Generalizations of the intermediate value theorem and applications

Michael N. Vrahatis

Department of Mathematics, University of Patras, GR-26110 Patras, Greece
vrahatis@math.upatras.gr

Abstract

Generalizations of the intermediate value theorem in several variables are presented. These theorems are very useful in various approaches including, among others, the existence of solutions of systems of nonlinear algebraic and/or transcendental equations, the existence of fixed points of continuous functions, the localization of extrema of objective functions, as well as the localization of periodic orbits of nonlinear mappings and periodic orbits (fixed points) on Poincaré's surface of section (Poincaré map). These theorems are of major importance for tackling problems with imprecise (not exactly known) information.

Based on the corresponding existence criteria emanated by the above theorems, methods, named "generalized bisection methods", are given. The only computable information required by the generalized bisection methods is the algebraic sign of the function value which is the minimum possible information (one bit of information) necessary for the purpose needed, and not any addi-tional information. Thus, these methods are of major importance for studying and tackling problems with imprecise (not exactly known) information. This kind of problems occurs in various scientific fields. This is so, because, in a large variety of applications, precise function values are either impossible or time con-suming and computationally expensive to obtain. Furthermore, these methods are particularly useful for studying and tackling various problems where the corresponding functions obtain very large and/or very small values.

It is well known that algebraic equations are very important in studying and solving problems in a variety of scientific fields. Regarding the algebraic signs of algebraic expressions there are various efficient approaches in obtaining this information.

Applications related to systems of nonlinear algebraic and/or transcendental equations, as well as fixed points of continuous functions are presented. Furthermore, an application is presented which concerns the computation of all the periodic orbits (stable and unstable) of any period and accuracy which occur, among others, in the study of beam dynamics in circular particle accelerators like the Large Hadron Collider (LHC) machine at the European Organization for Nuclear Research (CERN).

Keywords: Generalizations of the intermediate value theorem; existence theorems; fixed points; nonlinear equations; solutions of systems of nonlinear algebraic and/or transcendental equations; periodic orbits of nonlinear mappings.

References

[1] Vrahatis M.N. (1986) An error estimation for the method of bisection in $\mathbb{R}^n$, *Bulletin of the Greek Mathematical Society*, vol. 27, pp. 161–174.

[2] Vrahatis M.N. (1988) Solving systems of nonlinear equations using the nonzero value of the topological degree, *ACM Transactions on Mathematical Software*, vol. 14, pp. 312–329.

[3] Vrahatis M.N. (1988) A variant of Jung's theorem, *Bulletin of the Greek Mathematical Society*, vol. 29, pp. 1–6.

[4] Vrahatis M.N. (1989) A short proof and a generalization of Miranda's existence theorem, *Proceedings of the American Mathematical Society*, vol. 107, pp. 701–703.

[5] Vrahatis M.N. (1995) An efficient method for locating and computing periodic orbits of nonlinear mappings, *Journal of Computational Physics*, vol. 119, pp. 105–119.

[6] Vrahatis M.N. (2000) Simplex bisection and Sperner simplices, *Bulletin of the Greek Mathematical Society*, vol. 44, pp. 171–180.

[7] Vrahatis M.N. (2016) Generalization of the Bolzano theorem for simplices, *Topology and its Applications*, vol. 202, pp. 40–46.

[8] Vrahatis M.N. (2020) Intermediate value theorem for simplices for simplicial approximation of fixed points and zeros, *Topology and its Applications*, vol. 275, 107036, pp. 1–13.

[9] Vrahatis M.N. (2020) Generalizations of the intermediate value theorem for approximating fixed points and zeros of continuous functions, *Lecture Notes in Computer Science*, vol. 11974, pp. 223–238.