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# AN EFFICIENT METHOD FOR COMPUTING PERIODIC ORBITS OF CONSERVATIVE DYNAMICAL SYSTEMS

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## ABSTRACT

The accurate computation of periodic orbits and the precise knowledge of their properties are very important for studying the behavior of many dynamical systems of physical interest. In this paper, we present an efficient numerical method for computing to any desired accuracy periodic orbits (stable, unstable and complex) of any period. This method always converges rapidly to a periodic orbit independently of the initial guess, which is important when the mapping has many periodic orbits, stable and unstable close to each other, as is the case with conservative systems. We illustrate this method first, on the 2–Dimensional quadratic Hénon's mapping, by computing rapidly and accurately several periodic orbits of high period. We also apply our method here to a 3–D conservative mapping as well as a 4–D complex version of Hénon's map.

## 1. INTRODUCTION

It is well known that two dimensional (2–D) mappings of the $(x_1, x_2)$ plane onto itself defined by :

$$T : \begin{cases} \hat{x}_1 = \varphi_1(x_1, x_2), \\ \hat{x}_2 = \varphi_2(x_1, x_2), \end{cases} \tag{1.1}$$

can be used to study dynamical systems with two degrees of freedom. Such mappings can model conservative dynamical systems, if the determinant of the Jacobian of the map, i.e. $|det\, J_T| = 1$, or dissipative ones if $|det\, J_T| < 1$, [2, 3, 10, 11]. We say that $X = (x_1, x_2)$ is a fixed point of $T$ if $T(X) = X$ and a fixed point of *order p*, (or a periodic orbit of *period p*), if :

$$X = T^p(X) \equiv \underbrace{T\Big(T(\cdots(T(X))\cdots)\Big)}_{p \text{ times}}. \tag{1.2}$$

In general, it is difficult to find in the literature efficient methods for computing orbits of high period if the mapping is not decomposable into involutions [6, 20]. Also, traditional iterative schemes, such as Newton's method and related classes of algorithms [5, 16], often fail since they converge to a fixed point almost independently of the initial guess, while there may exist several different fixed points, close to each other, which are all desirable for the applications. Moreover, these methods are affected by the mapping evaluations taking large values in the neighborhood of *unstable* or *saddle-hyperbolic* periodic orbits, or may fail due to the nonexistence of derivatives or poorly behaved partial derivatives near fixed points [5, 16].

In this paper, we describe an efficient numerical method for rapidly computing periodic orbits (stable or unstable) of any period and to any desired accuracy. This method exploits topological degree theory to provide a criterion for the existence of a periodic orbit of an iterate of the mapping within a given region. More specifically, the method begins by constructing a polyhedron in such a way that the value of the topological degree of an iterate of the mapping relative to this polyhedron is $\pm 1$, which means that there exists a periodic orbit within this polyhedron. Then it repeatedly subdivides its edges (and diagonals) so that the new polyhedron also contains a periodic orbit within its interior, avoiding any computation of the topological degree. These subdivisions continue iteratively until the periodic orbit is computed to the desired accuracy.

This method is especially useful for the computation of high period orbits (stable or unstable) and is quite efficient, since the only computable information required is the algebraic signs of the components of the mapping. Thus it is not affected by the mapping evaluations taking large values in neighborhoods of unstable periodic orbits.

In the next section we start by giving a criterion for the existence of a periodic orbit inside a converging sequence of smaller and smaller characteristic polyhedra (the so-called CP–criterion). Also, in that section, we describe a generalized bisection method used in combination with the CP–criterion to compute the desired periodic orbit to any accuracy.

In Section 3, this procedure is applied to the calculation of stable and unstable periodic orbits of the 2–D Hénon's mapping for periods which reach up to the thousands. Then in Section 4, we apply our methods to periodic orbits of a 3–D conservative mapping and a 4–D complex version of the 2–D Hénon's map.

Finally we end, in Section 5, with some concluding remarks and a discussion of ongoing work on the application of these methods to symplectic mappings, which are known to model the behavior of Hamiltonian systems.

