"*General Seminar of Mathematics*",
Department of Mathematics,
University of Patras, Greece
vol. **13**, (1987), 107-151.

Gognition, Mathematics and Synthetic Reasoning

by
COSTAS A. DROSSOS

CONTENTS

This paper is what its title implies: Cognition, mathematics and synthetic reasoning. We discuss ideas end results about: Cognition and Intel1igence, Split brain research, the methodology of the analytic-elementwise vs. the ho1istic-structural and the ontology of Geometry vs. Logic. The synthesis of all of these is given by the "Basic dialectic cystem", which characterizes left and right brain hemispheres and implies the method of extension and intention. It also explains Piaget's and related theories of intellectual developrnent. Synthetic reasoning is best understood through the basic dialectic system, using categorical concepts, like the natural base categories, etc.

The paper is newly typeseted using LaTeX, and the content is the same as the original paper except some minor corrections. Due to the new typesetting the pages do not correspond to the original ones.

1. COGNITION AND INTELLIGENCE

We summarize here some basic principles, on which Piaget's theory is based. We are following closely [20].

Piaget's theory is a theory on creation, formation and development of intelligence, and has important consequences for Mathematics, Psychology, Pedagogy and Epistemology. Some of the salient features of Piaget's theory are:

(i) Knowledge is regarded as a dynamic rather than a static matter. "For Piaget, the direct internal representation of external reality is but one aspect of thought-the figurative aspect. The transformation is referred to as the operative aspect. Thus knowledge always has a sensory-motor or activistic part.

(ii) "The organism inherits a genetic program that gradually (through a process called 'maturation') provides the biological equipment necessary for constructing a stable internal structure out of its experiences with environment." These structures or schemes together with the functions of adaptation and organization form the basis of intelligence.

(iii) Another principle of Piaget's theory, which has a holistic categorical flavor as well, is the following:

"Everything is related to everything else."

These relationships and interdependencies reduce to the corresponding relationships and interdependencies of 'functions' and 'structures'. Let us briefly list the basic ingredients of the theory. Functions are biologically inherited modes of interacting with environment. There are two basic such functions: *adaptation* and *organization*; organization being the ability to form compound, higher order structures out of simpler, lower ones, while adaptation consists of a pair of dialectically opposing functions: *assimilation* and *accommodation*. Assimilation is the process to fit the 'objects' or 'events' to the existing 'structures' or 'schemes'[1].

Accommodation is the opposite process, that is, the attempt to adjust the existing structures to objects and events. The dialectic nature of adaptation is better understood through the following diagram [15, p. 174],

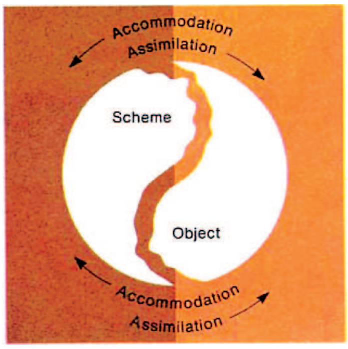


Figure 1: The "knowing circle": Assimilation and accommodation are reciprocal processes that continue working until a fit is achieved between a 'mental scheme' and an 'object'.

[1]Skemp, in [22, p. 39] best sums up structures (schemes, schemata) as follows: Scheme or schema is a "general psychological term for a mental structure. The term includes not only complex conceptual structures of Mathematics, but relatively simple structures which co-ordinate sensory-motor activity"…"The study of structures themselves is an important part of Mathematics; and the study of the ways in which they are built up and function, is at the very core of the psychology of learning mathematics"…"A schema has two main functions. It integrates existing knowledge, and it is a mental tool for the acquisition of new knowledge."

When accommodation and assimilation are in balance, we say that we have an equilibration. This balance is between the assimilation of objects to structures and the accommodation of structures to objects.

Functions are said to be invariant because they are present in every act of intelligence and they are characteristic of all biological systems. On the other hand structures vary with person's stages of development, and according to tasks that confront him during those stages". "Structures continually move attained, a scheme is sharper, more clearly delineated, than it h s been previously. However, equilibrium is always dynamic and therefore carries with it the seeds of its own destruction. For Piaget intelligence is constructed through the dialectics of adaptation (assimilation vs. accommodation) and it is developed through the changes and variability of structures, towards equilibration.

On the basis of different structures which he accepts, Piaget differentiates the corresponding four periods of development of intelligence. However, for the periods of development of the intelligence, we are following a different path. First we consider some basic ideas of split-brain research, connect them with some basic mathematical structures, finally taking the synthesis, for which the essential features of the periods follow.

