H.W. Broer
Vakgroep Wiskunde en Informatica, Universiteit Groningen
Postbus 800, 9700 AV Groningen
broer@math.rug.nl

The how and what

A review is given of the most well-known examples of dynamical systems with chaotic dynamics. After a phenomenological introduction, a definition of chaos is 'deduced'. One of the examples concerns the Hénon-attractor, that has only been recently characterized as the closure of a coiling curve. It will be made plausible that this characterization also holds in a more realistic example from mechanics. Also some links between chaos and probability are indicated.

1 Determinism, chaos and chance

Chaos is one of the intriguing features of nonlinear deterministic dynamics. Dynamical systems in principle model anything that moves in time. Examples are as various as simple contraptions of springs and pendula or the whole solar system or the world economy or even the atmospheric circulation which produces the weather. In this context, the only important thing is that it is deterministic: there is some *state space*, or *phase space*, containing all possible states of the system and an evolution law that prescribes the whole future when the present state is given. Of course it is not completely clear that all these examples should be deterministic.

Anyway, it turns out that even in extremely simple deterministic dynamical systems, i.e., with a low dimensional state space, the phenomenon of chaos can occur. Apart from the fact that this gives rise to interesting geometries in the state space, it also leads to basic unpredictability of the future dynamics.

Setting of the problem

The first entry to the subject matter is phenomenological, we just study a few examples in low–dimensional phase spaces, or with 'few degrees of freedom'. The interest is with the long–term, or asymptotic, dynamical behaviour, i.e., the regime where the initial and transient phenomena have disappeared.

As we shall see, there is a wide variety of possibilities. In the simplest cases the system is asymptotically at rest. We shall also encounter asymptotic periodic or multiperiodic behaviour. In those cases, in the phase space a point or (multi-)periodic attractor exists, where nearby evolutions converge to. All of this is still reckoned to be orderly dynamics. A first, rough, definition classifies all remaining cases to be 'chaotic', where the attractors in phase space usually are called 'strange'. After a while we shall arrive at a more sophisticated definition of chaos.

Unpredictable systems elsewhere often are described by probabilistic models. A good example are the dice, which by nature form a highly chaotic, deterministic system. The description in these terms is so complicated though, that we usually disguise our incompetence and ignorance by using statistics.

Outline

We start with a few classical examples, including the Logistic (or Quadratic) family, the Hénon family, the Baker transformation and the Tent map family. In these examples time is discrete and the dynamics mathematically amounts to the iteration of a map.

Next we present two examples with continuous time, generated by autonomous systems, of ordinary differential equations (ODE's). In particular we show the Lorenz en Rossler systems. After this we arrive at a definition of chaos and at the characterization of the strange attractor of a 2–dimensional diffeomorphism like the Henon map.

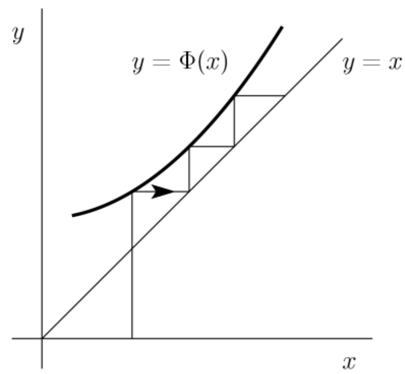

Figure 1 Graphical explanation of the iteration $x_{n+1} = \Phi(x_n)$.

# of chaos

The second part of the paper focuses on periodically driven systems. We shall describe some of the background physics and consider the Poincarémap of such a system. It seems that the strange attractors of these maps have the same kind of characterization as the Hénon map. We conclude by explicitly introducing some probabilistic aspects by defining an invariant distribution, hinting in the direction of Ergodic Theory. A special case is occupied by the conservative case without friction. Here the chaotic phenomena are still largely unknown, although some interesting comments are in order. Also we briefly mention the case of fractal basin boundaries, where we return to the dice.

2 The Logistic family

The first example is the well–known Logistic or Quadratic family, e.g., see [10]. It is a discrete dynamical system, given by the iterations

$$x_{n+1} = \mu x_n(1 - x_n), \tag{1}$$

$x_n \in [0, 1]$, $n = 0, 1, 2, \ldots$ Moreover $\mu \in [0, 4]$ is a parameter. The simplest interpretation of (1) is in terms of population dynamics. In that case an evolution $x_0, x_1, x_2, \ldots$, of this system is interpreted as the (scaled) size of a certain population as a function of the time n. Let us discuss this in some detail.

The linear growth model

We begin considering the linear growth model

$$x_{n+1} = \mu x_n, \quad x_n \in \mathbb{R},$$

$n = 0, 1, 2, \ldots$ The parameter μ in this model is the constant ratio between the sizes of successive generations, and can be interpreted as a netto growth rate of the population at hand. It is easily seen that in terms of the initial 'population' x_0 we have

$$x_1 = \mu x_0,\ x_2 = \mu^2 x_0,\ \ldots,\ x_n = \mu^n x_0,\ \ldots$$

$n = 0, 1, 2, \ldots$, which clearly is an exponential behaviour. Regarding the parameter μ we distinguish between the case where $0 < \mu < 1$, corresponding to exponential decay, and the case where $1 < \mu$, which corresponds to exponential growth. There are not too many circumstances in which this linear model is realistic. The Logistic family can be seen as the simplest nonlinear modification of it.

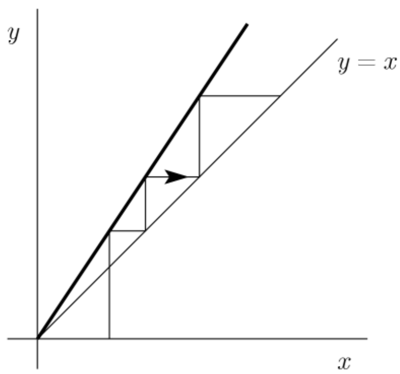

Figure 2 Graphical explanation of the linear growth model.