## 2. THE CP–CRITERION

In this section, we implement topological degree theory to give a criterion for the existence of a periodic orbit within a given region of the phase space of the system. This criterion is based on the construction of a sequence of "characteristic polyhedra" within a scaled translation of the unit cube. The concept of a characteristic polyhedron will be reviewed and a procedure for its construction will be presented. The theoretical development of the concepts employed here can be found in [23, 28].

As we said previously, the problem of finding periodic orbits of nonlinear mappings $T = (\varphi_1, \varphi_2, \dots, \varphi_n)$: $\mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ of period $p$ amounts to finding points $X^\star = (x_1^\star, x_2^\star, \dots, x_n^\star)^\top \in \mathcal{D}$ which satisfy the following equation :

$$T^p(X^\star) = X^\star. \tag{2.1}$$

Obviously, the problem of finding such a periodic orbit is equivalent to solving the following system :

$$F(X) = \mathcal{O}, \tag{2.2}$$

with $F = (f_1, f_2, \ldots, f_n) = T^p - I_n$, where $I_n$ indicates the identity mapping and $\mathcal{O} = (0, 0, \ldots, 0)^\top$ is the origin of $\mathbb{R}^n$.

Many problems require the solution of systems of nonlinear equations for which Newton's method and related classes of algorithms [5, 16] fail due to nonexistence of derivatives or poorly behaved partial derivatives. Also, Newton's method, (as well as Newton-like methods), often converge to a solution $X^*$ of $F(X) = \mathcal{O}$ almost independently of the initial guess, while there may exist several solutions nearby all of which are desired for the application. Because of these reasons, various approaches based upon topological degree theory and generalized bisection methods have been investigated in recent years [7, 10, 22, 23, 24, 28].

Bisection methods for finding solutions of systems of equations depend on a criterion, which guarantees that a solution lie within a given region. Then this region is subdivided in such a way that the criterion can again be applied to the new refined region.

In one dimension, this criterion consists of the product of the signs of the function evaluations at the endpoints of a given interval. Specifically, if one desires to locate a solution of a equation $f(x) = 0$ in the interval $(a, b)$ where $f: (a, b) \subset \mathbb{R} \to \mathbb{R}$ one examines whether the following relation is fulfilled :

$$sgn\, f(a) \cdot sgn\, f(b) = -1, \tag{2.3}$$

where $sgn$ is the sign function with values :

$$sgn\, \psi = \begin{cases} -1, & \text{if} \quad \psi < 0 \,; \\ 0, & \text{if} \quad \psi = 0 \,; \\ 1, & \text{if} \quad \psi > 0 \,. \end{cases} \tag{2.4}$$

If (2.3) holds, then we know that there is at least one solution within $(a, b)$. This is known as *Bolzano's existence criterion* and it can be generalized to higher dimensions [15, 25].

Instead of Bolzano's criterion however one may also use condition :

$$deg[f, (a, b), 0] = \frac{1}{2}\{sgn\, f(b) - sgn\, f(a)\}, \tag{2.5}$$

where $deg[f, (a, b), 0]$ is the *topological degree of f at the origin relative to* $(a, b)$. Now, if the value of $deg[f, (a, b), 0]$ is not zero, we know with certainty that there is at least one solution in $(a, b)$, since, in that case, Bolzano's criterion is fulfilled. The value of $deg[f, (a, b), 0]$ gives additional information concerning the behavior of the solutions of $f(x) = 0$ in $(a, b)$ relative to the slopes of $f$ [10]. For example, if $deg[f, (a, b), 0] = 1$ (which means that $f(b) > 0$ and $f(a) < 0$), then the number of solutions at points where $f(x)$ has a positive slope exceeds by one the number of solutions at points at which $f(x)$ has a negative slope.

