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SMALLER ALIGNMENT INDEX (SALI): DETECTING ORDER AND CHAOS IN CONSERVATIVE DYNAMICAL SYSTEMS

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Abstract. *We apply the method of the Smaller Alignment Index (SALI) for determining the ordered or chaotic nature of orbits, on some examples of Hamiltonian systems and symplectic maps. Finding the SALI for a sample of initial conditions allows us to distinguish clearly between regions of phase space where ordered or chaotic motion occurs. The computation of SALI is performed rather easily: For a given orbit we follow the evolution in time of two different initial deviation vectors and compute the norms of the difference (parallel alignment index) and the addition (antiparallel alignment index) of the two normalized vectors. The time evolution of the smaller alignment index reflects clearly the chaotic or ordered nature of the orbit. In general the SALI tends to zero for chaotic orbits, while it fluctuates around non-zero values for ordered orbits.*

1 INTRODUCTION

The distinction between ordered and chaotic motion of a dynamical system is fundamental in many areas of applied science. This distinction is particularly difficult in many degrees of freedom, basically because we cannot visualize their motion in phase space. So, we need fast and accurate tools to give us information about the chaotic or ordered character of an orbit, especially for conservative systems, where, in the absence of attractors, orbits can wander for very long times in uncharted domains of phase space.

Many methods have been developed over the years that try to give an answer to this problem. The inspection of the intersections of an orbit with a Poincaré surface of section has been used extensively mainly for 2 – dimensional (2D) maps and 2 degree of freedom dynamical systems. One of the most common methods is the computation of the maximal Lyapunov Characteristic Number (LCN)[1,2], which can be applied for systems with many degrees of freedom. Another efficient method is the frequency map analysis developed by Laskar[3,4]. In recent years new methods have been developed like the study of spectra of "short time Lyapunov characteristic numbers" [5,6] or "stretching numbers" [7,8] and the "spectral distance" of such spectra [9], as well as the study of spectra of helicity and twist angles [10-12]. In addition Froeschlé et al.[13] introduced the fast Lyapunov indicator (FLI), while Vozikis et al.[14] proposed a method based on the frequency analysis of "stretching numbers".

Recently a new, fast and easy to compute indicator of the chaotic or ordered nature of orbits, has been introduced, called the smaller alignment index (SALI)[15]. In the present paper, we first recall the definition of the smaller alignment index and show its effectiveness in distinguishing between ordered and chaotic motion, by applying it to a 2D and a 4D symplectic map as well as in a Hamiltonian system with two degrees of freedom.

2 DEFINITION OF THE SMALLER ALIGNMENT INDEX

Let us consider the n-dimensional phase space of a conservative dynamical system, which could be a symplectic map or a Hamiltonian flow. We consider also an orbit in that space with initial condition $P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$ and a deviation vector $\xi(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$ from the initial point P(0). In order to determine if this orbit is ordered or chaotic we follow the evolution in time of two different initial deviation vectors (e.g. $\xi_1(0)$, $\xi_2(0)$), noting that in maps the time is discrete and is equal to the number N of the iterations. In every time step, we compute the parallel alignment index (ALI):

$$d_-(t) \equiv \|\xi_1(t) - \xi_2(t)\| \qquad (1)$$

and the antiparallel ALI:

$$d_+(t) \equiv \|\xi_1(t) + \xi_2(t)\| \qquad (2)$$

where $\| \|$ denotes the Euclidean norm of a vector. It is evident from the above definitions that when $d_-=0$ the two vectors coincide and when $d_+=0$ the two vectors become opposite Then the smaller alignment index (SALI) is defined as the minimum value of the above alignment indices (ALIs) at any point in time, t:

$$\mathrm{SALI}(t) = \min(d_-(t) , d_+(t)) \qquad (3)$$

In 2D maps the ordered motion lies on a 1D torus, the so-called invariant curve and any two deviation vectors, after a transient period, become tangent to this curve, tending to coincide or become opposite to each other[9]. This means that one of the ALIs tends to zero. A similar behavior appears when the orbit tested is chaotic: any two deviation vectors eventually become tangent to the most unstable nearby manifold and so one of the ALIs tends to zero. If we consider the vectors $\xi_1(t)$ and $\xi_2(t)$ to be normalized in every time step with norm equal to 1, the two deviation vectors tend to coincide when $d_-(t)\to 0$ and $d_+(t)\to 2$ and tend to become opposite when $d_-(t)\to 2$ and $d_+(t)\to 0$. So in 2D maps the smaller alignment index (SALI) tends to zero both for ordered as well as chaotic orbits, following however very different time rates (as shown in the next section), which allows us to distinguish between the two cases.