Graphical analysis

As an intermezzo we study the dynamics of more general iterations

$$x_{n+1} = \Phi(x_n),$$

$n = 0, 1, 2, \ldots$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a smooth function. In figure 1 we sketched the graph of such a function as well as the graph of the identity map (i.e., of $y = \Phi(x)$ and $y = x$). It is easy to describe the iteration–step from x_n to x_{n+1} by use of this: indeed given x_n on the x–axis we read off the next iterate x_{n+1} on the y–axis. With help of the diagonal $y = x$ we next reflect the value x_{n+1} to the x–axis and we can repeat the process to obtain x_{n+2}. In figure 2 this idea has been applied to the linear growth model for $\mu > 1$. In this case we simply have $\Phi(x) = \mu x$.

Overpopulation

Let us now consider the dynamics of the Logistic family itself, which can be described as

$$x_{n+1} = \Phi_\mu(x_n) \quad \text{with } \Phi_\mu(x) = \mu x(1-x),$$

the Logistic map. First observe, that for small values of x_n the equation (1) is well–approximated by the linear growth model.

Next however, we observe that in the nonlinear model there is a maximal size of the population, namely $x_n = 1$. Indeed, if $x_n = 1$ it follows that $x_{n+1} = 0$, and so the next generation is extinct. This is one of the simplest ways to model the effect of overpopulation. Of course the maximal size 1 is artificial, but any other bound by an appropriate scaling may be brought to 1.

As before, we vary the parameter μ and consider the dynamics. To this end we perform the graphical analysis described in the previous subsection, see figure 3. As in the linear case, for $0 < \mu < 1$ the population gets extinct, where for larger values of n we almost are in the case of exponential decay. In a more geometrical language we say that $x = 0$ is a point, or equilibrium, attractor. This kind of dynamics is still quite uninteresting from the view point of population dynamics.

For $1 < \mu < 3$ we find a point attractor different from $x = 0$, indeed it is $x = p_\mu$, with

$$p_\mu := (\mu - 1)/\mu.$$

Observe that for $2 < \mu < 3$, the equilibrium p_μ is approached in a spiraling mode.

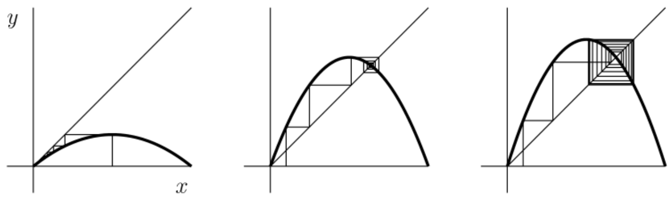

Figure 3 Dynamics of the Logistic family for different values of μ: a. $0 < \mu < 1$, b. $1 < \mu < 3$, c. $0 < \mu - 3 \ll 1$.

Then, for $3 < \mu$ none of the two equilibria 0 and p_μ is attracting, so where does the population size x_n go as $n \to \infty$? The answer for $0 < \mu - 3 \ll 1$ is given in figure 3c: the system now has a periode–2 attractor. Intuitively such a situation can be described as follows. Suppose the generations with even n are smaller than the ones with odd n. Then each odd generation exhausts the natural resources so much that the next generation is smaller, which allows the next generation to be larger again, etc.

For increasing values of μ an interesting scenario arises, which is described often in literature, see figure 4. We refer to, e.g., [10, 12]. To obtain it, for each value of μ, about 200 iterations are considered of which only the latter 100 are plotted. In this way the transient phenomena are ignored and only the asymptotic dynamics is observed. The first thing to be seen is that for increasing μ the system doubles its period. So the period–2 attractor is replaced by a period–4 attractor, then period–8, etc. It has been shown that this process goes with geometric progression: to be precise, the corresponding sequence $\mu_1, \mu_2, \mu_3, \ldots$ of parameter–values converges (almost) geometrically to a limit μ_∞. Names like Feigenbaum and Tresser are related to these phenomena, compare [9, 12]. For the parameter–value $\mu = \mu_\infty$ that limits this progression there is no longer any periodicity. For such aperiodic dynamics we shall use the adjective *chaotic*, but below we shall arrive at a more precise definition.

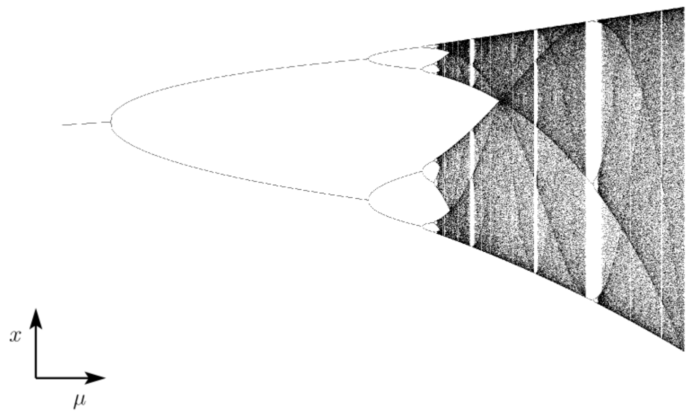

Figure 4 Bifurcation diagram of the Logistic family.

In the range $3 < \mu < 4$ many chaotic and periodic regimes can be distinguished and also many period doubling sequences.

Remarks

- One of the significant features of the present chaos is the stretching of the x–axis by the Logistic map $\Phi(x) = \mu x(1-x)$, except very near $x = 1/2$, for larger values of μ. By stretching alone the points all would leave the interval $[0, 1]$. However, the map also folds the interval over itself. It is this combination of stretching and folding that we shall see again when meeting other chaotic systems.
- The above (almost) geometric progression of the sequence μ_n can be expressed by saying

$$\lim_{n\to\infty} \frac{\mu_{n-1} - \mu_{n-2}}{\mu_n - \mu_{n-1}} = \delta,$$

for a positive constant $\delta = 4.6692016091029\ldots$ This constant δ turns out to be *universal* in the sense that it also occurs for similar unimodal maps Φ, again see [9, 12].
- The arguments of this section have the same character as in the derivation of the Volterra–Lotka systems for population dynamics. Their aim is not so much to precisely model the evolution of a given population, but rather to give as transparant as possible systems with certain qualitative properties, compare [11, 13, 24]. In Mathematical Biology, however, often more realistic models are used, with similar qualitative properties as the Logistic family.

3 The Hénon family

This family of maps was introduced by Hénon in 1976, intended as a simple (polynomial) example of a planar diffeomorphism with a strange attractor. Only quite recently the structure of this family has been understood better, mathematically speaking. Compare [3, 9–10, 12].