2. SPLIT BRAIN RESEARCH

For split brain research there are a number of excellent books and papers [2, 23]. Although the results on the asymmetry of the two halves of the brain permits many interpretations, there is a unanimous agreement that the functions of the two hemispheres are not symmetric.The two hemispheres are essentially two separate brains, connected with the corpus callosum. Every hemisphere is specialized, in a dynamic sense, in a definite type of processes. In normal brains, the left side of the body is controlled by the right hemisphere and the right side of the body by the left hemisphere.The following picture is adapted from [15],

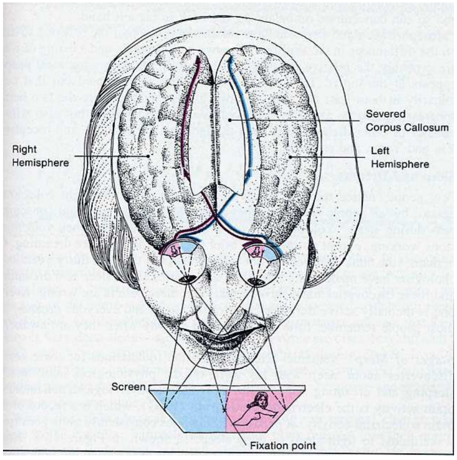


Figure 2: Split Brain

For the kind of specializations, various dichotomies have been proposed, for example:

Left Hemisphere	Right Hemisphere
Analytic	Holistic
Verbal	Non-verbal, visuo-spatial
Rational-Logical	Intuitive (Euristic)
Intellectual	Sensuous
Argumentative	Activistic, Empirical
Focal	Diffuse
Time	Space
Sequential	Simultaneous
Serial processing	Parallel processing
Western thought	Eastern thought

It have been said that the left hemisphere is something like a digital computer and the right like an analog one. We may go one step further and say that psychologically, Cognitive psychology is centered around the left hemisphere and Gestalt psychology around the right hemisphere.

From a philosophical point of view one can say that Analytic Philosophy is more appropriate and concentrated around the left brain, whether Husserl's Phenomenology is more appropriate to the right brain.

The above dichotomic schemes must not be understood in a static way but in a dialectic-dynamical sense. This is also related to the following question: *What is the reason for the brain to be developed into two hemispheres and for the human intelligence to be the unity of opposites of some basic dialectic schemes?*

One answer to this question might be the following: All the human sense-organs have been developed in such a way as to have an authentic perception of the world, perfectly simulating the corresponding laws of nature. For example, eyes function following the laws of optics with the main objective to have an optical perception of the world. Analogously, on the basis of the inherited initial genetic conditions, one would expect that the brain should have been developed through the function of adaptation into an equilibrium state, where the laws of thought are assimilation and accommodation to the objective events, phenomena and processes that take place. If we accept the fact that all physical phenomena are based on the dialectics of nature, then by adaptation it is natural for the brain to be developed in two hemispheres and for the human intelligence and thinking to be characterized by the laws of dialectics. On the other hand, if we consider mathematics as the most exact and well organized product of the human mind we expect that they should be dialectical in nature. Most mathematicians seem to be prejudiced for the deterministic, "zero-one", characteristics of mathematics. Aspects like "variable", "fuzzy","partially true", etc. are rather ignored. This adoration for the Law of the Excluded Middle and for Platonism, contradicts the recent developments in Topos theory, Fuzzy set theory, etc. Category and Topos theory unify in a dialectic fashion all aspects of Mathematics: Mathematics of constant and sharp objects, that is, "mathematics of being" and mathematics of variable and fuzzy objects, that is, "mathematics of becoming". Consequently, Category theory can be considered as the study of the "being of becoming" of mathematical objects.

Finally,taking into account the researches on the physiology of the human brain and the mathematical experiences and results, we can chose some basic dialectical schemes, which seem to characterize the brain and its products.

From the physiological point of view one tends to accept the analyses of Bradshaw-Nettleton [2], as having the highest consistency and harmony with the mathematical experience. Consequently, our first basic dialectic scheme will be the following variety of the one in [2]:

(*a*) *Analytic-elementwise vs. holistic-structural*

A second basic dialectic scheme comes from Lawvere's work [11, 10, 12, 13]:

(*b*) *Logic vs. Geometry*

Notice that in the term "Geometry" we can condense concepts like *the visual-spatial*, *the extensional*, *the sensuous and empirical*, *the semantical and the significate*. On the other hand, "Logic" summarizes concepts like *the rational*, *the intentional*, *the symbolic and the signifier*. Before we examine in more detail the above two schemes,we should like to make the following basic remarks: Since our approach is dialectical,we must emphasize that we consider two kinds of objects:

(a) *Constant objects*;

(b) *Variable objects*

Constant vs. variable objects is a scheme, which is the heart of all dialectics. In mathematics we encounter constant objects in the form of sets, and sets with structure. The concept of a variable object is based on the concept of function, and hence the original scheme, *constant vs. variable*, is transformed into the, *set vs. function*. This last dialectical scheme leads naturally to Category Theory. For the importance of this scheme, see [9, 1] and for a detailed construction of Infinitesimal Analysis and Boolean Analysis, based on this scheme, see [5]. In [11] we find: "...a 'set theory' for geometry should apply not only to abstract sets divorced from time, space, ring of definition, etc.., but also to more general sets which do in fact develop along such parameters. For such sets, usually logic is "intuitionistic" (in its formal properties) usually the axiom of choice is false, and usually *a set* is not determined by its points defined over 1 only".