The topological degree as well as Bolzano's criterion transfer all information regarding the roots to the boundary of the given region. Now, using the value of the topological degree, (or Bolzano's criterion), one can calculate a solution of $f(x) = 0$ by bisecting the interval $(a, b)$. So we subdivide $(a, b)$ into two subintervals $(a, c], [c, b)$, where $c = (a + b)/2$ is the midpoint of $(a, b)$, and keep the subinterval for which the value of the topological degree is not zero relative to itself, by checking the information on the boundaries. In this way, we keep at least one solution within a smaller interval. We can continue this procedure until the endpoints of the final subinterval differ from each other by less than a fixed amount. This method is called *bisection method* and can be expressed as follows [12, 23, 24, 27, 29] :

$$x_{n+1} = x_n + sgn\, f(a) \cdot sgn\, f(x_n) \cdot (b-a)/2^{n+1}, \quad x_0 = a, \qquad n = 0, 1, \ldots . \tag{2.6}$$

Of course, it converges to a solution $x^*$ in $(a,b)$ if for some $x_n$, $n = 1, 2, \ldots$, we have :

$$sgn\, f(x_0) \cdot sgn\, f(x_n) = -1. \tag{2.7}$$

Also, the minimum number of iterations $\gamma$, which are required to obtain an approximate solution $x'$ such that $|x' - x^*| \leq \varepsilon$ for some $\epsilon \in (0,1)$ is given by :

$$\gamma = \left\lceil \log_2\left((b-a) \cdot \varepsilon^{-1}\right) \right\rceil, \tag{2.8}$$

where the notation $\lceil \cdot \rceil$ refers to the smallest integer which is not less than the real number quoted.

Based on the relation (2.8), it has been proved in [21] that the bisection method is *optimal*, i.e. that it possesses asymptotically the best rate of convergence. Also, it is worth mentioning that the only computable information required by the bisection method is the signs of various function evaluations, and, as it always converges within the given interval $(a,b)$ independently of its length, it is a global convergence method. Using then the relation (2.8), one can easily find out the number of iterations needed for the attainment of an approximate solution to a predetermined accuracy.

It would be very desirable, of course, to generalize the above bisection method to higher dimensions. To do this we extend Bolzano's criterion in the following way : Let us define a characteristic $n$–polyhedron by constructing the $2^n \times n$ matrices $\mathcal{M}_n$ whose rows are formed by all possible combinations of $-1$, $1$. For example for $n = 1, 2, 3$ we have :

$$\mathcal{M}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathcal{M}_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \mathcal{M}_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{2.9}$$

Now, for $n = 1$ we consider the segment $[x_1, x_2]$ and evaluate the sign of $f(x)$ at the endpoints. Then, if the matrix :

$$\mathcal{S}(f;\, [x_1, x_2]) = \begin{bmatrix} sgn\, f(x_1) \\ sgn\, f(x_2) \end{bmatrix}, \tag{2.10}$$

agrees with $\mathcal{M}_1$, up to a permutation of the rows, then we say that $[x_1, x_2]$ is a *characteristic polyhedron.* Suppose now that $\Pi^n = \langle \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{2^n} \rangle$ is an oriented $n$–dimensional polyhedron with $2^n$ vertices, $\Upsilon_k \in \mathbb{R}^n$, (i.e. an orientation has been assigned to its vertices), and let $F = (f_1, f_2, \ldots, f_n)\colon \Pi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear mapping from $\Pi^n$ into $\mathbb{R}^n$. Then we call a *matrix of signs associated with $F$ and $\Pi^n$*, and denote it by $\mathcal{S}(F;\, \Pi^n)$, the $2^n \times n$ matrix whose entries in the $k$th row are the corresponding coordinates of the vector :

$$sgn\, F(\Upsilon_k) = \left(sgn\, f_1(\Upsilon_k), sgn\, f_2(\Upsilon_k), \ldots, sgn\, f_n(\Upsilon_k)\right)^{\top}. \tag{2.11}$$

The $n$–polyhedron $\Pi^n = \langle \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{2^n} \rangle$ in $\mathbb{R}^n$ is called a *characteristic $n$-polyhedron relative to* $F = (f_1, f_2, \ldots, f_n)\colon \Pi^n \subset \mathbb{R}^n \to \mathbb{R}^n$, if the matrix of signs associated with $F$ and $\Pi^n$, $\mathcal{S}(F;\, \Pi^n)$, is identical with the $n$–complete matrix $\mathcal{M}_n$.