Just iterating

Let us first describe what is the case. Given is the 2–dimensional map

$$\Phi_{a,b}(x,y) = (1 - ax^2 + y, bx),$$

with a and b real parameters. Iteration

$$(x_{n+1}, y_{n+1}) = \Phi_{a,b}(x_n, y_n),$$

$n = 0, 1, 2, \ldots$, then produces 2–dimensional evolutions of the form

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$$

In figure 5, one iteration has been plotted for parameter values $a = 1.4$ and $b = 0.3$. This is the celebrated Hénon attractor H, which attracts an open subset of initial points (x_0, y_0) in $\mathbb{R}^2$. The set H appears to be a fractal, the local geometry of which is the product of a curve and a Cantor set. Numerical estimates of a suitable fractal dimension (limit capacity, box counting dimension) of H give approximately 1.2. For background see [9], especially the chapters II and III. For general reference see also [12].

It is interesting to see what happens upon variation of the parameters (a, b), see figure 6. Indeed, the strange attractor H is not persistent for small perturbations, since for nearby values of (a, b) periodic attractors turn up. As we shall argue now, the complexity of the bifurcation diagram in figure 4 also is to be expected here: windows with periodic attractors, period doubling sequences, strange attractors, etc.

Mathematical intermezzo

Let us make this a bit more precise. To begin with, notice that the following two versions exist of the Logistic family:

$$\begin{aligned} x_{n+1} &= \mu x_n(1 - x_n) \text{ and} \\ x_{n+1} &= 1 - ax_n^2. \end{aligned}$$

The former we already know, and the latter can be derived from this by the affine scaling of x from $[0,1]$ to $[-1,+1]$ and by reparametrizing

$$\mu \in [0,4] \longleftrightarrow a \in [0,2] : a = \frac{1}{4}\mu(\mu - 2).$$

This gives rise to the following observation. Consider the Hénon family for $b = 0$:

$$\Phi_{a,0}(x,y) = (1 - ax^2 + y, 0).$$

Notice that this is no longer a diffeomorphism. Clearly the line $y = 0$, or the x–axis, is invariant[1] and the restriction $\Phi_{a,0}(x,0) = (1 - ax^2, 0)$ is nothing but a newly begotten version of the Logistic family. A natural question now would be which aspects of the dynamics of the Logistic family are inherited by the Hénon family for small (nonzero) values of b.

At the level of periodic attractors a lot is known about this, see [22], also see [8]. Indeed, it turns out that the periodic windows of the Logistic family as shown in figure 4 can be continued for small values of b.

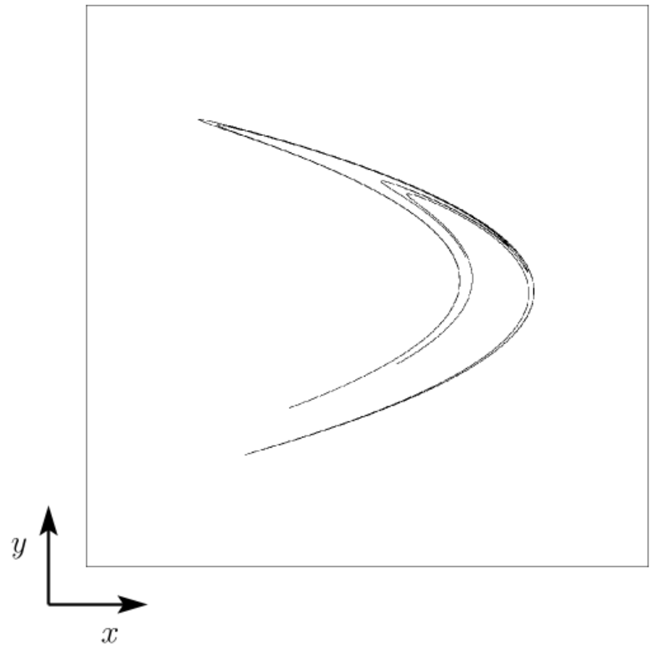

Figure 5 The Hénon attractor in the (x, y)–plane, for $a = 1.4$, $b = 0.3$.

The chaotic dynamics is a lot harder to continue. To understand what is going on here, we note that the Hénon map $\Phi_{a,b}$ has a saddle fixed point $p_{a,b}$. Such a point is unstable. Let us consider the set

$$W^u(p_{a,b}) = \{(x,y) \in \mathbb{R}^2 \mid \lim_{n\to\infty} \Phi_{a,b}^{-1}(x,y) = p_{a,b}\},$$

i.e., all points that by backward iteration of $\Phi_{a,b}$ go to $p_{a,b}$. Clearly one has $p_{a,b} \in W^u(p_{a,b})$. It can be shown that $W^u(p_{a,b})$ (locally) is a smooth curve, called the unstable manifold of $p_{a,b}$.

Remarks

- Similarly the stable manifold $W^s(p_{a,b})$ exists, consisting of all points converging to $p_{a,b}$ by forward iteration of $\Phi_{a,b}$. Both curves in $p_{a,b}$ are tangent to the corresponding eigenspaces of the derivative map.
- In this particular case the inverse $\Phi_{a,b}^{-1}$ can be directly computed as
$$\Phi_{a,b}^{-1}(x,y) = (\frac{1}{b}y, x + \frac{a}{b^2}y^2 - 1).$$

One of the features of the Hénon family is its constant Jacobian determinant: $|\det D\Phi_{a,b}| \equiv b$, meaning that for $|b| < 1$ the map $\Phi_{a,b}$ contracts area. This property is also called *dissipative*. From this fact one easily derives that the attractor $H_{a,b}$ of $\Phi_{a,b}$ is contained in the closure $\overline{W^u(p_{a,b})}$. All this was already known in the beginning of the eighties, proven by Tangerman (unpublished).

The difficult part is the converse, namely to show that $H_{a,b} = \overline{W^u(p_{a,b})}$. In 1991 Benedicks & Carleson [3], also see [12], showed that this indeed holds true for a subset of the (a, b)–plane of positive area. The proof again is based on perturbation arguments. One remaining question is whether the 'original' Hénon attractor $H = H_{1.4,0.3}$ has this property or not. Looking at figure 5 makes this very believable. A definite answer, however, may never be found.

