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Beyond the pole-barn paradox: How the pole is caught

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Hundred years after Einstein formulated the special theory of relativity [1], we review one of its famous paradoxes: the length contraction paradox, about a farmer who wants to catch a pole in a short barn. The original paradox describes the situation in which the velocity of the pole with respect to the barn is constant, but we go beyond that and focus upon the actual catch in which the pole is brought to a standstill. This natural follow-up question, rarely addressed in textbooks, turns out to have a very surprising outcome.

Key words: Special relativity, paradox, Lorentz contraction, shock wave

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1 Introduction

One of the most popular paradoxes of Special Relativity, appearing in almost every course on the subject, is the pole-barn paradox [2, 3, 4, 5, 6, 7]. It can be stated as follows: A farmer, who owns a barn 4 m long (ℓ_0) and a ladder of length 5 m (L_0), would like to fit the ladder inside the barn. To accomplish this, he plans to use the Lorentz contraction, i.e., the relativistic effect that objects 'shrink' when moving with constant velocity with respect to an observer. The farmer tells his son to take the ladder and run into the barn with speed v, as in Fig. 1, since then the pole's length will be reduced (in the rest frame of the barn) to:

$$L_0' = L_0\sqrt{1-\beta^2}\ , \quad \text{with} \quad \beta = \frac{v}{c}\ . \tag{1}$$

The square root term is the famous Lorentz contraction factor and $c = 3\cdot 10^8$ m/s denotes the speed of light. In order to make the pole fit in the barn, the farmer requires $L_0' \leq \ell_0$, or equivalently:

$$\beta \geq \sqrt{1-\left(\frac{\ell_0}{L_0}\right)^2}\ . \tag{2}$$

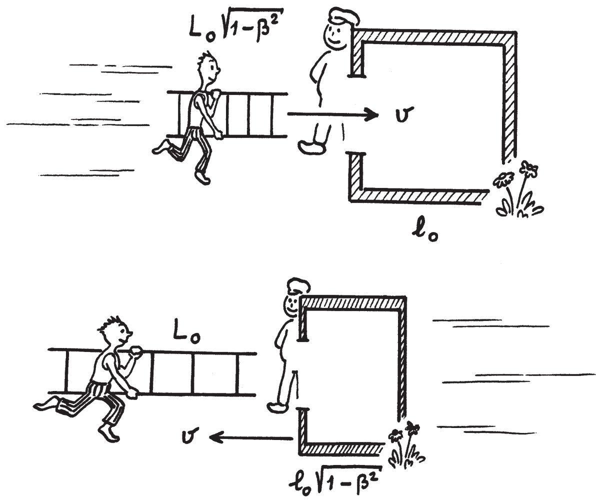

FIG. 1. The pole-barn paradox in the farmer's frame of reference (top), and in the son's frame of reference (bottom).

For the given values of ℓ_0 and L_0 this means that the son should run with speed $\beta \geq 0.6$.

Now, the first basic principle of special relativity is that the laws of physics are the same for two observers who move with a constant velocity with respect to one another [1]. Thus in the son's frame of reference it is the barn that is shortened (Fig. 1 bottom) and the pole does *not* fit. That is the paradox, and we briefly review its resolution in Section 2, which holds as long as the speed of the pole with respect to the barn (v) remains constant.

The actual catching of the pole, in which it is brought to rest, is discussed much more rarely [4, 5, 6]. Yet it is a natural follow-up question to the paradox and it illustrates the fact that the *outcome* of an experiment must be the same for all observers, even if the views on how this outcome is reached may (and will) differ from one inertial frame to the other. In Section 3 we therefore address the problem of bringing the ladder to a standstill. Intriguingly, it turns out that the speed required for a successful catch is *much smaller* than the one suggested by Eq. (2). Finally, in Section 4 we summarize our results and put the catch into its proper physical context.

