 1 & -2x_3 & 0 \\ 0 & 0 & 1 & 0 \\ -2x_3 & 0 & -2x_1 & 1 \end{pmatrix} . \qquad (29)$$

As is well known, for the p-periodic orbit to be stable the eigenvalues of $J_T^{(p)}$ must all lie on the unit circle. In that case, we call the orbit *elliptic-elliptic* (EE). If the first two eigenvalues λ_1, λ_2 are on the unit circle, while λ_3, λ_4 are real and are associated with eigenvectors whose projection predominantly lies in the x_3, x_y plane (with $|\lambda_3| > 1$, say and $|\lambda_4| < 1$), we call the orbit *elliptic-hyperbolic* (EH) or *hyperbolic-elliptic* (HE) if λ_1, λ_2 are interchanged with λ_3, λ_4. If all λ_i are real (with $|\lambda_1| > 1$, $|\lambda_3| > 1$, say, and $|\lambda_2| < 1$, $|\lambda_4| < 1$) the orbit is termed *hyperbolic-hyperbolic* (HH). Finally, if all λ_i are complex, with $|\lambda_1| = |\lambda_2| > 1$, say, and $|\lambda_3| = |\lambda_4| < 1$ we call the orbit *complex unstable* (CU).

Using our method, we have been able to make some interesting observations about the periodic orbits of our mapping (20). The first one concerns the arrangement of different orbits of the same period, in 4D space, according to their stability. The second one is connected with the existence of smooth, invariant surfaces, which extend around the origin, "enveloping" these periodic orbits in a way reminiscent of the 2D case of area-preserving maps [Greene, 1979; MacKay, 1982, 1992; Falcolini & de la Llave, 1992].

Let us now fix the frequencies of our map, $\omega_1/2\pi = 0.168$ and $\omega_2/2\pi = 0.201$, close to the resonance

$$(\sigma_1, \sigma_2) = \left(\frac{1}{6}, \frac{1}{5}\right), \tag{30}$$

where σ_1, σ_2 are the winding numbers of the orbit around the origin of the (x_1, y_1) and (x_2, y_2) planes, respectively. As the lowest common denominator of these fractions is 30, we expect to find that (20) possesses, at these parameter values, periodic orbits of period-30. Indeed, such orbits were located and computed (taking, typically, of the order of a few seconds CPU time on a HP 715) using the methods of Sec. 2. For example,

$$\begin{aligned} x_1 &= -0.135864506567776\,, \\ x_2 &= -0.228073409837263\,, \\ x_3 &= 0.343263653325621\,, \\ x_4 &= 0.038533043654550\,, \end{aligned} \tag{31}$$

with eigenvalues of the return Jacobian,

$$\begin{aligned} \lambda_1 &= 0.892417605195476 - 0.451210392097935i\,, \\ \lambda_2 &= 0.892417605195476 + 0.451210392097935i\,, \\ \lambda_3 &= 0.999696663152048 - 0.024628879037864i\,, \\ \lambda_4 &= 0.999696663152048 + 0.024628879037864i\,, \end{aligned} \tag{32}$$

is one such orbit of the EE type. We have also found period-30 orbits of other types which are arranged around the origin in a "constellation" of twelve groups consisting of ten points, each, as shown in Fig. 1(a): six of these groups are composed, alternatingly of five EE and five EH points and are separated by six other ten-point groups of alternating five HE and five HH points. Similarly, setting $\omega_1/2\pi \approx 1/6$ and $\omega_2/2\pi \approx 1/7$ led to the formation of seven distinct groups, in the x_3, x_4 projection, composed of six points from each period-42 orbit. The pairing of orbits according to their stability type, however, within these groups is not always the same as was found for the orbits of period-30. The distinction between EH and HE (and the corresponding ordering of the eigenvalues) follows from ordering the eigenvectors according to the magnitude of their projections in the x_1–x_2 and x_3–x_4 subspaces.

If we now perturb the initial conditions of one of the EH points, the orbit begins to wander away "enveloping" also the EE in each group, forming six 2D thin tori, whose projection in 3D space is shown in Fig. 1(b). As the perturbation increases, the size of these 2D toroidal "islands" grows [Fig. 1(c)], until they join into one 2D surface that "envelops" them all and extends around the origin, as shown in Fig. 1(d). This appears to be a rotational invariant surface, analogous to the ones studied by Greene and MacKay for 2D area-preserving maps [Greene, 1979; MacKay, 1982]. Unlike the 2D case, however, such surfaces do not divide the 4D phase space in disjoint regions. This is why Arnol'd diffusion phenomena in these maps are always possible [Bountis & Tompaidis, 1991; Bountis & Kollmann, 1994].