1 In fact, $\Phi_{a,0}$ maps $\mathbb{R}^2$ onto the x–axis, and on the x–axis we have 1–dimensional dynamics as before.

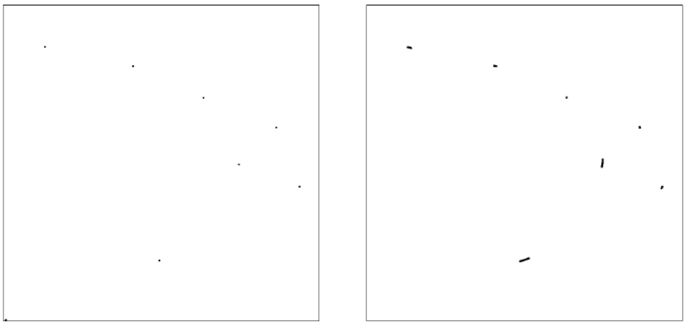

Figure 6 Attractors in the Hénon family for various (a, b), again depicted in the (x, y)–plane. The left case is a periodic attractor of period 7 and the right case a small scale strange attractor with 7 components.

4 The Baker transformation

One of the few ways to get entrance into chaos is that of symbolic dynamics. This can be best explained with help of a toy model called the Baker transformation. Indeed, this transformation is given by

$$x_{n+1} = 2x_n \pmod 1, \tag{2}$$

again with $x_n \in [0, 1)$, $n = 0, 1, 2, \ldots$ Clearly the map Φ in this example is given by $\Phi(x) = 2x \pmod 1$, for a graph see figure 7. Note that here the interval $[0, 1)$ firstly is stretched by a factor 2 and secondly cut, after which $[0, 1)$ twice covers itself. This reminds of a form of dough kneading, which explains the name. In this form the tranformation is discontinuous, but when $2\pi x$ is interpreted as an angle we can see the Baker transformation as the complex map

$$z \mapsto z^2,$$

restricted to the unit circle, in which case it is continuous.

Remark. The projection of this circle dynamics on the real axis gives rise to the Logistic map with $\mu = 4$, e.g., see [10].

Let us consider an evolution $x_0, x_1, x_2, \ldots$, given by iteration of the Baker transformation, so with $x_{n+1} = 2x_n \pmod 1$. In binary expansion the initial point x_0 has the form

$$x_0 = \sum_{j=1}^{\infty} a_j 2^{-j},$$

where $a_j \in \{0, 1\}$ for all $j = 1, 2, 3, \ldots$. In the usual notation $x_0 = 0.a_1a_2a_3\ldots$, the Baker transformation gets simple:

$$\begin{aligned} x_0 &= 0.a_1a_2a_3\ldots, \\ x_1 &= 0.a_2a_3a_4\ldots, \\ x_2 &= 0.a_3a_4a_5\ldots, \\ x_n &= 0.a_{n+1}a_{n+2}a_{n+3}\cdots, \end{aligned}$$

an operation named (*Bernoulli–*)*shift*. Observe that the above has the following geometric meaning. If $a_{n+1} = 0$ the n–th iterate x_n is in the left half interval $I_0 = [0, \frac{1}{2})$, while for $a_{n+1} = 1$ we have that $x_n \in I_1 = [\frac{1}{2}, 1)$, the right half interval. Conversely the positioning of the $x_0, x_1, x_2, \ldots$ with respect to the intervals I_0 and I_1 determines[2] the binary expansion of the initial point x_0. We now are in the realm of *symbolic dynamics*, a powerful tool for understanding chaotic dynamics.

One of the first surprising consequences of the above is the following. If the initial point x_0 is known only with finite precision, this precision decreases when iterating. After some time all precision even totally disappears! This is the heart of the unpredictability aspect of chaos, and it is of importance since in practice one is always dealing with finite precision.

A second direct consequence is the following. An initial condition $x_0 \in [0, 1]$ has a periodic evolution of period N, precisely if the binary expansion $x_0 = 0.a_1a_2a_3\cdots a_Na_{N+1}\ldots$ repeates with period N. We now can formulate

Theorem 1 (Denseness of periodic points). *The set of periodic points of the iterated Baker transformation is dense in* $[0, 1)$.

Proof. Let $p \in [0, 1)$ be an arbitrary number, with a binary expansion $p = 0.a_1a_2a_3\cdots a_Na_{N+1}\ldots$. Then consider the repeating binary expansion $q = 0.a_1a_2a_3\cdots a_Na_1a_2a_3\ldots a_Na_1\ldots$. Clearly the point q is periodic with period N, where $|p - q| \le 2^{-N}$. □

All these periodic points are unstable: the system in practice will behave aperiodic. This can be seen by observing the graph of the Baker transformation and its iterates, compare figure 7.

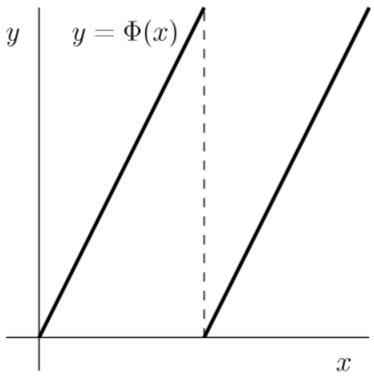

Figure 7 Graph $y = \Phi(x)$ of the Baker transformation.

Speaking of dense properties, in general a point $x_0 \in [0, 1]$ is called eventually periodic for the map Φ, if some finite iterate $\Phi^N(x_0)$ is a periodic point under Φ. In the present case, where Φ is the Baker transformation, the set of points that are eventually periodic with period 1, i.e., the eventual fixed points, also is dense in $[0, 1]$. Indeed, the eventual fixed points are exactly the finite binary expansions, which evidently fill the interval densely. The reader is invited to find other dense properties for him– or herself.

A less intuitive consequence of the binary expansions is the following

Theorem 2 (Existence of a dense orbit). *There exists a point* $x_0 \in [0, 1)$ *such that its orbit* $\{x_0, x_1, x_2, \ldots\}$ *under iteration of the Baker transformation is dense in* $[0, 1)$.

2 For the moment leaving alone ambiguities like $0.01111\ldots = 0.10000\ldots = \frac{1}{2}$.

Proof. Consider the binary expansion

$$\begin{aligned} x_0 &= 0.\,0\,1 \\ &\quad 00\ 01\ 10\ 11 \\ &\quad 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111 \\ &\quad \dots \end{aligned}$$

where all possible truncated expansions are listed. By a similar argument as in the previous proof, this point comes arbitrarily close to any given point $p \in [0,1)$. □

The last theorem says that the 'attractor' $[0,1)$ of the Baker transformation is connected: it cannot be decomposed in smaller sub-attractors.