2 Resolution of the paradox

The paradox disappears as soon as one analyzes the situation in the relativistically correct way, recognizing that it is not only a question of length (space) but also of time. That is, we have to work in space-time and we have to consider "events", an event being anything that happens at a given point in space-time. In the present case there are two relevant events:

(A) the front of the ladder hits the back wall,

and

(B) the rear end of the ladder enters the door.

According to the farmer event B comes first (so there is an instant when the ladder is completely

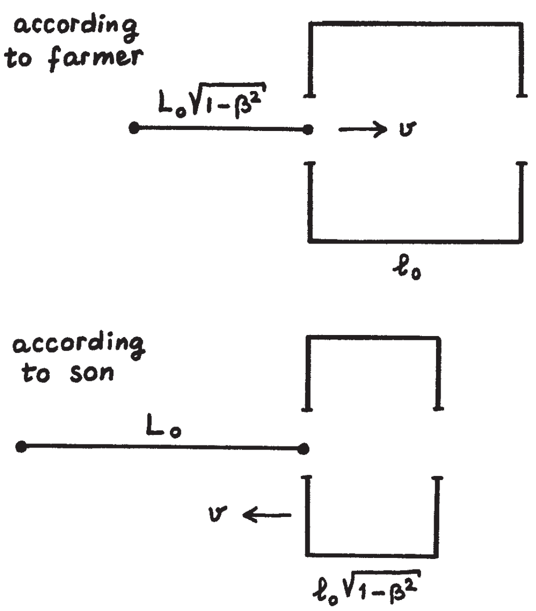

FIG. 2. The situation at $t = t' = 0$, in the reference frames of the farmer (top) and the son (bottom). The back door at the right-hand side has been included to allow the pole to pass through the barn uninterruptedly with constant velocity.

inside the barn and can be caught) but according to the son event A comes first. This is confirmed by a straightforward inspection of Fig. 2, where the views of the farmer and the son are put side by side. The clocks have been set such that $t = t' = 0$ when the front of the ladder enters the barn ("event zero"). We have also introduced a back door at the right-hand side of the barn, so the pole can pass through the barn uninterruptedly with constant velocity, which is necessary for the present analysis.

Event A: In the farmer's frame of reference, the front of the ladder goes through the back door at time

$$t_A = \frac{\ell_0}{v} \,, \tag{3}$$

while according to the son this happens at [8]

$$t'_A = \frac{\ell_0\sqrt{1-\beta^2}}{v} \,. \tag{4}$$

Event B: According to the farmer, the back end of the ladder passes through the front door at

$$t_B = \frac{L_0\sqrt{1-\beta^2}}{v} \tag{5}$$

and according to the son at

$$t'_B = \frac{L_0}{v} . \tag{6}$$

It is easily shown from Eqs. (3)-(6) that at sufficiently large velocities ($v \geq 0.6c$ for $\ell_0/L_0 = 4/5$) the order of A and B is not the same for the farmer and the son; e.g., $v = 0.7c$ gives $t_A = 5.71/c$, $t_B = 5.10/c$, $t'_A = 4.08/c$, and $t'_B = 7.14/c$. So according to the farmer event B comes first ($t_A > t_B$), while for his son event A comes first ($t'_A < t'_B$).

This resolves the paradox: There is no contradiction at all, only a different perspective in space-time for the farmer and his son. The fact that an event does not occur simultaneously for observers moving with respect to one another is a common feature of special relativity, and if two events are not causally connected (as is the case for A and B, their separation in space-time being such that $|\Delta x| > c|\Delta t|$) even the *order* in which they occur may be reversed. This makes it possible that in the son's frame of reference (with the shortened barn and the normal-sized ladder) the ladder will always stick out from one side or the other, while in the farmer's frame (normal-sized barn, shortened ladder) the ladder is completely inside the barn between t_B and t_A.