Thus, we decided to take a closer look at these surfaces approximating them with periodic orbits in the following way: Following a method introduced by Greene [1979] for 2D maps, we have located periodic orbits whose (rational) rotation numbers $\sigma_n^{(i)}$, limit on two rationally independent rotation numbers σ_1, σ_2 of an invariant surface, as

$$\sigma_n^{(i)} = \frac{p_n^{(i)}}{q_n^{(i)}} \underset{n\to\infty}{\longrightarrow} \sigma_i\,, \qquad i = 1, 2\,. \tag{33}$$

The period of these orbits will be sought as the product, or the lowest common multiple of the denominators of their winding numbers $\sigma_n^{(i)}$. Let us choose, for example:

$$\begin{aligned} \sigma_1 &= \frac{\sqrt{5}-1}{2} = 0.618033989, \ldots\,, \\ \sigma_2 &= \sqrt{2} - 1 = 0.414213562, \ldots\,, \end{aligned} \tag{34}$$

as the winding numbers of the invariant surface we wish to approximate.

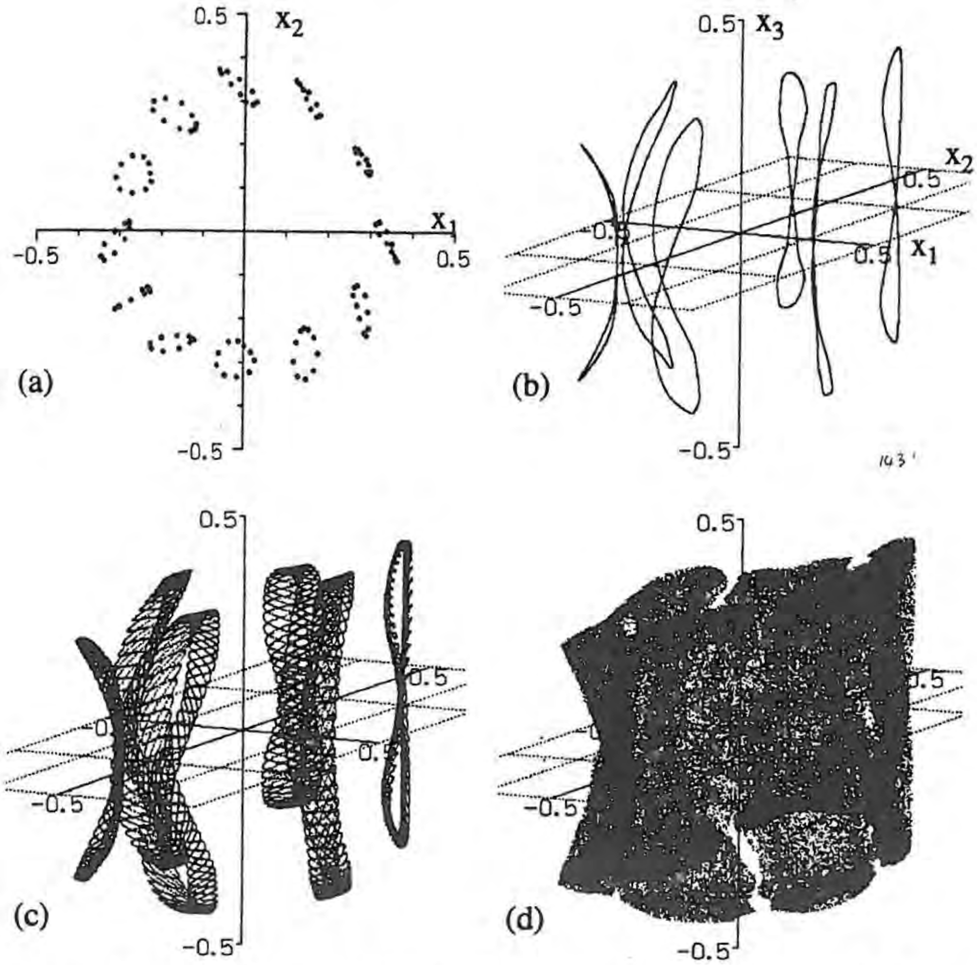

Fig. 1. (a) 2D projection of four orbits of period-30: EE (violet), EH (black), HE (orange), and HH (blue) forming 12 groups of 10 points. Six of these groups are composed, alternatingly of five EE and five EH points, and are separated by six other groups of alternating five HE and five HH points. Here, $q_x = 0.168$, $q_y = 0.201$. (b) Projection in 3D space of six 2D thin tori, formed by slightly perturbing the initial conditions of one of the EH points of Fig. 1(a). The orbit wanders away enveloping also the EE in each group. (c) The size of the 2D toroidal islands of Fig. 1(b) grows as the perturbation increases. (d) A rotational invariant 2D surface will be formed as the size of the 2D toroidal islands of Fig. 1(c) grows with increasing perturbation (see also Fig. 2(b)).

We shall now use two "linear" frequencies, $\omega_1/2\pi$, $\omega_2/2\pi$, which are very close to the irrationals σ_1 and σ_2 of (34). In fact, following Greene, we shall look for periodic orbits whose rotation numbers are rational approximants of σ_1, σ_2 from their continued fraction expansions [Bazzani *et al.*, 1988; Corless, 1992; Peirone, 1993]. More specifically, let $/a_0, a_1, \ldots/$ denote the coefficients of the continued fraction expansion of an irrational number $\alpha \in \mathbf{R}\backslash\mathbf{Q}$, recursively determined by $a_j = [1/\alpha_j]$ and $\alpha_j = \{1/\alpha_{j-1}\}$ for all $j \geq 1$, $\alpha_0 = \alpha - [\alpha]$ and $a_0 = [\alpha]$, where $[x]$ denotes the integer part of the real number x while $\{x\}$ denotes the nearest integer of x, i.e., $\{x\} = [x + \frac{1}{2}]$. The partial continued fractions of α are thus constructed by