Remarks

- Consider the family of Tent maps
$$\Phi_a(x) = \begin{cases} 2ax, & 0 \le x \le \frac{1}{2} \\ 2a(1-x), & \frac{1}{2} \le x \le 1, \end{cases}$$
where a is a real parameter. The Tent map Φ_1, i.e., with $a = 1$ behaves strikingly similarly to the Baker transformation. In particular it has a strongly related symbolic dynamics. Moreover, Φ_1 has the same dynamics as the Logistic map for $\mu = 4$, i.e., of $\Phi_\mu(x) = 4x(1-x)$, see [10].
- In general the Tent maps Φ_a, as well as the Logistic maps Φ_μ can be attacked succesfully with symbolic dynamics. Special care has to be taken with the orbit of the central point $x_0 = \frac{1}{2}$. Moreover, general symbolic dynamics is defined on Cantor sets, which is nicely illustrated by considering the Tent map family for $a > 1$, in which case the invariant set
$$\bigcap_{n=0}^{\infty} \Phi^{-n}([0,1]),$$
consisting of the initial points, the orbits of which stay inside $[0,1]$ for all time, is a Cantor set. Compare [9–10, 12, 18].

5 Mathematical intermezzo: Towards a definition of chaos

Summarizing, in the above we met dynamical systems with discrete time, generated by a map Φ by iteration $\vec{x}_{n+1} = \Phi(\vec{x}_n)$, $n = 0,1,2,\dots$ Here $\vec{x}$ varies over some finite dimensional state space. Let $\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots$ be an evolution of this system, then

$$\begin{aligned} \vec{x}_1 &= \Phi(\vec{x}_0), \\ \vec{x}_2 &= \Phi(\vec{x}_1) \\ &= (\Phi \circ \Phi)(\vec{x}_0) = \Phi^2(\vec{x}_0) \\ &\dots \end{aligned}$$

The examples we saw are the Logistic family $\Phi_\mu(x) = \mu x(1-x)$, the Hénon family $\Phi_{a,b}(x,y) = (1 - ax^2 + y, bx)$ and the Baker transformation $\Phi(x) = 2x \pmod 1$.

Until now we discussed chaos in a heuristic manner, saying that it is aperiodic, or just 'weird'. Now we shall give a somewhat more rigid definition of a chaotic attractor, compare [10], although we shall not go all the way to Bourbaki. First we have to say what will be understood by an attractor.

Definition 3 (Attractor, [9]). *A subset H of the phase space of the system generated by Φ is called an attractor iff*

1. *H is invariant, i.e., if $\vec{x}_0 \in H$, then also $\Phi(\vec{x}_0) \in H$;*
2. *There exists a neighbourhood U of H, such that for all initial states $\vec{x}_0 \in U$, for the corresponding evolutions $\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots$ (with $x_n = \Phi^n(\vec{x}_0)$, $n = 0,1,2,\dots$), one has that $\vec{x}_n \to H$;*
3. *In H the iterated map Φ has a dense orbit.*

We repeat that the existence of a dense orbit means that H is indecomposable, or that it is minimal with respect to the former two properties. We met this property before in the Baker transformation.

Next consider an evolution $\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots$, with $\vec{x}_{n+1} = \Phi(\vec{x}_n)$, $n = 0,1,2,\dots$, inside the attractor H. For nearby initial points $\vec{y}_0 \in H$, we also consider the evolution $\vec{y}_0, \vec{y}_1, \vec{y}_2, \dots$, with $\vec{y}_{n+1} = \Phi(\vec{y}_n)$, $n = 0,1,2,\dots$

Definition 4 (Sensitivity). *The restriction $\Phi|_H$ has sensitive dependence on initial values iff there exists a positive constant δ, such that the following holds. For any initial value $\vec{x}_0 \in H$ and any (small) positive number ε, there exists an initial value $\vec{y}_0 \in H$, with $|\vec{y}_0 - \vec{x}_0| < \varepsilon$, and there exists a number $N \in \mathbb{N}$, such that $|\vec{y}_N - \vec{x}_N| > \delta$.*

The property of sensitive dependence also holds for the Baker transformation, it has to do with the stretching that leads to loss of information in the practical case when working in finite precision, leading to unpredictability of the evolutions on a longer–term range. For further discussion of this notion and possible refinements, see [9], chapters I and III. Also see [7], chapter VI. For more background see [18]. Finally we have:

Definition 5 (Chaos). *The attractor H of the dynamical system generated by the map Φ is chaotic iff the restriction $\Phi|_H$ has sensitive dependence on initial values.*

It is mathematically more or less reassuring to have the above conditions available. However, we then carry the burden of proving that in the above examples these definitions play a significant role. In the case of the Baker transformation this can be done. (Hint: try $\delta = \frac{1}{4}$.) In the other cases this is not easy at all, and sometimes only vague statements are proven like 'for a set of parameter values of positive measure the system has a chaotic attractor'. See above.

Remarks

- In the literature many variations exist on the above definitions. For example, an attractor does not necessarily have to attract a full neighbourhood, but a set of full (or just positive) measure will do. Also sometimes it is required that the periodic points should be dense in the attractor, again see the example of the Baker transformation.
- One reason is provided by the unwieldy examples, that look chaotic, but can only be forced to satisfy alternate definitions.
- Another reason is that there is at least one other entrance to chaotic dynamics, which uses probabilistic or measure theoretical methods. At the end we shall come back to this.

6 Two examples with continuous time

The scope of the above seems to be somewhat limited, although we shall see that the case of the Hénon family represents a large class of systems that occur in applications.

One of the icons of the theory is the Lorenz attractor, see figure 8. It is generated by the following autonomous system of ODE's in $\mathbb{R}^3$ in the same way as before: let the system run and omit all transient information. An evolution now simply is a solution or integral curve.