On the other hand, in the case of 4D maps and Hamiltonian systems with 2 degrees of freedom the distinction between ordered and chaotic motion is even easier. In these cases, the ordered motion occurs on a 2D torus on which any initial deviation vector becomes almost tangent after a short transient period. In general, two different initial deviation vectors become tangent to different directions on the torus, producing different sequences of vectors, so that both quantities $d_-(t)$ and $d_+(t)$ do not tend to zero but fluctuate around positive values in the interval (0,2). For chaotic orbits, any two initially different deviation vectors tend to coincide in the direction defined by the most unstable nearby manifold and hence either coincide with each other, or one vector tends to the opposite of the other. This means than one ALI (the smaller one we call SALI) tends to zero when the orbit is chaotic and to a non-zero value when the orbit is ordered. Thus, the completely different behaviour of the SALI helps us distinguish between ordered and chaotic motion in 4D maps and in Hamiltonian systems with 2 degrees of freedom, but more importantly in systems of higher dimensionality also.

3 APPLICATION OF THE ALIGNMENT INDICES

3.1 Symplectic mappings

Following [15] we compute the SALI in some simple cases of ordered and chaotic orbits in symplectic maps with two and four dimensions. In particular we use the 2D map:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \cdot \sin(x_1 + x_2) \end{aligned} \quad \mathrm{mod}(2\pi) \qquad (4)$$

and the 4D map:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \cdot \sin(x_1 + x_2) - \mu \cdot [1 - \cos(x_1 + x_2 + x_3 + x_4)] \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 - \kappa \cdot \sin(x_3 + x_4) - \mu \cdot [1 - \cos(x_1 + x_2 + x_3 + x_4)] \end{aligned} \quad \mathrm{mod}(2\pi) \qquad (5)$$

which is composed of two 2D maps of the form (4), with parameters ν and κ, coupled with a term of order μ. All variables are given (mod 2π), so $x_i \in [-\pi, \pi)$, for i=1,2,3,4. The map (5) is a variant of the 4D map studied by Froeschlé[16]. Some dynamical structures on the phase space of this map, were examined in detail in [17] for small values of the coupling parameter μ.

In the case of the 2D map (4) we consider the ordered orbit A with initial conditions $x_1=2$, $x_2=0$ and the chaotic orbit B with initial conditions $x_1=3$, $x_2=0$ for ν=0.5. The initial deviation vectors used are $\xi_1(0)=(1,0)$ and $\xi_2(0)=(0,1)$ for both orbits. These vectors eventually coincide in both cases, but at completely different time rates. This is evident from figure 1(a), where the SALI (coinciding with d_- for both orbits) is plotted as a function of the number N of iterations for the ordered orbit A (grey line) and the chaotic orbit B (black line) in log-log scale. For the ordered orbit A the SALI decreases as N increases, following a power law and becomes SALI$\cong 10^{-13}$ after 10^7 iterations, which means that the two deviation vectors almost coincide. On the other hand the SALI of the chaotic orbit B decreases abruptly, reaching the limit of accuracy of the computer (10^{-16}) after only about 200 iterations. After that time, the two vectors are identical since the same numbers in the computer represent their coordinates. So, it becomes evident that the SALI can distinguish between ordered and chaotic motion in a 2D map, since it tends to zero following completely different time rates.