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_{n-2} + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}\,, \quad a_j \in \mathbf{Z}\,, \tag{35}$$

where

$$\begin{aligned} p_n &= a_n p_{k-1} + p_{k-2}\,, \\ q_n &= a_n q_{k-1} + q_{k-2}\,, \end{aligned} \tag{36}$$

for all $k \geq 0$, with initial conditions $q_{-2} = p_{-1} = 1$, $q_{-1} = p_{-2} = 0$. It can be proved that the even subsequences, p_{2j}/q_{2j}, are increasing, while the odd ones, p_{2j+1}/q_{2j+1}, are decreasing and that both converge to α verifying the following inequalities:

$$\frac{1}{q_j + q_{j+1}} \leq |q_j\alpha - p_j| \leq \frac{1}{q_{j+1}}. \qquad (37)$$

Consider now the two quadratic irrationals σ_1, σ_2 of (34). These numbers have the continued fraction expansions

$$\begin{aligned} \sigma_1 &= /1,\, 1,\, 1,\, 1,\, 1, \ldots / \equiv /1^\infty/\,, \\ \sigma_2 &= /2,\, 2,\, 2,\, 2,\, 2, \ldots / \equiv /2^\infty/\,. \end{aligned} \qquad (38)$$

The first rational approximants of σ_1, σ_2, using (35), are listed in Table 1 together with the period of the orbits that correspond to them.

Using $q_x = \omega_1/2\pi = \sigma_1 + 10^{-3}$ and $q_y = \omega_2/2\pi = \sigma_2 + 10^{-3}$, we have succeeded in computing the corresponding periodic orbits for periods up to the thousands, within an accuracy of $\varepsilon = 10^{-15}$. In Table 2 we list some of these orbits, in order of increasing period, together with their four eigenvalues, indicating their stability properties. We have computed, of course, for each period, all the orbits with different stability characteristics.

It is important to note, however, that we cannot claim with certainty that the periodic orbits we find correspond to the winding numbers listed in Table 1. In fact, in some cases, using Laskar's frequency analysis [Laskar *et al.*, 1992; Laskar, 1993], we have found that the rotation numbers of our periodic orbits differed (though not by a big amount) from the values listed in Table 1 [Vrahatis *et al.*, 1996].

Observe, now, that there are some periodic orbits in Table 2, which have been listed in parentheses in Table 1. These are orbits with periods which are lowest common multiples of the denominators of the corresponding rotation numbers.

