Universität Rostock

Traditio et Innovatio

HOCHSCHULE WISMAR
UNIVERSITY OF APPLIED SCIENCES
TECHNOLOGY, BUSINESS AND DESIGN

11th Summer Workshop on Interval Methods

July 25-27, 2018

Book of Abstracts

Ekaterina Auer[1]
Julia Kersten[2]
Andreas Rauh[2]

[1]Hochschule Wismar
Department of Electrical Engineering University of Technology, Business and Design

[2]University of Rostock
Chair of Mechatronics

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WISSENSCHAFTEN
1419

An Interval Median Algebra

Stavros P. Adam[1,3], Gerasimos C. Meletiou[2,3] and Michael N. Vrahatis[3]

[1] Department of Computer Engineering,
Technological Educational Institute of Epirus, Arta 47100, Greece
adamsp@teiep.gr
[2] Department of Agricultural Technology,
Technological Educational Institute of Epirus, Arta 47100, Greece
gmelet@teiep.gr
[3] Computational Intelligence Laboratory, Dept. of Mathematics,
University of Patras, GR–26110 Patras
adamsp@upatras.gr – vrahatis@math.upatras.gr

Keywords: Intervals, Median Algebra, Median Structure

Introduction

Intervals and interval computation owe much to the seminal work of Sunaga [1] who defined intervals and interval arithmetic as a result of an Interval Algebra. As discussed in [2], the set of real intervals $\mathbb{IR}^*$ with the operations Min, Max is a distributive lattice with a number of interesting algebraic properties. In this paper, we are interested in endowing this lattice of intervals with a median structure thus obtaining a median algebra. We show how this algebra is defined, its properties, its compatibility with the interval arithmetic and how intervals (proper, improper and modal) are treated in this framework. Finally, we discuss some theoretical issues as well as some potential applications.

Background and Hypotheses

Since the early times of interval computation [1], the concept of interval and related arithmetic gave rise to the elaboration of a significant number of theoretical and computational concepts and tools under the title of interval analysis as well as important developments in various scientific and engineering fields. Hence, the initial interval concepts were gradually enriched with the concepts of improper intervals and the related Kaucher arithmetic as well as modal intervals together with related operations. In addition to the above, one should mention set-membership approaches which adopted an interval based view of sets by means of unions of axis aligned boxes. This view, which can be significantly represented by SIVIA [3] and its ramifications, resulted in important research and development work in the area of control systems and their applications. Among all the important foundational approaches it seems that the work of Sunaga [1] was the first that introduced an algebraic structure for the intervals and yet proved its contribution to the definition of interval arithmetic operations and to the perspective of using intervals in numerical analysis.

In algebra the concept of median structure and the resulting median algebra constitutes a well established algebraic construct with constantly increasing appeal from both theoretic and practical points of view of algebraic notions [4,5]. A median algebra is a ternary algebraic structure consisting of a set M together with a ternary operation $(\alpha, \beta, \gamma) \mapsto (\alpha\beta\gamma)$ on M such that the following axioms hold:

- $(\alpha\alpha\beta) = \alpha$ (Idempotency)
- $(\alpha\beta\gamma) = (\beta\alpha\gamma) = (\alpha\gamma\beta)$ (Symmetry)
- $(\alpha\beta(\gamma\delta\epsilon)) = ((\alpha\beta\gamma)\delta(\alpha\beta\epsilon))$ (Distributivity)

If M is a distributive lattice $(M, \vee, \wedge)$ one can define a ternary operation $m : M^3 \to M$ such that for $\alpha, \beta, \gamma \in M$

$$m(\alpha, \beta, \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma),$$

where the symbols $\wedge$ and $\vee$ denote the *join* and *meet* operations of the lattice M. This operation is referred as the median ternary operation in M and satisfies the axioms above. Moreover, this ternary operation is self dual in the sense that

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \vee (\beta \wedge \gamma).$$

Since every chain is a distributive lattice, so are the real numbers and therefore $\mathbb{R}$ is a median algebra. Median structures have been successfully applied in various areas of interest such as metric spaces giving rise to median spaces and measured wall spaces, consensus theory, taxonomy, majority decision making, median graphs and trees which often occur as almost-natural representations of structured data in various applications of machine learning, etc.

Main Results

In this paper, we show how the set of real intervals $\mathbb{IR}$ becomes a median algebra and so does its extension $\mathbb{IR}^*$ of modal intervals. Using some of the foundational concepts defined in [1] and extensions presented in [2] we show that the median structure of the set of intervals is an algebraic framework supporting the representation of proper, improper and modal intervals. Moreover, we show how arithmetic, containment and relational operations between intervals are formulated in this context without contradicting the initial definitions. Finally, we discuss some theoretical issues and potential applications.

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