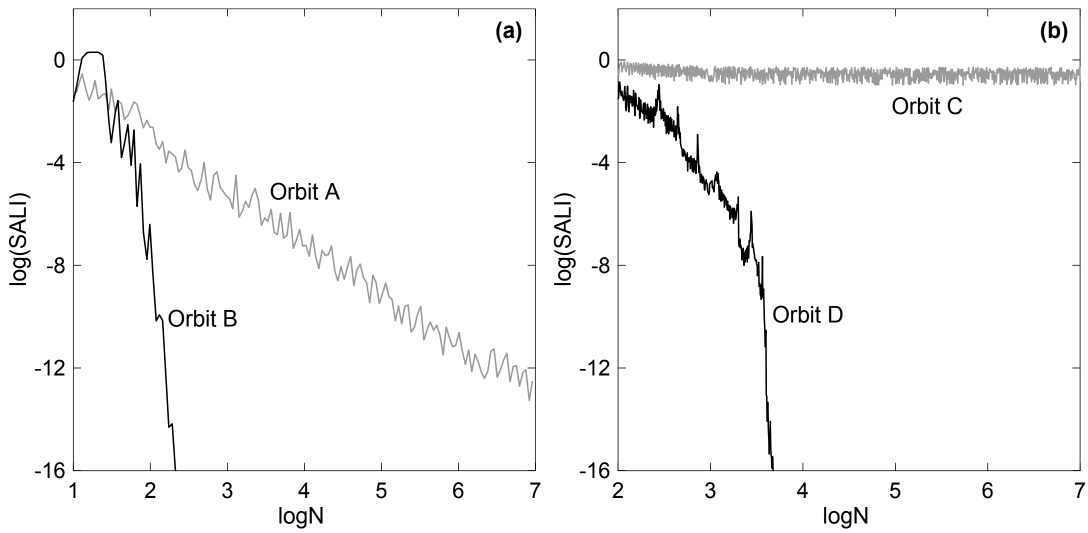

Figure 1. The evolution of the smaller alignment index SALI, with respect to the number N of iterations, in log-log scale, for (a) the 2D map (4) with ν =0.5, for the ordered orbit A with initial conditions $x_1=2$, $x_2=0$ (grey line) and for the chaotic orbit B with initial conditions $x_1=3$, $x_2=0$ (black line) and (b) the 4D map (5) with ν =0.5, κ =0.1, $\mu=10^{-3}$, for the ordered orbit C with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$ (grey line) and for the chaotic orbit D with initial conditions =3, $x_2=0$, $x_3=0.5$, $x_4=0$ (black line).

In the case of the 4D map (5) for ν =0.5, κ=0.1 and $\mu=10^{-3}$ we consider the ordered orbit C with initial conditions x1=0.5, x2=0, x3=0.5, x4=0 and the chaotic orbit D with initial conditions x1=3, x2=0, x3=0.5, x4=0. The initial deviation vectors used are (1,1,1,1) and (1,0,0,0). As we see in figure 1(b) the SALI of the ordered orbit C remains almost constant (grey line), fluctuating around SALI≅0.28. On the other hand, the SALI of the chaotic orbit D decreases abruptly reaching the limit of accuracy of the computer (10^{-16}) after about 4.7 10^3 iterations (black line). After that time the coordinates of the two vectors are represented by opposite numbers in the computer (since the SALI coincides with d_+). So, in 4D maps the SALI tends to zero for chaotic orbits, while it tends to a positive non-zero value for ordered orbits. Thus, the different behaviour of SALI clearly distinguishes between ordered and chaotic orbits.

3.1 The case of a Hamiltonian system of two degrees of freedom

In order to illustrate the effectiveness of SALI in determining the chaotic or ordered nature of orbits in a flow, we consider the two degrees of freedom Hamiltonian

$$H(x, y, p_x, p_y)=\frac{1}{2}(p_x^{\,2}+p_y^{\,2})+\frac{1}{4}(x^4+y^4+\eta(x-y)^4) \tag{6}$$

where x, y are the generalized coordinates, the p_x, p_y conjugate momenta and η a real parameter. The possible integrability of this Hamiltonian, for different values of the parameter η, was studied in [18], using singularity analysis in complex time.

For η=0.15 and H=1,000 we consider the ordered orbit E with initial conditions x=0, y=0, p_x=15, p_y=42.13 and the chaotic orbit F with initial conditions x=0.5, y=0, p_x=0.5, p_y=44.72 and the initial deviation vectors (dx(0), dy(0), d p_x (0), d p_y (0)) are (1,0,0,0) and (0,0,1,0). As we see in figure 2 the SALI of the ordered orbit E remains almost constant, fluctuating around SALI≅0.97 (grey line), while the SALI of the chaotic orbit F decreases abruptly reaching the limit of accuracy of the computer (10^{-16}) after about 1,000 time units (lack line). This behaviour is similar to the one encountered in the case of the 4D map (5) (figure 1(b)), since the phase space of the Hamiltonian system is four-dimensional as in the 4D map.