3 Beyond the paradox: The pole is brought to rest

3.1 In the reference frame of the son

Now we come to the actual catching of the ladder. Let us assume that the farmer closes the barn at some moment between t_B and t_A, when the ladder (in his reference frame) is inside the barn [9]. Bringing the ladder to a standstill requires a tremendous deceleration, inducing considerable internal stresses in the ladder [2], starting when its front end hits the wall and comes to an abrupt stop [10] (Fig. 3): A shock wave carries the information of this collision through the successive parts of the ladder, agitating them violently; the part to the left of the shock remains at constant velocity until the shock wave passes. Assuming that the ladder survives the shock, at the end of the deceleration process (when the entire ladder has come to rest with respect to the barn), it attains its rest length L_0 again. The ladder will of course not fit nicely: it will bend, or break, or poke through the door. But the fact is that it is caught.

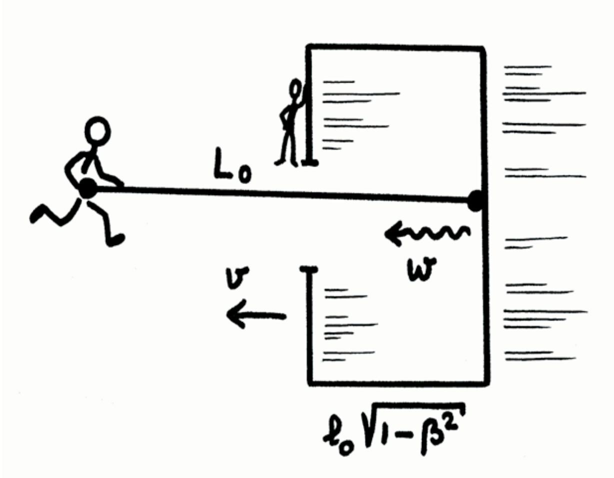

FIG. 3. The catch in the son's frame of reference: a race between shock wave and barn door towards the son. The situation is sketched at $t = t'_A$, when the ladder first hits the back wall.

How can this result be explained from the point of view of the son? The crucial point to note here is that the seeming difficulty (i.e., that according to the son the ladder is never completely inside the barn) only holds as long as the relative speed between ladder and barn is constant. And this is not the case anymore. Indeed, during the deceleration process even the different parts of the ladder itself are in motion with respect to each other, so the ladder temporarily does not represent one single inertial frame, but a range of (accelerated) frames. The length of the ladder during its deceleration is therefore no longer defined by the Lorentz contraction Eq. (1). There is no such thing as a rigid body in special relativity when speed changes.

Let us take up the series of events for the son at t'_A [Eq. (4)] when the front of the ladder hits the back wall, as sketched in Fig. 3. At this instant the son at the rear end of the ladder has no way of knowing yet that the front has already hit the wall. A shock wave is just starting to come towards him, at a speed w, which cannot exceed c (information cannot travel faster than the speed of light) and typically will be much smaller. The part of the ladder that has already been affected by the shock wave will be in turmoil, but for the son - until the wave reaches him - the situation is unchanged, with the door of the barn coming towards him at speed v. If v is large enough, the door reaches him before the shock wave does so, and the catch will be successful.

The door reaches him at

$$t' = t'_A + \frac{L_0 - \ell_0\sqrt{1-\beta^2}}{v} \;, \tag{7}$$

and the shock wave at

$$t' = t'_A + \frac{L_0}{w} \;. \tag{8}$$

So we require, for the catch to succeed,

$$\frac{L_0 - \ell_0\sqrt{1-\beta^2}}{v} \leq \frac{L_0}{w} \;. \tag{9}$$

This can also be written as $1 - f\sqrt{1-\beta^2} \leq v/w$ (with $f = \ell_0/L_0$ the barn-to-pole ratio), or equivalently $1 - \beta/(w/c) \leq f\sqrt{1-\beta^2}$. Squaring both sides this gives a quadratic inequality for β:

$$\left[\frac{1}{(w/c)^2} + f^2\right]\beta^2 - \frac{2\beta}{w/c} + (1-f^2) \leq 0. \tag{10}$$