What is also interesting, about the orbits of Table 2 is their stability characteristics, i.e., their eigenvalues λ_i. They indicate that these orbits have either just turned unstable (if some $|\lambda_i| > 1$ in the EH, HE and CU cases), or are very close to doing so, in the EE cases, where all λ_i's are still on the unit circle. In fact, we were able to find EE-stable orbits with period up to 33490.

In Fig. 2(a) we display this "last" EE orbit of period 33490, projected in 3D space. Perturbing then slightly its initial conditions, as we did in Fig. 1(d) for the period-30 orbit, we allow the iterations of the map to "draw" in 3D an invariant surface, which "envelops" the period-33490 orbit and extends all around the origin of the 4D space; see Fig. 2(b).

Table 1. Rational approximants in the continued fraction expansions of the quadratic irrationals σ_1 and σ_2 and the full period of the corresponding periodic orbit.

$\sigma_1 = 0.61803398874989$	$\sigma_2 = 0.41421356237310$	Full Period
$\frac{1}{1} = 1.00000000000000$	$\frac{1}{2} = 0.50000000000000$	2
$\frac{1}{2} = 0.50000000000000$	$\frac{2}{5} = 0.40000000000000$	10
$\frac{2}{3} = 0.66666666666667$	$\frac{5}{12} = 0.41666666666667$	36 (12)
$\frac{3}{5} = 0.60000000000000$	$\frac{12}{29} = 0.41379310344828$	145
$\frac{5}{8} = 0.62500000000000$	$\frac{29}{70} = 0.41428571428571$	560 (280)
$\frac{8}{13} = 0.61538461538462$	$\frac{70}{169} = 0.41420118343195$	2197 (169)
$\frac{13}{21} = 0.61904761904762$	$\frac{169}{408} = 0.41421568627451$	8568 (2856)
$\frac{21}{34} = 0.61764705882353$	$\frac{408}{985} = 0.41421319796954$	33490

Table 2. Periodic orbits and their stability for the frequencies $q_x = 0.61903$ and $q_y = 0.4152$ and the corresponding eigenvalues λ_i, $i = 1, \ldots, 4$ of the return Jacobian.

Period (and Stability)	X_1^*	$\mathrm{Re}(\lambda_i)$	$\mathrm{Im}(\lambda_i)$
145	0.096877938749679	1.019517904049305	0.000000000000000
(HE)	−0.188433136165055	0.980855751554931	0.000000000000000
	−0.065986723717374	0.999970570695045	−0.007671880071613
	−0.498425728580339	0.999970570695045	0.007671880071613
169	0.239296782521746	1.000000306611101	0.000000000000000
(HE)	0.179006460223186	0.999999693389011	0.000000000000000
	0.244613198268430	0.984072545932447	−0.177767332043995
	0.469208035078025	0.984072545932447	0.177767332043995
280	−0.039860143419013	1.000006046931921	−0.000007791459929
(CU)	0.250884444427295	1.000006046931921	0.000007791459929
	−0.035255366469458	0.999993953043960	−0.000007791365721
	0.262007851011220	0.999993953043960	0.000007791365721
560	−0.026376447925352	0.999999999735146	−0.000023021534995
(EE)	−0.234209016364679	0.999999999735146	0.000023021534995
	0.228044643838823	0.999999996274356	−0.000086320834290
	−0.316821393627015	0.999999996274356	0.000086320834290
2197	0.033101838907224	0.999999974078013	−0.000227726495332
(EE)	0.142194126750058	0.999999974078013	0.000227726495332
	0.104500919077146	0.999999991299542	−0.000131922959818
	0.450101672498364	0.999999991299542	0.000131922959818
2856	0.192381310629934	1.001820069528981	−0.003621342321534
(CU)	0.082763340106664	1.001820069528981	0.003621342321534
	0.244979110845414	0.998170194874210	−0.003608149002697
	0.482789954959600	0.998170194874210	0.003608149002697
8568	0.102245970330149	1.022595637604861	0.000000000000000
(HE)	0.083069392211058	0.977903639611024	0.000000000000000
	0.242963697128183	0.999975739172172	−0.006965528396915
	0.483232739573672	0.999975739172172	0.006965528396915
33490	0.454977773558536	0.999999997244185	−0.000092411272156
(EE)	0.238699840858798	0.999999997244185	0.000092411272156
	0.112352720179551	0.999999999503046	−0.000033565151946
	0.151048107914082	0.999999999503046	0.000033565151946