Suppose now that $\Pi^n$ is a characteristic $n$–polyhedron and that $F = (f_1, f_2, \ldots, f_n)$: $\Pi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then, under suitable assumptions on the boundary of $\Pi^n$ the value of the topological degree of $F$ at $\mathcal{O}$ relative to $\Pi^n$ is given by :

$$deg[F, \Pi^n, \mathcal{O}] = \sum_{X \in F^{-1}(\mathcal{O})} sgn\, det\, J_F(X) = \pm 1 \neq 0, \tag{2.12}$$

(see [28, Theorem 2.9]), which implies the existence of a periodic orbit inside $\Pi^n$. For a detailed description of how to construct a characteristic $n$-polyhedron to locate a desired periodic orbit see [23, 24, 26].

Next, we turn to the description of a generalized bisection method, used in combination with the CP–criterion outlined above, for computing periodic orbits of any period and any desired accuracy. This method has all the advantages of one–dimensional bisection and is particularly useful in cases where the period of the orbit is high since it always converges within the initial specified region. Moreover, it is very efficient, since the only information it requires is the algebraic signs of the components of the mapping.

This method is based on the refinement of a characteristic $n$–polyhedron and may be called *characteristic bisection.* In the literature several bisection methods are available [7, 10, 22] that require the computation of the topological degree in order to secure its nonzero value. In the bisection method outlined here, however, the computation of the topological degree is avoided by making sure that it retains a nonzero value at every iteration.

The method bisects a characteristic $n$–polyhedron $\Pi^n$ in such a way that the new refined $n$–polyhedron is also a characteristic one. To do this it computes the midpoint of a proper 1–simplex (edge) of $\Pi^n$ and uses it to replace that vertex of $\Pi^n$ for which the vectors of their signs are identical (see [23, 24, 26, 28] for details).

The number of characteristic bisections of the proper 1–simplexes of a $\Pi^n$ required to obtain a new refined characteristic $n$–polyhedron $\Pi^n_\star$ such that the length of its longest edge $\Delta(\Pi^n_\star)$ satisfies $\Delta(\Pi^n_\star) \leq \varepsilon$, for some $\varepsilon \in (0,1)$, is given by :

$$\zeta = \left\lceil \log_2 \left( \Delta(\Pi^n) \cdot \varepsilon^{-1} \right) \right\rceil, \tag{2.13}$$

(see [28] for a proof). Based on the above formula we conjecture that our generalized bisection method is "optimal". We plan, however, to address the complete proof of this in a future publication.

## 3. An Application to Hénon's 2–D Mapping

We shall first illustrate the method of Section 2 on a quadratic area–preserving 2–D mapping originally due to Hénon [11] :

$$T : \begin{cases} \hat{x}_1 = x_1 \cos a - (x_2 - x_1^2) \sin a, \\ \hat{x}_2 = x_1 \sin a + (x_2 - x_1^2) \cos a. \end{cases} \tag{3.1}$$

For a point $X^\star$ which has period $p$ this means that $T^p(X^\star) = X^\star$ so we consider the following mapping :

$$F = (f_1, f_2) = T^p - I_2, \tag{3.2}$$

where $I_2$ is the identity mapping, and solve, for any period $p$, the following system of 2 equations in 2 unknowns :

$$F(X) = \mathcal{O} = (0,0)^\top. \tag{3.3}$$

To do this we choose a starting point :

$$X^0 = (x_1^0, x_2^0)^\top, \tag{3.4}$$

and two stepsizes in each coordinate direction :

$$H = (h_1, h_2)^\top, \tag{3.5}$$

in such a way that the box thus constructed forms a domain within which we will attempt to locate and compute a solution of the system $F(X) = \mathcal{O}$, which is a periodic orbit of the mapping $T^p$.

Suppose now that a periodic point $X_1^\star$ has been computed within a predetermined accuracy $\varepsilon$ such that :

$$\|T^p(X_1^\star) - X_1^\star\| \leq \varepsilon. \tag{3.6}$$

Then, in order to compute all the other points $X_i^\star$, $i = 2, \ldots, p$ with the same accuracy $\varepsilon$ we iterate the mapping $T$ as follows : First we obtain an approximation $\hat{X}_2$ of the next point of the orbit $X_2^\star$ by the following relation :

$$\hat{X}_2 = T(X_1^\star), \tag{3.7}$$

and check if the following relation is fulfilled :

$$\|T^p(\hat{X}_2) - \hat{X}_2\| \leq \varepsilon. \tag{3.8}$$

Let us illustrate the above procedure on the mapping (3.1). In general, a visualization of the orbits of the mapping is very helpful for choosing the starting point $X^0$ and the stepsizes $H$. In any case, if such a visualization is not available, one can search within various boxes taking a suitable grid for the domain of interest. The phase plots (Figures 1, 2) which are shown here are drawn using the new software package GIOTTO of [29].