Let us now examine a little closer the two basic dialectical schemes.

3. THE ANALYTIC-ELEMENTWISE VS. THE HOLISTIC-STRUCTURAL

The physical world, the world of "being", is clearly organized in *hierarchical levels*. For example we have the molecule level, the atomic level, the quantum and subquantum levels etc.

Every level has its own structure,laws and methodology and is mainly related to the neighboring ones. The relationships between neighboring levels, the explanations of the properties of the objects of the one in terms of the other, etc. are the most important problems. What appears to be "elements" of one level are usually structured wholes of elements, or gestalts of the preceding lower level. Usually the human brain works at two successive levels: The *analytic-elementwise* lower level and the *holistic-structural* higher one. We regard the elements of the lower level as constant, realized entities, without internal structure, usually called “urelements” or “atoms”. Structured wholes of such urelements give rise to elements of the higher level. It has been said that category theory works with three levels of abstraction: Analytic- elementwise, holistic-structural and categorical. However if we introduce variable or generalized elements, we may reduce the levels of abstraction back to two.

An object A may be constant or variable. If it is constant then it can be determined using two methods:

(i) Analytically-elementwise, by specifying its urelements, that is its extension or

(ii) Holistic-structurally or element-free, by specifying a structure S to which A is an element.

If A is variable,then the analytic-elementwise determination of A is not sufficient to determine A. However the holistic-structural determination of A, which happens to coincide with the determination of its “generalized extension” that is by specifying all its variable elements, suffices for a complete determination of A.

Let us see some standard examples.

EXAMPLES: (1) Sets. Let X be a set and $A \subseteq X$. We consider the elements of X as urelements. We can then describe A by specifying its extension, that is by specifying which urelements of X belong to A. This is the analytic-elementwise way. However we can consider an object of a higher hierarchical level than the one of A, i.e. $\mathscr{P}(X)$ and its Boolean structure $\langle \mathscr{P}(X), \cup, \cap, \emptyset, X\rangle$. Then every subset $A \subseteq X$, becomes an element of $\mathscr{P}(X)$ and is determined by the Boolean structure and axioms. For example we may have element-free characterizations,like:

(i) $A \cup B = A \cup B \Longrightarrow A = B$;

(ii) $A \subseteq B \Longleftrightarrow A \cap B = A \Longleftrightarrow A \cup B = B$;

(iii) If there exists $Y \in \mathscr{P}(X)$ such that, $A \cup Y = B \cup Y$ and $A \cap Y = B \cap Y$ then $A = B$, etc.

(2) Real Numbers. Consider the elements of $\mathbb{Q}$, as urelements. Then the real number π can be determined analytically-elementwise by determining for example an equivalence class of Cauchy sequences of urelements. From the holistic-structural point of view, π can be determined by the axioms of the structure of the real numbers. It is worth of notice that π, presents itself as an actual infinity at the level of $\mathbb{R}$ and as a potential infinity at the level of $\mathbb{Q}$. Thus we can compactly say,that when we consider variable urelements we essentially speak about "the becoming of being" and when we consider the holistic-structural essentially we speak about something stable "the being of becoming". In addition the constant and crisp is usually more susceptible to quantitative and extensional characterizations through a classical type of logic, whereas the variable and fuzzy is usually more susceptible to qualitative and intentional characterizations, through a non-classical logic. Finally, it seems that all paradoxes and ambiguities in mathematics, have something to do with an unconscious passing and confused mixing of one with another hierarchical level.

(3) Variable objects-Functions.

Let $f : A \to B$ be a function, A, B sets. Usually f can be considered from two dual points of view:

(a) **As a variable element of** B**.**

$$(f(a))_{a\in A}\,;$$

in which case A is construe as "time" and $f(a)$ as the state of the system f. Under this interpretation, the defining relation for a function, simply says that, at a concrete time a, the system cannot be at two different states at the same time!

(b) **As a fuzzy element of** B.