Taking the equality sign, this is an ordinary quadratic equation, whose two solutions define the bounds of the interval of β-values for which Eq. (10) holds. For our problem (which is to find the minimally required value of β) we are interested in the lower bound of the interval, i.e.,

$$\beta \geq \frac{1 - \sqrt{1-(1-f^2)\{1+(w/c)^2 f^2\}}}{\{1+(w/c)^2 f^2\}}\,\frac{w}{c} \;. \tag{11}$$

This is the condition for a successful catch, written in dimensionless form.

Figure 4 shows the required velocity of the pole as a function of the shock speed w/c for two different values of f. For small w/c the required velocity is seen to follow a straight line (dashed in Fig. 4) from which it curves upward for growing w/c. This is confirmed by expansion of Eq. (11): $\beta \geq (1-f)(w/c) + \frac{1}{2}f(1-f^2)(w/c)^3 + O(w/c)^5$.

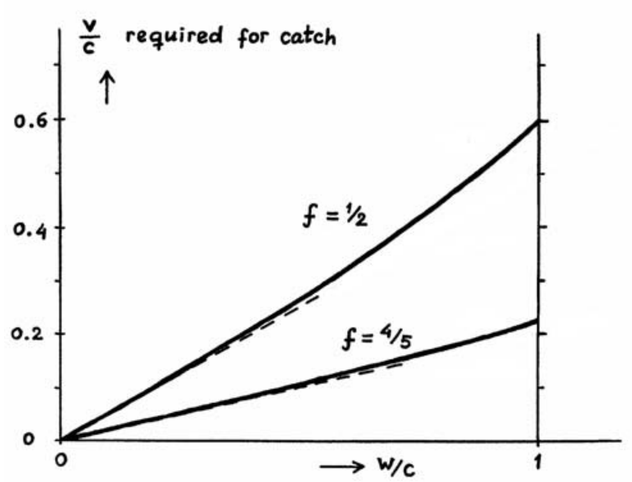

FIG. 4. The velocity of the pole (v/c) required for a successful catch, as a function of the shock speed w/c, for two representative values of the barn-to-pole ratio $f = \ell_0/L_0$ [Eq. (11)]. The required velocity gently curves upwards from the dashed line $v/c = (1-f)w/c$ and ends [at $w/c = 1$] at the value $v/c = (1-f^2)/(1+f^2)$, cf. Eq. (12). Realistic shock speeds of several km/s lie towards the extreme left of the diagram.

The condition (11) is considerably less demanding than that of Eq. (2), $\beta \geq \sqrt{1-f^2}$, which was based upon the erroneous notion that the pole should fit into the barn while moving at uniform speed v. Even in the worst-case scenario $w = c$, which makes the catch as hard as possible, the new condition requires a velocity that is not nearly as large [Eq. (11) with $w/c = 1$]:

$$\beta \geq \frac{1-f^2}{1+f^2} \;. \tag{12}$$

For $f = \ell_0/L_0 = 4/5$ the uniform-speed reasoning would have $\beta \geq 3/5 = 0.60$, whereas Eq. (12) shows that only $\beta = 9/41 \approx 0.22$ is really needed to accomplish the catch.

For realistic shock wave speeds ($w \ll c$) the required velocity will be much smaller. Shock waves through a solid typically have speeds in the order of several km per second [11]: Taking $w = 10$ km/s (i.e., $w/c = 3.33 \cdot 10^{-5}$), the required velocity according to Eq. (11) is only $\beta \geq 6.67 \cdot 10^{-6}$, or $v \geq 2$ km/s. So not only is the catch successful according to the son, but the required speed is *much* less than foreseen by the farmer based on Eq. (2). This of course means that the analysis in the frame of reference of the farmer can be sharpened. That is the topic of the next subsection.