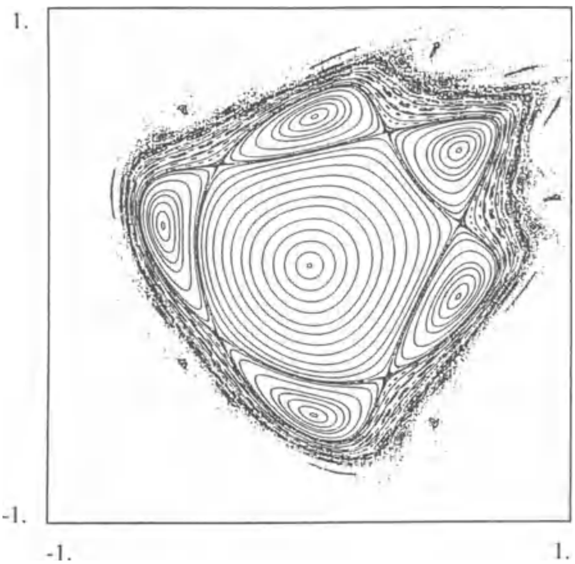


Figure 1. Hénon's mapping for $\cos a = 0.24$.

Taking for example $a = \cos^{-1}(0.24)$, we can see in Figure 1 that there is a chain of five big islands around the origin. So, in this case, we shall search for five elliptic and five hyperbolic fixed points of period $p = 5$. The reason for this choice of $a$ is that the corresponding phase plot has a large region of stability around the origin (see [3, p.1868], [11, p.298]) and may be of interest to applications in beam dynamics.

Now in order to find the elliptic periodic orbit of this period we choose one island, include it into a box by taking appropriate values for $X^0$ and $H$, e.g. :

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ -0.3 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix},$$

and apply our method with accuracy $\varepsilon = 10^{-16}$. Thus, we compute the following stable periodic orbit :

$$\begin{aligned}
X_1^5 &= (\ \ 0.5672405470221847, \ -0.1223202134278941)^\top, \\
X_2^5 &= (\ \ 0.5672405470221847, \ \ \ 0.4440820516139216)^\top, \\
X_3^5 &= (\ \ 0.0173925844399303, \ \ \ 0.5800185952239573)^\top, \\
X_4^5 &= (-0.5585984457571741, \ \ \ 0.1560161118011652)^\top, \\
X_5^5 &= (\ \ 0.0173925844399305, \ -0.5797160932304572)^\top,
\end{aligned}$$

utilizing about 6.5 *m*secs of CPU time on the CERN VAX 9000–410 system.

It is well known that periodic orbits are identified by their *rotation number* $\sigma$ :

$$\sigma = \frac{\nu}{2\pi} = \frac{m_1}{m_2},$$

where $\nu$ is the *frequency* of the orbit and $m_1$, $m_2$, are two positive integers (see [4, 9]). So from the sequence with which the above points are created on the $x_1, x_2$ plane, we can infer the rotation number of this orbit $\sigma = m_1/m_2 = 1/5$, indicating that it has produced $m_2 = 5$ points by rotating around the origin $m_1 = 1$ times.

Now to compute the unstable periodic orbit of period $p = 5$ we may choose the initial values :

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.7 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix},$$

and apply our method with the same accuracy $\varepsilon = 10^{-16}$. Thus, the following orbit is computed :

$$\begin{aligned}
X_1^5 &= (\ \ 0.2942106885737921, \ -0.4274862418615337)^\top, \\
X_2^5 &= (\ \ 0.5696326513533602, \ \ \ 0.1622406787439296)^\top, \\
X_3^5 &= (\ \ 0.2942106885737916, \ \ \ 0.5140461711325987)^\top, \\
X_4^5 &= (-0.3443814883177751, \ \ \ 0.3882084578625210)^\top, \\
X_5^5 &= (-0.3443814883177746, \ -0.2696098483665559)^\top.
\end{aligned}$$

also utilizing around 6.5 *m*secs of CPU time.