The function f can be expressed dually as follows:

$$(A_b)_{b\in B}$$

where $A_b := \{\, a \in A \;:\; f(a) = b \,\}$. Then to every $b \in B$ we can assign a degree of belongingness of b to the fuzzy element f of B, by the following Boolean-fuzzy element of B,

$$f^{\triangleleft} : B \to \mathscr{P}(A) \;\; /\!\!/ \;\; b \mapsto A_b$$

Now if A and B are objects with their elements defined only over 1, then analytic-elementwise determination of f is possible through its action on these elements. On the other hand the holistic-structural way of determining f in the general case, presents some difficulties. If B has some structure, then usually we get a similar structure on B^A, even if we need to consider some ultraproduct or Boolean product construction, and this structure is sufficient for determining f. If however B does not have any structure, but is just an abstract set then, B as a set is a "constant" entity, but does not have any geometrical form or structure and in this respect it is completely chaotic.

For example if $B = \mathbb{R}$ then $\mathbb{R}^A$ is a vector-lattice and $f \in \mathbb{R}^A$ is determined by this dominating structure. Similarly if B an ordered set. In the case however that A and B are abstract sets, then B cannot be determined structurally in the usual way.

In order to determine f in a holistic-structural way, we have to determine the action of f, not only on the urelements of A, but also on its variable elements. This is necessary since if A and B are abstract sets we can only discriminate the elements of the sets and say what are their cardinals. But this information is not enough to determine structurally the function f. Considering variable elements is the same as putting some structure and form on the sets. If we exhaust all this structural information, only then it is possible to determine f in a holistic-structural way.

For this purpose, let T be a set, playing the role of time- domain for the variable elements. If $T = 1 = \{\emptyset\}$ then every urelement of A corresponds to exactly one urelement of B:

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, dashed, "f \circ a \equiv f(a)"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

Now for a general T we have that, to every variable element of A there corresponds exactly one variable element of B:

$$\begin{tikzcd} & T \arrow[dl, "x"'] \arrow[dr, dashed, "f \circ x \equiv f(x)"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

This line of thought leads to the concept of category. Indeed, in this case we can identify A and B with the slice categories,

$$\mathbf{Set}{\downarrow}A \quad \text{and} \quad \mathbf{Set}{\downarrow}B$$

and the function f with a higher type object, the functor $\mathrm{Hom}(-, f)$ such that for every $T \in \mathbf{Set}$, we have:

$$\begin{array}{rcl} H^T(-) : \mathscr{C} & \to & \mathbf{Set} \\ A & \mapsto & H^T(A) \equiv \mathrm{Hom}(T, A), \\ (A \xrightarrow{f} B) & \mapsto & (H^T(A) \xrightarrow{H^T(f)} H^T(B)), \end{array} \qquad \text{with,} \quad \begin{tikzcd} T \arrow[d, "a"'] \arrow[dr, dashed, "b = f \circ a := H^T(f)(a)"] & \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

see also [13]. This analysis is essentially the substance of Yoneda's Lemma and leads finally to the doctrinal diagram see [25].

Finally, one possible way to express the dialectical scheme of the analytic-elementwise vs. holistic-structural is through the concept of a "geometric morphism" see e.g. [25]. For example if $\mathscr{E}$ is a topos then a geometric morphism $g : \mathscr{E} \longrightarrow \mathbf{Set}$ is given by,

$$g^*(A) := Hom(1, A), \quad A \in \mathscr{E}$$

and

$$g(T) := \coprod_T 1, \quad T \in \mathbf{Set}.$$

Thus $g^*(A)$ are the urelements of A and $g(T)$ expresses a kind of structural sum of elements of T. Next we come to the other basic dialectical scheme (Lawvere's Scheme).

4. LOGIC VERSUS GEOMETRY

In [11] we find: "The unity of opposites in the title (Quantifiers and sheaves) is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect." and "...in a sense logic is a special case of geometry".

According to Piaget, "there is a logic of co-ordination of actions, deeper in roots than the logic of language and much more earlier than the logic of "propositions", with the strict sense of the term".

Logic, language and the semiotic function, arise out of an activistic, "sensory-motor intelligence" which we encounter in the early stages of development of the human intelligence, and in cases where we have fuzzy, unsharp and variable objects and situations. Thus geometry and dynamical geometric activity is the primary, and logic and language is the secondary and the byproduct, contrary to the school of Logicism.

If this is so, then we need a kind of logic which should be language-free and completely expressible in terms of arrows (the actions) and diagrams of arrows. Such a logic is the Categorical Logic, in which the logical operators arise as morphisms and functors.