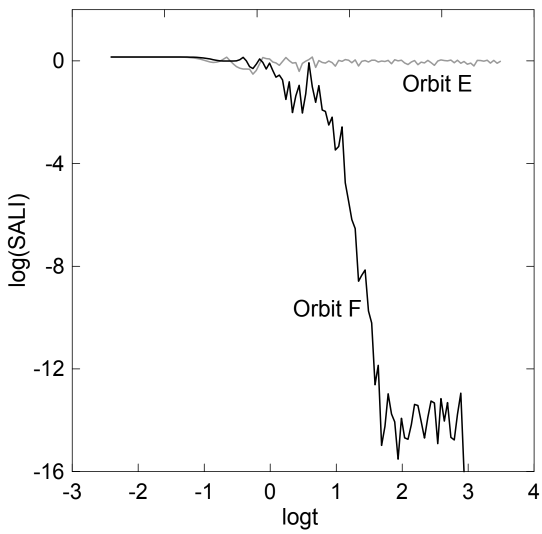

Figure 2. The evolution of the smaller alignment index SALI as a function of time t, in log-log scale, for the Hamiltonian (6) with H=1000 and η=0.15, for the ordered orbit E with initial conditions x=0, y=0, p_x =15, p_y=42.13 (grey line) and for the chaotic orbit F with initial conditions x=0.5, y=0, p_x=0.5, p_y=44.72 (black line).

For H=1000 and η=0.15 there exist regions in phase space where the motion is chaotic. These regions correspond to the area filled with scattered points in the Poincaré surface of section (x, p_x) (figure 3(a)), while ordered motion corresponds to the islands formed by the invariant smooth curves. Since the values of the SALI tend to completely different values for ordered and chaotic orbits, its computation for a sample of initial conditions can be used to distinguish between regions of phase space where ordered (or chaotic motion) occurs. So for any point on the Poincaré surface of section (x, p_x) we compute the value of the SALI for t=3,000 time steps. The points for which SALI<10^{-8} are marked with black dots in figure 3(b), while for 10^{-8}≤SALI<10^{-4} they are marked with grey dots. The resulting image indicates the regions where the motion is chaotic (black points) or slightly chaotic (grey points) leaving blank the region where ordered motion occurs. The resemblance with the region seen as chaotic in figure 3(a) is obvious. So, starting with any initial condition, the computed value of the SALI rapidly gives a clear view of chaotic vs. ordered motion even in conservative systems described by ordinary differential equations, where surface of section plots are already time consuming for 2 degrees of freedom and practically useless for systems of higher dimensionality.

4 CONCLUSIONS

In this paper, we have given some examples of Hamiltonian systems and symplectic maps where the computation of the smaller alignment index SALI allows us to distinguish in a “cost – efficient” way between ordered and chaotic orbits.

An advantage of using the SALI in Hamiltonian systems or in multidimensional maps is that usually the chaotic nature of an orbit can be established beyond any doubt. This happens because when the orbit under consideration is chaotic, the SALI becomes equal to zero, in the sense that it reaches the limit of the accuracy of the computer. After that time the two deviation vectors are identical (equal or opposite), since their coordinates are represented by the same or opposite numbers in the computer. Thus they have exactly the same evolution in time and cannot be separated.