Similarly we can use this procedure to compute periodic orbits of higher periods. Looking at the phase plot (Figure 1) of Hénon's mapping for the same value of $a$ as before, we are able to distinguish 16 islands across the "boundary" of the mapping. Hence, in this case, we search for 16 stable and 16 unstable periodic orbits of period $p = 16$. To do this we choose one island and enclose it into a box by choosing appropriate values for $X^0$ and $H$ :

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.1 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

and apply our method (with accuracy of $\varepsilon = 10^{-16}$). Note that, from the sequence with which these points are created on the $x_1, x_2$ plane, we can infer the rotation number of this orbit :

$$\sigma = m_1/m_2 = 3/16,$$

indicating that it has produced $m_2 = 16$ points by rotating around the origin $m_1 = 3$ times.

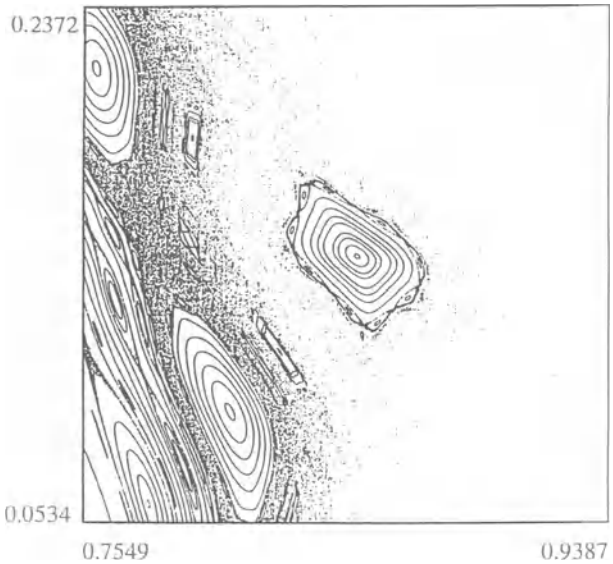


Figure 2. Magnification of a island of period 16 of Hénon's mapping for $\cos a = 0.24$.

Enlarging the vicinity of an island of period 16, we distinguish, in Figure 2, a chain of 8 islands around it, so the period of each of these islands is $p = 16 \times 9 = 144$. To compute all the points of this period, we start again with the computation of one which is stable, taking, e.g.

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.865 \\ 0.131 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.005 \\ 0.005 \end{pmatrix}.$$

Enlarging further the vicinity of an island of period 144 a chain of 9 islands appears around it, with period $p = 144 \times 9 = 1296$. Finally, by enlarging the vicinity of an island of this period we distinguish a chain of 8 islands around it, with period $p = 1296 \times 8 = 10368$, where each one has a chain of 6 islands around it, with period $p = 10368 \times 6 = 62208$, etc. The rotation numbers of these orbits, respectively, are $\sigma = 243/1296$, 1944/10368, 11664/62208 and the coordinates of some of their points are listed in Table 1. The calculation of the corresponding unstable orbits requires CPU times of the same, if not smaller magnitude.

It is now clear, from the above, that one can similarly proceed to calculate orbits of higher and higher period in CPU times which do not grow so rapidly as to make the calculation of these orbits impractical (see Table 1).

Table 1 . Rotation Numbers and Periodic Orbits of Hénon's 2–D Mapping

| Rotation number $\sigma$ | $x_1$ | $x_2$ | CPU time |
|---|---|---|---|
| 1/5 | 0.5672405470221847 | 0.4440820516139216 | 6.5 *m*sec |
| 3/16 | 0.8504309709743801 | 0.1490801034942473 | 11.1 *m*sec |
| 27/144 | 0.8685006897387088 | 0.1341772208865609 | 46 *m*sec |
| 246/1296 | 0.8656539452140570 | 0.1320418594308142 | 2.2 sec |
| 1944/10368 | 0.8655111610927791 | 0.1321457880746314 | 5.6 sec |