We shall give some flavor of the dialectics of geometry and logic, considering again the basic concept of function. This concept is basic since it encompasses all the important characteristics of mathematics. For example,a constant set A may be represented by its indicator function I_A. Therefore more general functions, represent variability and fuzziness, as we have seen before. A function $f : A \longrightarrow B$ may be also interpreted in a logical way. Suppose that $A = \{x \in V : p(x)\}$ and $B = \{y \in V : q(y)\}$, then the function-arrow may be interpreted as *the logical implication* and A and B as the propositions p and q. Hence $x \mapsto f(x) = y$ essentially means $p(x) \Longrightarrow q(y)$.

Functions also of the form $f : A \longrightarrow \Omega$ where Ω is a set of truth values, like 2, a general Boolean algebra $\mathbb{B}$, or a Heyting algebra $\mathbb{H}$, etc., represent pure properties or intentional characteristics. If in addition we have also a measure $m : \longrightarrow \mathbb{R}$ then $m \circ f$ represent quantitatively expressed properties or measurements.

Incoming functions to A,

$$f : T \longrightarrow A,$$

have a geometric or extensional meaning. If T has an appropriate cardinality, then such a function is a list $(f(t))_{t \in T}$ of the elements of A and hence an extensional abstract determination and presentation of A. We can consider T not only as an index set but also as time domain and the function as *a path*, *a figure of type* T, etc., see [14].

Increasing the number of incoming functions to A, we essentially increase the geometrical form and structure of A.

Finally getting into account the above remarks, and the existing similar literature, one can convinced himself that the notion of elementary topos is a generalization of the static set theory, into one that embodies *the dialectics of the constant and sharp* versus *the variable and fuzzy* and the logical and geometrical aspects of mathematics, as well. We can summarize these remarks in the following picture:

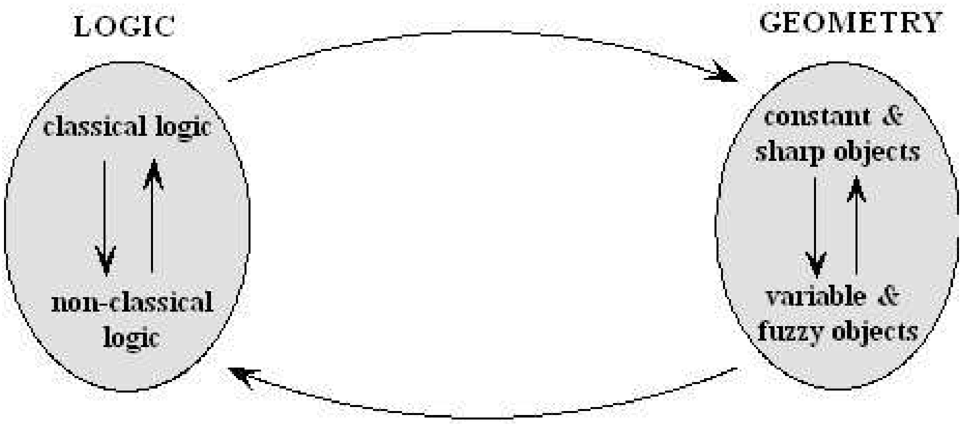


Figure 3: Logic vs. Geometry

5. THE BASIC DIALECTICAL SYSTEM

Combining the two basic dialectical schemes one arises at the following system:

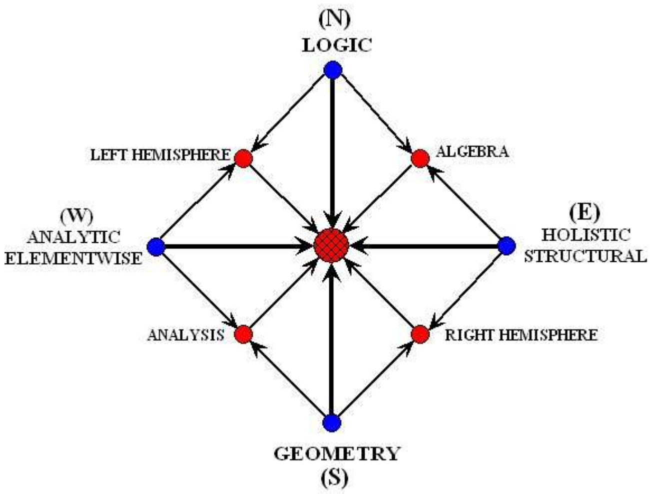


Figure 4: Mathematics as quintessence

For easy of reference, but having also some unexpected geographic interpretation, we shall denote with W (west) the analytic-elementwise, with E (east) the holistic-structural, with S (south) the geometric and with N (north) the logical aspect. We can consider the following "adjunctions":

(i) $NW \rightleftarrows SE$

(ii) $SW \rightleftarrows NE$

The first "adjunction" represents the dichotomy of *left hemisphere vs. right hemisphere*, and is highly consistent with the physiological experimental data:

Left Hemisphere (NW)	vs.	**Right Hemisphere (SE)**
analytic-logical-verbal		holistic-visuo-spacial
formal		conceptual
sharp, rigorous		fuzzy, intuitive
sequential time		instant time
algorithmic, discrete		dialectic, continuous
constructiveness		non-constructiveness

The second "adjunction" represents a more familiar dichotomy:

Geometry & Analytic-Elementwise (NW)	vs.	**Logic & Holistic-Structural (NE)**
extensive		intensive
concrete		abstract
Analysis		Algebra

Next we should like to treat the basic characteristics of the dialectics of extensive vs. intensive.