We have also computed the analogue of Greene's residue:

$$R_p = \frac{1}{8}[4 - \mathrm{Tr}(J_T^{(p)})]\,, \qquad (39)$$

[cf. (28)] for our p-periodic orbits, which has already been studied by other researchers in connection with higher-dimensional, symplectic twist maps [Tompaidis, 1994]. Interestingly enough, we have found that for all our periodic orbits (even the purely hyperbolic ones: HH), $0 < |R_p| \ll 1$, at least at the parameter values used in this paper. This suggests that even though the invariant "surfaces" we seek to approximate may not be "smooth", we

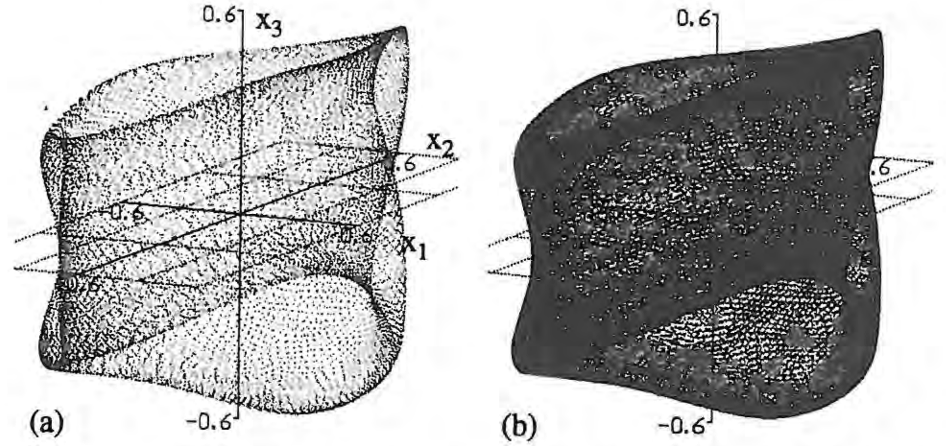

Fig. 2. (a) The EE-stable orbit of period-33490, projected in x_1, x_2, x_3 space, for $q_x = 0.61903$, $q_y = 0.4152$. (b) By perturbing slightly the initial conditions of the orbit of Fig. 2(a) the iterations of the map draws in 3D an invariant surface, which envelops the period-33490 orbit and extends all around the origin of the 4D space.

are still far from strong instabilities and large scale regions of bounded motion around the origin are expected to exist, as verified also by all our numerical results.

To further test these ideas we have introduced a factor $(1+\varepsilon)$ in Eqs. (20) and considered the one-parameter family of symplectic maps:

$$T_\varepsilon : \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 + x_1^2 - (1+\varepsilon)x_3^2 \\ x_3 \\ x_4 - (1+\varepsilon)2x_1x_3 \end{pmatrix}. \quad (40)$$

Then by slightly varying ε away from zero we have observed that it is possible to move the eigenvalues of the orbits of Table 2 away from bifurcation and find stable (EE) orbits with period higher than 33490. Indeed, it now becomes possible, to find elliptic-elliptic orbits of period-130790, using the methods of Sec. 2. Perturbing then slightly their initial conditions one can find new invariant surfaces further away from the origin, which might be better approximations of the one we originally set out to find.

All this serves to strengthen the argument that an analogue of Greene's 2D criterion may hold for 4D symplectic maps as well. However, the situation in 4D is somewhat less clear, since it is not always easy to find the exact periodic orbits whose rotation numbers belong to a prescribed sequence, like that of Table 1, [Vrahatis et al., 1996].

Consider, for example, periodic orbits of period-169 and different stability type. They are arranged in 13 groups of 13 points each, in the x_1, x_2 plane, in a similar way as was found for the period-30 orbits. What is more interesting here, however, is that although orbits of type EH and HH, when perturbed, describe similar invariant surfaces [see Figs. 3(a) and 3(b)], perturbations of the HE orbit lead to a surface of entirely different morphology [see Fig. 3(c)].