## 4. Periodic Orbits of Higher–Dimensional Maps

Recently, we have begun to study the sequence of period–doubling bifurcations in a 3–D mapping given by the equations :

$$T \; : \; \begin{cases} \hat{x}_1 = x_2 , \\ \hat{x}_2 = x_3 , \\ \hat{x}_3 = -x_1 + x_2^2 + x_3^2 + \mu \left( x_2^2 x_3 + x_2 x_3^2 \right) , \end{cases} \tag{4.1}$$

where $\mu$ is a free parameter. Such mappings are known to arise as special solutions of certain discretized lattice equations [17, 18], as well as in some problems of fluid dynamics [8]. The system (4.1) has the properties of being volume–preserving and reversible [20].

We have followed the bifurcations of an orbit of period 2 to an orbit of period 4 and then to orbits of period 8, 16, 32 etc., for several values of $\mu$, and have observed that they occur in a very similar way as in the case of 2–D area–preserving mappings : In other words, periodic orbits of period $2^n$ are "born" stable on a symmetry plane $x_1 = x_3$ of initial conditions $x_1, x_2$ at points where the "mother" orbit of period $2^{n-1}$ destabilizes.

This destabilization occurs with two of the eigenvalues of the period $2^{n-1}$ orbit "colliding" at $-1$ while the 3d eigenvalue is always $+1$. In Table 2 we list some of these bifurcation points computed by our method to an accuracy of $10^{-16}$.

Table 2 . Bifurcation Points for $\mu = 4$

| Bifurcation | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $2 \leftrightarrow 4$ | 0.9276675561404863 | -0.5446097058809972 | 0.9276675561404863 |
| $4 \leftrightarrow 8$ | 1.1388838325396091 | -0.5565872257732393 | 1.1388838325396091 |
| $8 \leftrightarrow 16$ | 1.0758404065131902 | -0.5017191586916615 | 1.0758404065131902 |
| $16 \leftrightarrow 32$ | 1.0912618737438293 | -0.5163084493440770 | 1.0912618737438293 |

These results lead us to believe that (4.1) may have one analytic integral for all $\mu$ and thus be reducible to a 2–D mapping [19]. Such an integral can be easily found for $\mu = 0$ :

$$\mathcal{I}_0 = \left(x_2 - x_1 - x_3 + x_2^2\right)^2 = \text{const.}, \tag{4.2}$$

and $\mu = \infty$,

$$\mathcal{I}_\infty = x_1^2 + x_2^2 + x_3^2 = \text{const.}, \tag{4.3}$$

but is as yet unknown for general values of $\mu$.

Furthermore, the reversibility and perhaps also integrability of (4.1) may be destroyed by multiplying the cubic terms in (4.1c) by two different parameters $\mu_1 \neq \mu_2$, thus breaking the symmetry of the equations under $x_1 \leftrightarrow x_3$. All this can be studied by following the bifurcation properties of period $2^n$ orbits, using the methods described in this paper and is currently under investigation [12].

Finally, let us apply our method to the computation of periodic orbits of Hénon's mapping (1.3) with complex components, which is an example of a real 4–D conservative mapping. In order to compute such complex periodic orbits we expand a given 2–D mapping to 4 dimensions by separating real and imaginary parts. Thus we construct a 4–D real map $T_4$ and solve the following system of 4 equations :

$$F = T^p - I_4 = \mathcal{O} = (0, 0, 0, 0)^\top. \tag{4.4}$$

where $I_4$ is the identity mapping.

Let us illustrate this procedure on Hénon's mapping (3.1) : Replacing $x_1$, $x_2$ by their complex form, $x_j + i\,y_j$, $j = 1,2$, $x_j, y_j \in \mathbb{R}$, yields the following system of equations :

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a & 0 & 0 \\ \sin a & \cos a & 0 & 0 \\ 0 & 0 & \cos a & -\sin a \\ 0 & 0 & \sin a & \cos a \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 - x_1^2 + y_1^2 \\ y_1 \\ y_2 - 2x_1 y_1 \end{pmatrix}, \tag{4.5}$$

cf. (4.4), for a given value of the frequency $a$.