Extension And Intention

The philosophical categories of extension and intention have appeared throughout the historical development of philosophy where one can find similar terms like "connotation" and "denotation". In general extensive means relative to matter and space,whereas intensive means relative to properties of the matter.Specifically the term extension means that an object occupy a definite place in space and is related with concepts like, "set", "class", "part", "element", "belongingness" etc. Thus if p is a property then,

$$A = \{\, x\,;\, x \in V \& p(x)\,\}$$

is the extension of the property p. In this sense, Boolean algebras is the appropriate frames to represent extensions of sharp or crisp and constant sets. On the other hand intensive means relative to the properties of the matter. Specifically intention is referred to concepts like "property "attribute", "proposition", "character", "characteristic" etc. We find in [16] the following opinion of Gauss:

> "The subject of mathematics is all extensive magnitudes (those in which parts can be conceived); intensive magnitudes (all non-extensive magnitudes) insofar as they depend on the extensives. To the former class of magnitudes belong space (or geometrical magnitudes which include lines, surfaces, solids and agles), time, number; to the later: velocity, density, rigidity, pitch of tone, intensity of tones and of light, probability, etc.

"From a geometrical point of view we have Grassmann's basic contribution: "The Science of Extensive Quantities", see [16], and from a logical point of view we have Carnap's contribution to the dialectics of extensive and intensive, see [3]. In Carnap we find:

> "...the method of extension and intention, is developed by modifying and extending certain customary concepts, especially those of class and property. The method will be contrasted with various other semantical methods used in traditional philosophy or by contemporary authors. These other methods have one characteristic in common: They all regard an expression in a language as a name of a concrete or abstract entity. In contradistiction, the method here proposed takes an expression not naming anything, but as possessing an intention and extension.
>
> "The method of extension and intention needs only one expression to speak about both the property and the class and, generally one expression only to speak about an intention and the corresponding extension".

We shall divide the theory of extensive-intensive into a qualitative and quantitative one. An important contribution to the later is Lawvere's study see [14], whose ideas and insights have been a constant source of inspiration. Next we shall be briefly concerned with the qualitative aspects of extension and intention, giving also a summary of Lawvere's ideas.

Before we continue, we should like to make the following basic remark. Under the influence of the other pair (W-E) of the basic dialectic system, each entity can be considered at least from two hierarchical levels: The lower analytic and elementwise (the elementary!) and the higher holistic and structural. For example at the level of $\mathbb{Q}$, π is an equivalence class, and as such has an extension. However in the level of $\mathbb{R}$, π is considered as an atom or a urelement obeying the dominating structure of $\mathbb{R}$, and thus does not have an extension.

Now, let $A \subseteq V$ be a set. Then a function of the type,

$$\emptyset \longrightarrow A,$$

presents A as an abstract set, without indicating any of its points. In this case A is rather an urelement without any reference to its extension, and an element-free presentation would be more appropriate:

$$1 \longrightarrow \mathscr{P}(V) \;\; /\!\!/ \;\; \emptyset \mapsto A$$

The abstract extension of A, coincides with the set $\mathrm{Hom}(1, A)$. However this extension seeing from a holistic level, does not posses any external geometrical form or any internal algebraic structure but simply exists in a chaotic form. In a similar way, a physical body with an external geometrical form and an internal algebraic structure cannot be describe using only its urelements or atoms, $1 \longrightarrow A$, but we have to use, externally curves to structure its geometric form and internally varying particles, of various types T, i.e. motions that are parametrized by a time domain T. If we want to perceive the abstract extension of A, in a concrete geometric form, we must consider all incoming to A functions i.e. $\mathbf{Set}{\downarrow}A$ and in general $C{\downarrow}A$ where C is a small category the objects of which act as time-domains. Each function $T \longrightarrow A$ is the subextension of the extension of A, of a geometrical aspect of A (figure, curve, listing, etc.). All these aspects constitute the "variable elements of A" with domains of variation $T \in \mathbf{Set}$. Therefore $\mathbf{Set}{\downarrow}A$ is the extension of A presented in a geometrical form, in the environment of **Set**, and constitutes all extensional aspects of the set A, with respect to **Set**.