We have also studied the structure of invariant surfaces by perturbing several other periodic orbits (among those listed in Table 1) and of various stability types. Some of these orbits when perturbed yield invariant surfaces of similar morphology, as seen, e.g., in Fig. 4(a) for periods 169, 2197, 8568. Others produce, upon perturbation, more familiar surfaces [see Fig. 4(b) for periods 2197, 2856, 8568], which tend to rotate around the origin (at least in x_1, x_2 projections) in a more or less uniform way and are reminiscent of what was found for the period-169 orbits [see Figs. 3(a) and 3(b)]. Still, one finds again at high periods, like the 2197-EH periodic orbit of Fig. 4(a), the same peculiar morphology of the invariant surface observed in perturbations of the 169-HE orbit, in Fig. 3(c). What

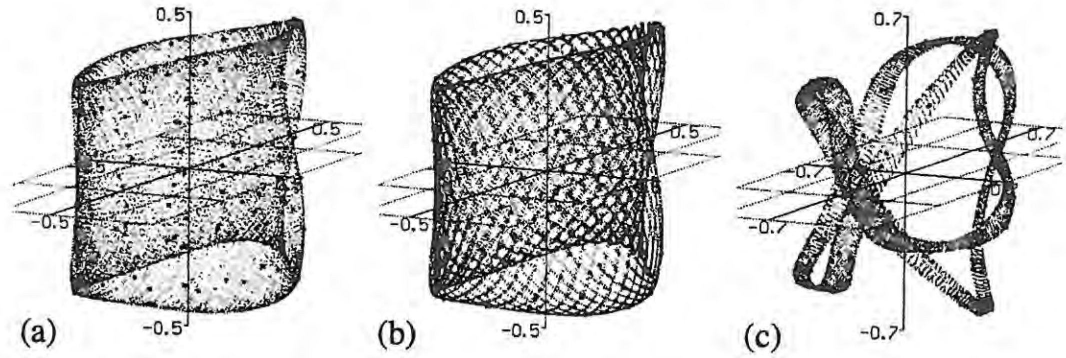

Fig. 3. The morphology of the invariant surface obtained by perturbation of a (a) 169-EH periodic orbit, (b) 169-HH periodic orbit, and (c) 169-HE periodic orbit, for the same q_x, q_y as in Fig. 2. Fat dots: the periodic orbit itself.

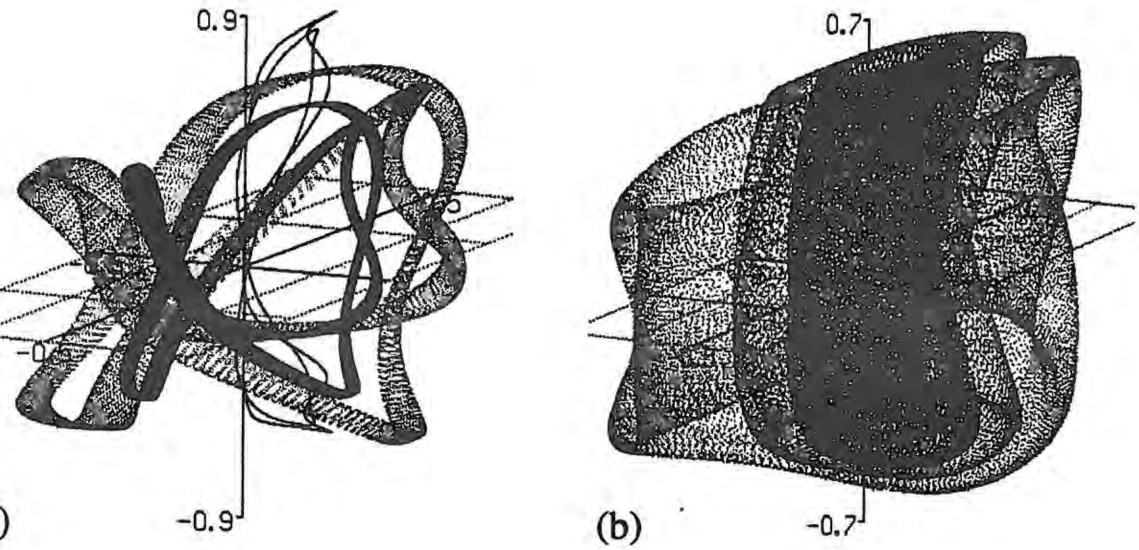

Fig. 4. (a) The morphology of the invariant surface obtained by perturbation of a period-169-HE (violet), a period-2197-EH (black), a period-8568(408)-HE (orange) and a period-8568(72)-EE (blue). (b) The morphology of the invariant surface obtained by perturbation of a period-2197-EE (violet), a period-2856-CU (orange), a period-8568-HE (black), and the period-33490-EE (blue) of Fig 2. Parameter values as in Fig. 2.

is also surprising is that periodic orbits of different periods with similar morphology do not necessarily share the same stability properties.