3.2 In the reference frame of the farmer

In order to make the farmer's criterion agree with that of the son, we take up the series of events at t_A [Eq. (3)] when the ladder hits the wall (just as in the previous subsection, but now from the point of view of the farmer). At t_A the shock wave starts to travel towards the rear end of the ladder, but this end (unaware of the oncoming shock) is still travelling at speed v to the right and keeps doing so until it meets the shock wave, see Fig. 5. Thus, the ladder keeps shrinking until at the meeting point, and *that* is the moment the farmer should close the door. Obviously, the ladder does not yet have to fit inside the barn in the constant-velocity situation (as the farmer had argued initially) and that is why in Fig. 5 we have sketched a pole that is much less contracted than in Figs. 1(top) and 2(top).

The shock wave travels at a speed w in the son's frame of reference. Its speed in the farmer's frame of reference is, by the well-known velocity addition rule [12, 13]:

$$w_f = \frac{w - v}{1 - vw/c^2} \ . \tag{13}$$

The ladder will be entirely in the barn after

$$t = t_A + \frac{L_0\sqrt{1-\beta^2} - \ell_0}{v} \tag{14}$$

and the shock wave arrives at the ladder's rear end at

$$t = t_A + \frac{\ell_0}{w} = t_A + \frac{\ell_0(1 - vw/c^2)}{w - v} \ . \tag{15}$$

This now yields the following condition:

$$\frac{L_0\sqrt{1-\beta^2} - \ell_0}{v} \leq \frac{\ell_0(1 - vw/c^2)}{w - v} \tag{16}$$

which, after some rewriting, proves to be exactly the same as the son's criterion Eq. (11). [14]

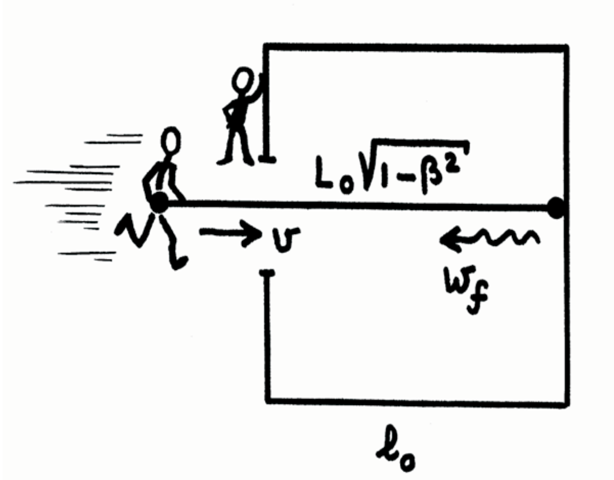

FIG. 5. The catch in the farmer's frame of reference: a race between shock wave and son towards the door of the barn. Analogously as in Fig. 3, the situation is sketched at $t = t_A$, when the ladder just hits the back wall. Note that the son might be inside or outside the barn, depending on the value of β $(= v/c)$; here we have chosen to depict the situation for a relatively small value of β, in agreement with the catch condition Eq. (11).

4 Conclusion

In conclusion, the ladder is caught if v is sufficiently large to "outrun" the shockwave: The required velocity, Eq. (11), is much less than one would expect by naively requiring that the pole should fit into the barn while moving at uniform speed [i.e., Eq. (2)]. In fact, the catch is completely beyond the paradox and is accomplished not thanks to the Lorentz contraction, but thanks to the fact that shock waves have a finite speed of propagation. There is hardly anything relativistic about this: The situation may be compared to a squash ball that is flattened during its impact with a wall. At the right moment, it can be caught in a box that is much shorter than the rest diameter of the ball.