Now if $f : A \longrightarrow B$ is a function then,

$$f^{\triangleright} : \mathbf{Set}{\downarrow}A \longrightarrow \mathbf{Set}{\downarrow}B$$

is a covariant functor, since for every extensional aspect,

$$x : T \longrightarrow A$$

of A corresponds an extensional aspect $f(x) := f \circ x$ of B.

According to Lawvere, covariance of $f^{\triangleright}$ is the crucial property of extensionality.

Now for the intentional aspects of A, we consider instead of ingoing functions, outgoing ones. That is functions of the type,

$$A \longrightarrow T$$

Thus, $A \longrightarrow 1$, presents A as an abstract set, that is without any property except of being a set. This situation corresponds to the extensive aspect $0 \longrightarrow A$, or

$$1 \longrightarrow \mathscr{P}(V) \;\; /\!\!/ \;\; \emptyset \mapsto A$$

Functions of the type,

$$A \longrightarrow 2$$

divide the elements of A according to one property. The comma category $A{\uparrow}\mathbf{Set}$ expresses all intentional aspects of A, with respect to $\mathbf{Set}$.

If $f : A \longrightarrow B$ is a function then,

$$f^{\lhd} : A{\uparrow}\mathbf{Set} \longrightarrow B{\uparrow}\mathbf{Set}$$

is a contravariant functor, since for every intentional aspect of B, $p : B \longrightarrow T$, corresponds an intentional aspect $q : A \longrightarrow T$ of A defined as follows:

$$f^{\lhd}(p) := p \circ f$$

From the above discussion it is clear that the dialectics of *extension vs. intention* coincides with the duality principle in category theory. In many books this principle is usually founded on the corresponding duality of logic and Boolean algebras. However here this principle is based, not only on logic, but also on all relevant components of *the dialectics of SW vs. NE.*

As a simple example we take the concepts of monomorphism and epimorphism. To say that $f : A \longrightarrow B$ is a monomorphism is the same as saying that the extensional aspects of A and B are in one-to-one correspondence i.e.,

$$\begin{tikzcd} T \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[rr, bend left, dashed, "f \circ x"] \arrow[rr, bend right, dashed, "f \circ y"'] & A \arrow[r, "f"] & B \end{tikzcd}, \quad f \circ x = f \circ y \;\Rightarrow\; x = y$$

whereas an epimorphism $f : A \longrightarrow B$, says that the intentional aspects of B and A are in one-to-one correspondence i.e.,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, dashed, "p \circ f"] \arrow[rr, bend right, dashed, "q \circ f"'] & B \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & V \end{tikzcd}, \quad p \circ f = q \circ f \;\Rightarrow\; p = q$$

The following example taken from [7], is instructive in separating the quantitative and qualitative aspects of the extension and the intention.

Let $(\Omega, \mathscr{A}, P)$ be a probability space, e.g. $([0,1], \mathscr{B}, \mu)$, which we intend to use as the basis of time-domains in considering variable elements. For simplicity we consider the real numbers $\mathbb{R}$. Any intended set of real numbers $\mathbb{R}$, has a well defined *algebraic structure* and *geometrical form*. Suppose now we consider $\mathbb{R}$, in an environment allowing random interactions. We would like to see what are the changes in extension and intention of $\mathbb{R}$, in this new environment. To this end, there are compelling reasons -one being the possibility of element-free formulation of random concepts- to consider the probability $\sigma - algebra$, $(\mathbb{B}, p)$, where $\mathbb{B} := \mathscr{A}/\approx$ and

$$C \approx D \quad \text{iff} \quad P(C \bigtriangleup \quad D) = 0 \quad C, D \in \mathscr{A} \tag{$*$}$$

Each random variable (r.v.) induces an at most countable positive partition on Ω and this in turn, through the equivalence relation $(*)$, induces an at most countable positive partition of the unity $1 \in \mathbb{B}$, $T = \{t_i : i \geq 1\} \subseteq \mathbb{B}$, i.e. $t_i \geq 0$, $t_i \wedge t_j = 0$ and $\bigvee_{i \geq 0} t_i = 1$ giving thus rise to an elementary or discrete r.v. which we denote with the same symbol,

$$X : T \longrightarrow \mathbb{R} \;/\!\!/\; t_i \mapsto x_i$$

Such partitions of unity are termed "*random experiments*". Let $\mathscr{T} = \mathscr{T}(\mathbb{B})$ is the set of all random experiments, and consider on $\mathscr{T}$ the relation,

$$T_1 \preccurlyeq \quad T_2 \quad \text{iff} \quad (\forall t_{1i} \in T_1)(\forall t_{2j} \in T_2)[t_{1i} \leqslant \quad t_{2j}]$$

i.e. T_1 is a finer partition of unity than T_2. Also,

$$T_1 \wedge T2 := \{t_{1i} \wedge t_{2j} \neq 0 \,|\, t_{1i} \in T_1, t_{2j} \in T_2\}$$