In this case, of course, a visualization of these four dimensional orbits is quite complicated. For our purposes, it is convenient to consider two dimensional projections of these orbits and use them to obtain good initial values of $X^0$ and $H$. If that is not easily achieved, we may also take an appropriate grid of initial conditions in 4–D space to start the computation.

Let us apply our method to mapping (4.5) to compute some of its periodic orbits for $a = \cos^{-1}(0.24)$ and $p = 2$. Selecting a proper grid we thus calculate within an accuracy of $\varepsilon = 10^{-16}$ the following point of period 2 :

$$X_1^2 = \begin{pmatrix} -1.2773327473170111 \\ -1.0000000000000000 \\ 1.9056702094980712 \\ -2.4341749641783542 \end{pmatrix}.$$

Proceeding in the same way as before, we have been able to compute periodic orbits of various periods. For example, with $a = \cos^{-1}(0.24)$ and $p = 3$ we have computed within an accuracy of $\varepsilon = 10^{-16}$ the period–3 point :

$$X_1^3 = \begin{pmatrix} -1.2773327473170111 \\ 0.0611205432937182 \\ 1.2285511493682861 \\ -1.5692685842514041 \end{pmatrix}.$$

Note that the mapping (4.5), while conservative, it is not symplectic [13, 14]. Our method, however, can also be easily applied to periodic orbits of $2N$–dimensional symplectic mappings (with $N > 1$) which are analogous to $N$–degree of freedom Hamiltonian systems. In fact, we have already started such a generalization and computed some of the low order periodic orbits of a 4–dimensional quadratic mapping, which has two frequencies $a_1$ and $a_2$ and is closely related to (4.5) :

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{pmatrix} \cos a_1 & -\sin a_1 & 0 & 0 \\ \sin a_1 & \cos a_1 & 0 & 0 \\ 0 & 0 & \cos a_2 & -\sin a_2 \\ 0 & 0 & \sin a_2 & \cos a_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1 x_3 \end{pmatrix}. \tag{4.6}$$

This is an interesting model, since mappings of this type are of direct relevance to the dynamics of particle beams (in a 4–Dimensional phase space) passing repeatedly through FODO cells of magnetic focusing elements [4]. In this context, the computation of high period orbits can be quite useful in helping us study the existence and structure of nearby invariant surfaces, by investigating e.g. the validity of Greene's criterion in these higher–dimensional systems [9]. More detailed results on this question, however, are expected to appear in a future publication [30].

## 5. CONCLUDING REMARKS

An efficient method for rapidly and accurately computing periodic orbits of nonlinear mappings has been described in this paper. This method exploits topological degree theory to construct a characteristic polyhedron and locate the periodic orbit within a given region, without making any computation of the topological degree. Then it repeatedly subdivides this polyhedron to compute the periodic orbit rapidly and to any accuracy.

The method is very efficient, since the only computable information that is required is the algebraic signs of the components of the mapping. Thus it is not affected by the mapping evaluations taking large or imprecise values. Moreover, it always converges rapidly to a periodic orbit within the initial specified region independently of the initial guess, which is particularly useful in cases where the period of the periodic orbit is very high and the mapping has many periodic orbits, close to each other.

It is also a globally convergent method, it can be applied to nondifferentiable continuous functions and does not involve derivatives or approximations of such derivatives. Furthermore, using this method the number of iterations needed to find out a periodic orbit to a predetermined accuracy is known. Based on this we have conjectured that the method of the present paper is optimal (i.e. that it possesses asymptotically the best rate of convergence).

We have illustrated this method first to Hénon's 2–Dimensional mapping, used in the study of beam dynamics in particle accelerators and have succeeded in overcoming the difficulties of other schemes in the computation of periodic orbits which are strongly unstable and/or of very high period.

Moreover, we have applied our method to higher dimensions, by using it to calculate periodic orbits of 3–D and 4–D mappings of conservative dynamical systems.

Since the method is especially suited for the calculation of orbits of high period, it might be used to approximate quasiperiodic orbits which lie on invariant tori of nonlinear mappings. Thus, we may be able to check in the case of 4 (and higher) dimensions the validity of the famous Greene's conjecture concerning the break-up of invariant surfaces and the onset of large scale chaos [9]. Such topics will be addressed, however, in a future publication.

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