As mentioned in Section 3.1, the typical speed of shock waves through a solid is in the order of 10 km per second, i.e., $w \ll c$. Accordingly, the required velocity of the pole is well approximated by the first term in the expansion of Eq. (11):

$$v \approx (1 - f)w \ . \tag{17}$$

For pole-to barn ratios close to 1 this is well within experimental reach: If $f = 0.995$ the required velocity v will be just about 50 m/s. One might for instance shoot a 100-cm blunt arrow (with a characteristic speed of 70 m/s) into a 99.5-cm box equipped with a trap door. By closing the door at the proper moment, specified in the first paragraph of Section 3.2, the catch should be fairly easy to accomplish.

As a final remark, it may be noted that an instantaneous transmission of the shock ($w_f \to \infty$) would bring us back to the original uniform-speed condition Eq. (2). An infinite shock wave velocity in the frame of the farmer would correspond to a finite (but superluminal) shock wave velocity in the son's frame: From the velocity addition rule, Eq. (13), one finds $w = c^2/v$. Inserted in the catch criterion (9) this superluminal shock speed yields $\ell_0/L_0 \geq \sqrt{1-\beta^2}$, i.e., precisely the original, much too severe condition (2) formulated by the farmer. This underlines once again the central point of the paper: The catch condition is determined solely by the speed of the shock wave that carries the information of the oncoming standstill from one end of the pole to the other.

Acknowledgements – We gladly seize the opportunity to congratulate Prof. Dr. Siegfried Grossmann on the occasion of his 75$^{\text{th}}$ anniversary. We wish him many more years to come, in good health, and always with the admirable kindness and mastery of the subject that are so characteristic of him.

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[7] For a valuable list of relativistic paradoxes, see G.P. Sastry, *Is length contraction really paradoxical?*, Am. J. Phys. **55**, 943-946 (1987). Also on the world wide web many references to these paradoxes can be found.

[8] These times are related to each other via the Lorentz transformation $t'_A = (t_A - vx_A/c^2)/\sqrt{1-\beta^2}$, with $x_A = \ell_0$ (and analogously for the times of event B).

[9] Several other stopping scenarios, using a series of clamps, are treated in [6].

[10] A non-instantaneous stop of the front end can be treated along the lines of E.F. Taylor and A.P. French, *Limitation on proper length in special relativity*, Am. J. Phys. **51**, 889-893 (1983). See also: H. Nikolić, *Relativistic contraction of an accelerated rod*, Am. J. Phys. **67**, 1007-1012 (1999).

[11] The shock wave speed is related to the speed of sound waves (typically 4 km/s in a solid) via the so-called Hugoniot relation, $v_{\text{shock}} = v_{\text{sound}} + Cu_p$, where C is a material constant (typically 1.5) and u_p the local velocity of the displaced particles in the solid (also in the order of a few km per second). See e.g. H.C. Pant *et al.*, *Equation-of-state studies using laser-driven shock wave propagation through layered foil targets*, Current Science **82**, 149-157 (January 2002); R.S. Hixson, G.T. Gray, and D.B. Hayes, *Shock compression techniques for developing multiphase equations of state*, Los Alamos Science **28**, 114-119 (2003).

[12] This relation, being the relativistically correct version of the more intuitive Galilei addition formula $w_f = w - v$, ensures that the speed of light is the same for all observers, i.e., $c_f = (c-v)/(1-v/c)$. In fact, this invariance of c is one of the two postulates on which Einstein based the theory of relativity; see [1].

[13] From Eq. (12) it may be inferred that $w-v$ should be positive, since the shock wave must travel in the proper direction (from the wall to the rear end

of the pole) also in the farmer's frame of reference. Stated differently, the required velocity v of the pole never exceeds the shock wave speed w, in full agreement with the catch criterion [Eq. (11)] as can be seen clearly in Fig. 4.

[14] It may be checked that the worst-case criterion Eq. (12) of the son is found if one takes $w = c$. In this case also the farmer observes the same shock speed $w_f = c$ (cf. footnote [12]), so the right hand side in Eq. (16) becomes ℓ_0/c, reproducing the son's worst-case criterion Eq. (12).