In a very recent work [Vrahatis *et al.*, 1996] we have examined in more detail these two types of surfaces — the "tube" tori of Fig. 4(a) and "rotational" tori of Fig. 4(b) — the dynamics in their vicinity and how the former result from the latter upon perturbation.

4. Concluding Remarks

An efficient method for rapidly and accurately computing periodic orbits of nonlinear mappings has been described in this paper. This method exploits topological degree theory to construct a characteristic polyhedron and locate the periodic orbit within a given region, without making any computation of the topological degree. Then it repeatedly subdivides this polyhedron to compute the periodic orbit rapidly and to any accuracy.

We have applied our method in this paper to higher-dimensional maps, by using it to calculate periodic orbits of certain 4D symplectic mappings of interest to accelerator dynamics. One such orbit of particularly long period, is the one shown on the cover of this issue. Since the method is especially suited for the calculation of orbits of high period, it may be used to approximate quasiperiodic orbits which lie on invariant tori of nonlinear mappings, in the following way: Choosing sequences of rational approximants to a pair of irrational winding numbers, we have computed the corresponding periodic orbits and their stability properties by calculating the eigenvalues of the Jacobian of the return map.

Ordering the eigenvectors, in pairs, by the magnitude of their projection in the x_1, x_2 and x_3, x_4 planes and numbering the eigenvalues λ_i, $i = 1, 2, 3, 4$, accordingly, we have been able to distinguish five stability types: EE (Elliptic-Elliptic) if all λ_i's of the orbit are on the unit circle, EH (Elliptic-Hyperbolic) if λ_1, λ_2 are on the unit circle and λ_3, λ_4 real, HE, with λ_1, λ_2 real and λ_3, λ_4 on the unit circle, HH, with all λ_i real and CU (Complex Unstable) if all λ_i are complex but not on the unit circle.

An analytical approach to the structure of resonances in phase space, for a class of near-integrable mappings, has been presented in [Todesco, 1994].

In pursuit then of invariant surfaces of bounded motion around the origin we fixed the linear frequencies of the map near two irrationals, $\omega_1/2\pi \approx \sigma_1 = (\sqrt{5}-1)/2$ and $\omega_2/2\pi \approx \sigma_2 = \sqrt{2}-1$, and computed periodic orbits, with period $q_i^{(n)}$ and rotation number close to the continued fraction approximants $\sigma_i^{(n)} = p_i^{(n)}/q_i^{(n)}$ of σ_i, $i = 1, 2$, as n increases. Although we were thus able to compute orbits with periods in the hundreds of thousands, we could not find EE orbits with period greater than 33490.

Furthermore, by perturbing slightly the parameters of the map, or the initial conditions of these high period orbits we have found large invariant tori of bounded motion "surrounding" the origin of 4D space. For many of the orbits, of different period and stability type, these tori turned out to have similar geometry structure (when projected in 3D space). In some other cases, however, these perturbations approximate a "twisted tube"-like surface of different morphology than the others.

Finally, introducing a small parameter ε in our mapping equations we have found that, by varying ε, it is possible to change the eigenvalues of the orbit sequence we have followed in a systematic way. Thus, we have been able to stabilize periodic orbits of period longer than 33490 and hopefully make the motion around the origin of the map globally more stable.

These results suggest that if an analogue of Greene's criterion for studying invariant tori for symplectic 4D maps exists, it is considerably more complicated to show than the corresponding one for the 2D case. On the other hand, in torus maps which globally satisfy the twist condition, there is evidence that Greene's criterion can be directly generalized in higher dimensions [Bollt & Meiss, 1992].

Acknowledgments

We gratefully acknowledge many stimulating and useful discussions with Professor G. Turchetti and Dr. E. Todesco. One of us (MNV) wishes to express his appreciation to many members of the Mathematical Physics group of the Bologna University for their help. He also wishes to thank the CERN and INFN for their support as well as the CERN and Physics Department of Bologna University for their hospitality and use of computing facilities, both of which proved quite helpful for the completion of the work reported in this paper. Part of this work was also supported by "Human Capital and Mobility" contracts no. CHRX. CT93-0107 and CHRX. CT94-0480.

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