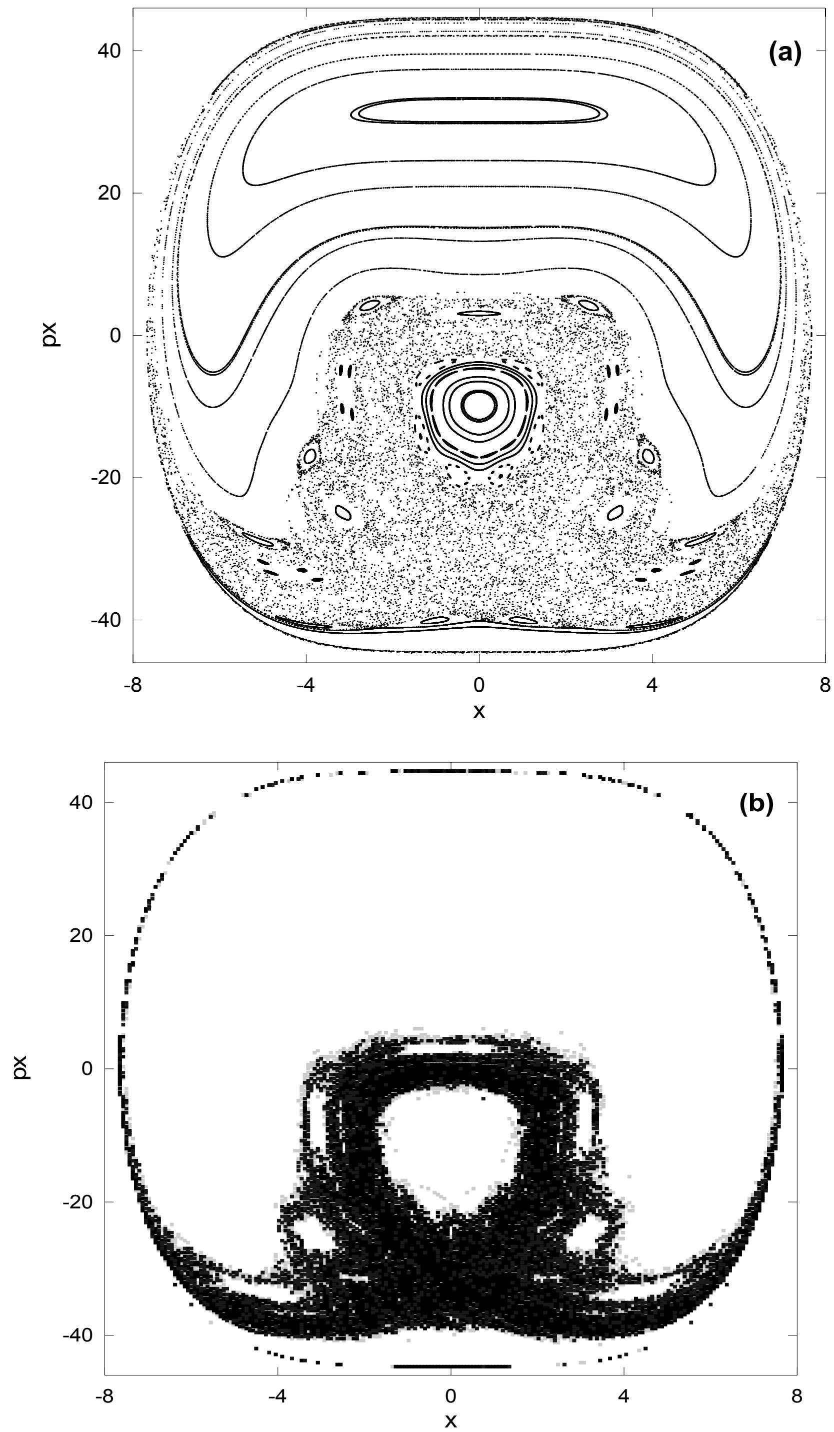

Figure 3. (a) The Poincaré surface of section (x, px) of Hamiltonian (6) for H=1,000 and η=0.15. (b) Regions of different values of the SALI on the Poincaré surface of section (x, px). Black points correspond to initial conditions that give SALI<10^{-8} after 3,000 time steps (chaotic motion), while grey points to initial conditions that give 10^{-8}≤SALI<10^{-4} (slightly chaotic motion). The resemblance of the region of chaotic motion in the two panels is evident.

In 2D maps the SALI tends to zero along both ordered and chaotic orbits, for reasons that we have explained. However, they do so at very different time rates and hence the distinction between these two behaviours can still be accurately made. Our approach, in fact, begins to be truly valuable for Hamiltonian systems of 2 degrees of freedom (where detailed surface of section plots are too costly) and promises to become extremely useful for higher than 2 degree of freedom Hamiltonians and higher dimensional symplectic maps.

Indeed, it is in such cases that the dynamics becomes particularly complex and there exist regions with varying “degrees” of chaotic behaviour, where orbits are seen to diffuse at very different rates [19]. Furthermore, there are periodic orbits of remarkable spatial complexity and varying numbers of stable and unstable directions in their tangent space [20], which may very well turn out to seriously affect the behaviour of the orbits in their vicinity.

It seems to us, therefore, that our previously established and well tested methods for computing long periodic orbits for these systems [21], combined with the evaluation of the SALI can constitute a useful quantitative indicator of the different properties of chaotic motion in these higher dimensional systems. In fact, we have already started to compare the values of our index with those of diffusion rates and Lyapunov exponents in the neighbourhood of such unstable periodic orbits and results will be reported soon in a future publication.

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