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= \quad Rx - y - xz \\ \dot{z} &= -Bz + xy, \end{aligned}$$

where $\sigma = 10$, $B = 8/3$ and $R = 28$. Here and elsewhere we use the notation $\dot{x}(t) = \frac{d}{dt}x(t)$. The example was published in 1963 in a meteorological journal, for details and background, e.g., see [12, 16]. We just mention here that the system has to do with long term behaviour of the atmospheric circulation and indicates that even at the level of simplified climate models chaos may occur. Again chaos is related to unpredictability.

Figure 8 The Lorenz attractor projected on a plane.

This attractor has been a challenge for mathematicians for a long time, only quite recently it has been shown to be chaotic according to the definitions, see [23].

The unwieldiness of the Lorenz system was one reason for Rössler to introduce a simpler system in 1976 with similar properties as the Lorenz system:

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= \quad x + \frac{1}{5}y \\ \dot{z} &= \quad \frac{1}{5} + z(x-\mu). \end{aligned}$$

For $\mu = 4.9$ the attractor is given in figure 9. In the present case it is easier to extract mathematical information.

In both cases the attractor locally seems to be the product a surface and a Cantor set. Numerically estimated fractal dimensions are between 2 and 3. For background see [9, 12, 16, 21].

7 Hénon everywhere

The examples of dynamical systems we met until now all had a somewhat artificial character. In this section we introduce a simple mechanical context, around the swing, in which we shall produce Hénon–like attractors of 2–dimensional diffeomorphisms. We aim to illustrate that the theorem of Benedicks and Carleson regarding strange attractors for such maps,

$$H = \overline{W^u(p)},$$

seems to be quite a universal characterization. Also see [19–20] for extensions. Here p is an appropriate saddle point and $W^u(p)$ is its unstable manifold, see above. It has to be added here that the mathematics of the matter is not so far yet that these illustrations also can be made precise.

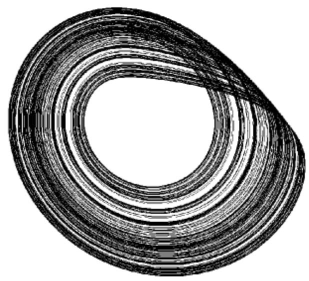

Figure 9 The Rössler attractor projected on a plane.

Mechanical background

As a starting point we take the planar pendulum without friction, as it occurs in almost all Mechanics textbooks, also see [4–5]. Consider a mathematical pendulum of length ℓ with mass m that moves in a vertical plane, under the influence of graviation with acceleration g. Let x be the deviation from the downward vertical equilibrium position, measured in radians. As before $\dot{x}$ and $\ddot{x}$ denote the derivatives with respect to the time t, in this case called velocity and acceleration, respectively. The equation of motion then reads

$$\ddot{x} = -\omega^2 \sin x, \qquad (3)$$

where $\omega^2 = g/\ell$. This is a direct consequence of Newton's second law $F = m \times \ell\ddot{x}$, where $F = F(x)$ is the component of the gravitational force in the direction of motion. We note the established fact that this equation of motion is independent of the mass m, which was known experimentally already to Galileo.

The above considerations hold in the *conservative* case without friction. However, it is easy to introduce some friction or damping in the equation of motion as follows

$$\ddot{x} = -\omega^2 \sin x - c\dot{x}, \qquad (4)$$

where $c > 0$ is a positive damping coefficient.

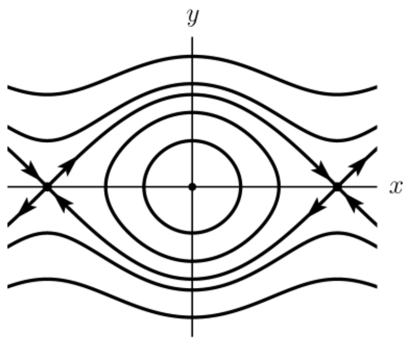

Figure 10 Phase portrait of the undamped pendulum.

The phase plane

One tool when dealing with nonlinear systems like the pendulum, is the phase plane. Indeed, introducing $y = \dot{x}$ as an independent variable we consider the Cartesian (x, y)–plane. The equations of motion (3) and (4) then turn into a system of ODE's

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x - cy, \end{aligned} \tag{6}$$

compare the 3–dimensional systems from the previous section. These vector fields give rise to integral curves, the projection of which to the x–axis corresponds to the physical dynamics within these models. In figures 10 and 11 we depicted the corresponding phase portraits respectively. In the latter, damped, case almost all evolutions end in the lower equilibrium point $(x, y) = (0, 0)$, which is the attractor in that case.

The Poincaré map

In the damped case generally all energy dissipates away and the pendulum comes to complete rest. Now what happens if we introduce energy to the system, e.g., by also varying the pendulum–length periodically in time? Just consider the equation of motion

$$\ddot{x} + c\dot{x} + (\omega^2 + \varepsilon \sin(\Omega t)) \sin x = 0 \tag{7}$$

which leads to the 3–dimensional vector field

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(\omega^2 + \varepsilon \sin(\Omega t)) \sin x - cy \\ \dot{t} &= 1, \end{aligned} \tag{8}$$

which is $2\pi/\Omega$–periodic in t. Let $(x(t), y(t), t)$ be an integral curve of the system (8) and consider instants $t_n = 2\pi n/\Omega$, $n = 0, 1, 2, \ldots$. We call $(x(t_n), y(t_n))_{n=0}^{\infty}$ a *stroboscopic* sequence. From the theory of ordinary differential equations, e.g., see [11, 13, 24], it follows that there exists a map $(x, y) \mapsto \Phi(x, y)$ such that a stroboscopic sequence is formed by iterations under Φ. We call Φ the stroboscopic map or Poincarémap of the time periodic vector field (8).

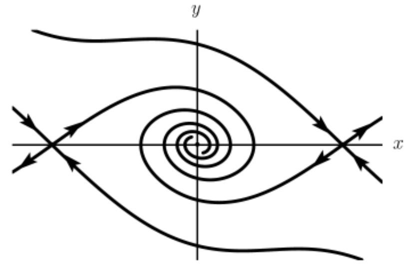

Figure 11 Phase portrait of the damped pendulum.

In figure 12 we depicted the attractor H for a certain value of c, ω, Ω and ε. In the particular case we are in, a strong resonance is taking place in the sense that $\omega : \Omega \approx 1 : 2$, in which case $(x, y) = (0, 0)$ is a saddle–point of the Poincarémap Φ. This means that the downward 'equilibrium' of the pendulum has been destabilized by the resonant excitation. It is easy to guess now that the $H = \overline{W^u((0,0))}$, compare the Hénon case, although at the moment it won't be easy to prove. Locally the geometry also resembles the Hénon case: as before it 'is' the product of curve and Cantor set. For a more extensive discussion, see [4–6, 19–20].

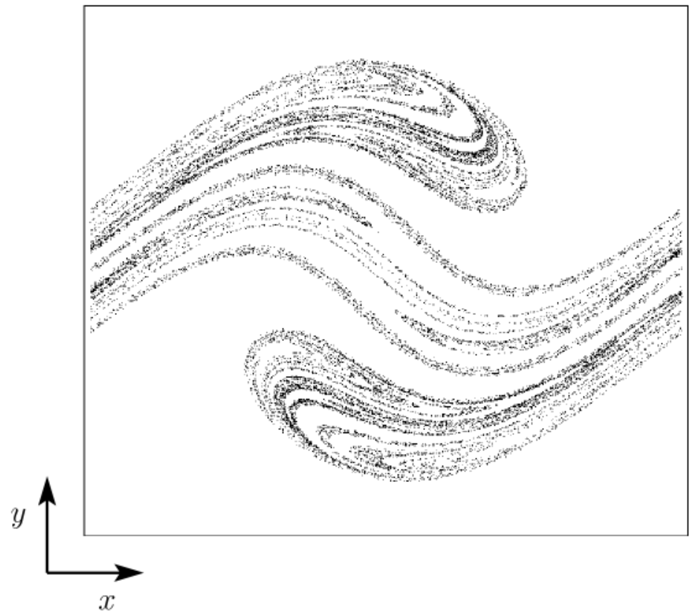

Figure 12 Hénon–like strange attractor in the Poincarémap of a forced pendulum: the damped case.

Remarks

- The mathematical conjecture behind all this goes much further, namely that in families of 2–dimensional dissipative diffeomorphisms strange attractors are *characterized* by $H = \overline{W^u(p)}$, for a suitable periodic saddle–point p.
- The Poincarémap also has a nice geometric meaning. Indeed, consider the vector field (8) in the 3–dimensional (x, y, t)–space. By periodicity in t, the whole 'block' of dynamics between $t = 0$ and $t = 2\pi/\Omega$ repeats itself periodically.

  The Poincarémap now follows the integral curves from the section $t = 0$ to $t = 2\pi/\Omega$, thus defining the smooth planar map Φ, compare [11, 13, 24].

Without friction

The undamped case with $c = 0$ also is interesting. In that case the system is *conservative* and the Poincarémap Φ can be shown to preserve area, see [1, 7]. In figure 13 several evolutions (or orbits) have been plotted. Here regular and irregular, or chaotic, behaviour occur for the same parameter values. However, the 'chaotic cloud' here consists of only one orbit.

Its structure is not yet well–understood. The general conjecture is that the chaotic evolution densely fills a set with *positive* area (the so–called chaotic sea) and that the dynamics on this sea is sufficiently 'mixing'. This problem was raised in the sixties of the 20-th century, see [2], and to this day remains unsolved. For an elementary discussion also see [6]. Recently however, in a simpler case of the so–called area preserving standard map, a result in this direction was proven [15].

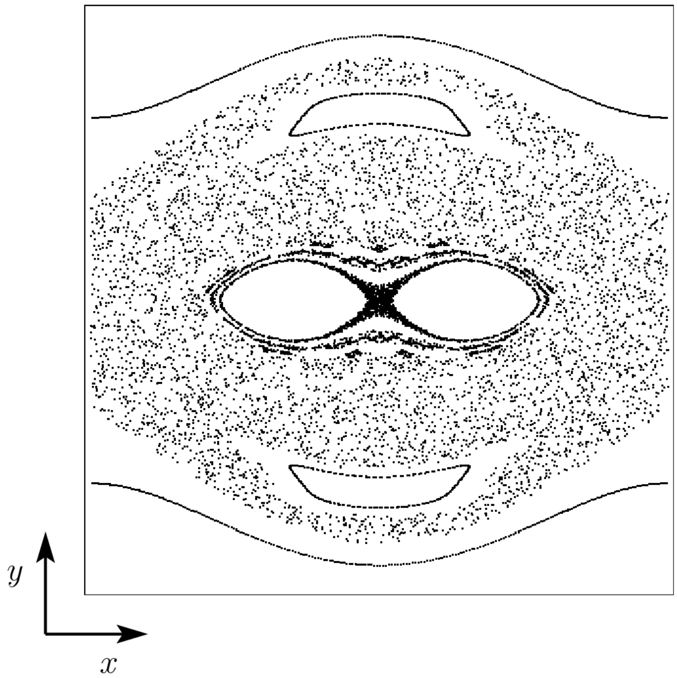

Figure 13 Hénon–like strange attractor in the Poincarémap of a forced pendulum: the undamped case.

Remark. In figure 13, due to the 1:2-resonance, the origin $(x, y) = (0, 0)$ is unstable: it is a saddle fixed point of the map Φ. Its unstable manifold approximately forms a figure 8 loop. Inside there exists a stable period–2 evolution. The motion of the incense container in the church of Santiago de Compostela can be explained in this way. See [5].

8 Random aspects

In this section we indicate some connections of the above with Probability Theory. Compare [1–2, 12].

The dynamical distribution

Let H be a (strange) attractor of the discrete dynamical system $x_{n+1} = \Phi(x_n)$, with initial state x_0. This leads to the Φ–evolution $x_0, x_1, x_2, \ldots$ For any (reasonable) subset $A \subseteq H$ consider the probability

$$P(A) := \lim_{N\to\infty} \frac{\#\{n \mid 0 \le n \le N \,\&\, x_n \in A\}}{N+1},$$

which is nothing but the relative visiting frequency of the evolution to A. In many cases $P(A)$ is independent of the initial state x_0. Assuming this independence, we obtain a probability distribution on H, usually called the *dynamical* (or *physical*) *distribution*. It is easy to see that the distribution is dynamically invariant, in the sense that $P(\Phi^{-1}(A)) = P(A)$, for (reasonable) subsets $A \subseteq H$. This invariant distribution allows an important connection with Ergodic Theory, a subdiscipline of Probability Theory, see [14, 17]. The remark in section 7 about 'mixing' belongs to this theory.

For the Baker transformation the dynamical distribution is just the uniform distribution on $[0, 1]$, while for the Logistic map with $\mu = 4$ the density is given by the Bèta–function

$$\frac{1}{\pi\sqrt{x(1-x)}}.$$

For the Hénon attractor such a simple formula cannot be given, nevertheless it is worthwile to think in this way. The strange attractor H, as depicted in figure 5 can be seen as the support of the invariant distribution. The sensitivity means that a small cloud of initial states is smeared out over the whole attractor H. Again, more precise statements can be made with help of Ergodic Theory.

In the general chaotic setting one can say that the sensitivity of definition 4 means that iterates $\Phi^N(\vec{x}_0)$ and $\Phi^M(\vec{x}_0)$ are almost uncorrelated for large values of $N - M$. This seems to imply that for individual evolutions (orbits) only the short term future allows for deterministic predictions and that for the longer term future statistical methods have to be applied. Also see [21].

Final remarks

Another way to deal with uncertainty is by the so–called fractal basin–boundaries. Suppose a discrete dynamical system $x_{n+1} = \Phi(x_n)$ is given, with initial state x_0. It may occur that the system has a finite number of point attractors, but that the basins of attraction have strange boundaries. Indeed, in the case of three attractors the boundaries may have the following property: as soon as two of the boundaries meet, also the third is joining in. In that case, the total boundary certainly is a fractal set. This kind of subsets of the plane was studied extensively by L.E.J. Brouwer.

An example of this phenomena can be found in [12], where the computer experiment is presented of a magnetic spherical pendulum that is attracted by three different magnets.

Such a fractal basin boundary often is a strange repellor. A class of examples can be found when iterating complex analytic maps: the so–called Julia set belongs to this kind. See, e.g., [9, 12].

With some good will, also the good old die can be seen in this perspective. Indeed, we can view this system to have six different point attractors with complicated basin boundaries. As we said before, in such a case 'randomness' seems to be only introduced for the sake of convenience.

All this leads to a well–known philosophical question [25], namely whether the latter is not always the case: that randomness is a convenient short–cut in deterministic dynamics that are just too complicated to deal with in another way. In this respect one may think of systems like the atmospheric circulation (producing the weather). Another example comes from Statistical Physics and concerns the hard spheres approximation model for a gas, which has sensitive dependence on initial states.

On the other hand it seems that Quantum Mechanics takes care of 'true' randomness at very small precision.

Acknowledgements

The author thanks Aernout van Enter, Igor Hoveijn and Floris Takens for their help in preparing this paper.

References

1 V.I. Arnol′d: *Mathematical Methods of Classical Mechanics*. Springer–Verlag 1980.

2 V.I. Arnol′d, A. Avez: *Ergodic Problems of Classical Mechanics*. Benjamin Inc. 1968.

3 M. Benedicks, L. Carleson: *The dynamics of the Hénon map.* Ann. Math. **133** (1991) 73–169.

4 H.W. Broer: *Huygens' isochrone slinger*, Euclides **70**(4) (1995) 110–117.

5 H.W. Broer: *De chaotische schommel*, Pythagoras **35**(5) (1997) 11–15.

6 H.W. Broer, J. vd Craats, F. Verhulst: *Het einde van de voorspelbaarheid? Chaostheorie, ideeën en toepassingen*. (Aramith Uitgevers &) Epsilon Uitgaven **35** 1995.

7 H.W. Broer, F. Dumortier, S.J. van Strien, F. Takens: *Structures in Dynamics*. Studies in Mathematical Physics 2, North–Holland 1991.

8 H.W. Broer, C. Simó, J.C. Tatjer: *Towards global models near homoclinic tangencies of dissipative diffeomorphisms*. Nonlinearity **11** (1998) 667–770.

9 H.W. Broer, F. Verhulst (eds.): *Dynamische Systemen en Chaos, een revolutie vanuit de wiskunde*. Epsilon Uitgaven **14** 1990.

10 R.L. Devaney: *An Introduction to Chaotic Dynamical Systems*. (2nd edition), Addison-Wesley (1986), 1989.

11 J.J. Duistermaat, W. Eckhaus: *Analyse van Gewone Differentiaalvergelijkingen*. Epsilon Uitgaven **33** 1995.

12 H.–O. Peitgen, H. Jürgens, D. Saupe: *Chaos and Fractals, New Frontiers of Science*. Springer–Verlag 1992.

13 M.W. Hirsch, S. Smale: *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press 1974.

14 A. Katok, B. Hasselblatt: *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia of Mathematics and its Applications **54**. Cambridge University Press 1995.

15 O. Knill: *Positive Kolmogorov–Sinai entropy for the standard map*. Peprint 00–203 at mp.arc, available at www.ma.utexas.edu /mp.arc/c/99/99-293.ps.gz.

16 E.N. Lorenz: *The Essence of Chaos*. University of Washington / University College London Press 1993.

17 R. Mañé: *Ergodic Theory and Differentiable Dynamics*. Ergebnisse der Mathematik und ihre Grenzgebiete **8**, Springer–Verlag 1987.

18 W.C. de Melo, S.J. van Strien: *One–Dimensional Dynamics*. Ergebnisse der Mathematik und ihre Grenzgebiete **3**(25), Springer–Verlag 1993.

19 L. Mora, M. Viana: Abundance of strange attractors. *Acta Math* **171** (1993) 1–71.

20 J. Palis, F. Takens: *Hyperbolicity & Sensitive Chaotic Dynamics at Homoclinic Bifurcations.* Cambridge Studies in Advanced Mathematics **35**, Cambridge University Press 1993.

21 D. Ruelle: *De Wetten van Toeval en Chaos*. Aramith Uitgevers 1991.

22 J.C. Tatjer: *Invariant manifolds and bifurcations for one and two dimensional maps*. PhD Thesis, Universitat de Barcelona, 1991.

23 W. Tucker: *The Lorenz attractor exists*. PhD Thesis, Uppsala Universitet, 1999. Preprint at http://www.math.uu.se/ ~warwick/papers.html .

24 F. Verhulst: *Nonlinear Differential Equations and Dynamical Systems* (2nd ed). Universitext, Springer–Verlag 2000.

25 D. Wick: *The Infamous Boundary, Seven Decades of Controversy in Quantum Physics.* Birkhäuser 1995.