Let $\mathscr{E}(\mathbb{B})$ be the set of all elementary r.v. For each r.v. $X : T \longrightarrow \mathbb{R}$ let,

$$f_X : \mathbb{R} \longrightarrow T, \quad \text{with} \quad f_X(x) := \begin{cases} t_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases}$$

It can be seen that f_X is a Boolean qualitative analog of a probability density which coincides in our case with a Boolean probability. Let,

$$\mathbb{R}^{(\mathbb{B})} := \{f_X : X \in \mathscr{E}\}$$

Then extending the operations and relations from $\mathbb{R}$ into $\mathbb{R}^{(\mathbb{B})}$, we see that the structures $(\mathscr{E}, +, \cdot, \leq, 0, 1)$ and $(\mathbb{R}^{(\mathbb{B})}, +, \cdot, \leq, 0, 1)$ are isomorphic. The second structure is the Boolean power of $\mathbb{R}$, and both structures constitute nonstandard (in a Boolean sense) Archimedean models of $\mathbb{R}$. Since $\mathscr{T}$ is a small category then $\mathbf{Set}^{\mathscr{T}}$ is a topos which is isomorphic with $\mathbf{Set}^{(\mathbb{B})}$, which has as objects the Boolean powers of each set in **Set** and morphisms the Boolean extensions of the corresponding arrows in **Set**. In this example one can see that there is a one-to-one correspondence between intentional and extensional aspects. Thus if $T \longrightarrow \mathbb{R}$ is a variable element of $\mathbb{R}$, there is a corresponding $\mathbb{B}$-fuzzy element of $\mathbb{R}$, which is the opposite function $\mathbb{R} \longrightarrow T$. A generalized extensionality principle should be: *two objects are equal iff they have the same urelements and variable or fuzzy elements.*

Concluding, we can say that: $\mathscr{E}(\mathbb{R})$ is the stochastic qualitative extension (with twofold meaning!) of $\mathbb{R}$ and $\mathbb{R}^{(\mathbb{B})}$ is the Boolean qualitative intention of $\mathbb{R}$. If we employ also the measure $p : \mathbb{B} \longrightarrow [0, 1]$ then we get quantitative expressions for the extension and intention of $\mathbb{R}$, in a random environment.

Let us finally summarize the purely quantitative aspects of Lawvere's theory of extension and intention[14]:

EXTENSIVE QUANTITIES	vs.	**INTENSIVE QUANTITIES**
• are covariant		• are contravariant
• constitute a linear space		• constitute an algebra
• have "total value" $\mathscr{M}(X \to 1)$		• include the"constants" $\mathscr{C}(Y \to 1)$
• include "Dirac δ's" associated with points of the space: $\mathscr{M}(1 \to X)$		• have "values" associated with the points of the space $\mathscr{C}(1 \to Y)$

L I N K S

Billinear Pairing (integration Processes)
Module structure of extensive quantities over intesive ones

E X A M P L E S

In Top: $M(X)$ Randon measures	C(X) continuous functions on X
In $\mathscr{C}^{\infty}$: Currents	Differential forms, etc.

6.THE NATURAL CATEGORIES OF THE BRAIN: Synthetic Thought

Coming back to the brain structure we claim that,is characterized by the basic dialectical system,the unity of opposites or synthesis of which is essentially the substance of the synthetic thought.

In Kock's book [8], we find: "The aim of the present book is to describe a foundation for synthetic reasoning in deferential geometry. We hope that such a foundational treatise will put the reader in a position where he, in his study of differential geometry, can utilize the synthetic method freely and rigorously, and that it will give him notions and language by which such study can be communicated." Next Kock describes the "synthetic" reasoning as follows:

> " It deals with space forms in terms of their structure i.e. the basic geometric and conceptual constructions that can be performed on them. Roughly these constructions are morphisms which constitute the base category in terms of which we work; the space forms themselves being objects of it.
>
> This category is *cartesian closed*, since, whenever we have formed ideas of "spaces" A and B, we can form the idea of B^A, the space of all functions from A to B."

In the paper [21] of G. Reyes we find a description of synthetic arguments:

> **1)** They are intrinsic; no mention is made of atlas, coordinates, etc., even when manifolds are mentioned.
>
> **2)** Infinitesimals are freely used and they substitute limits...
>
> **3)** No "logic" is used in as much as the statements concerned are equations.

The following table attempts to sum up this discussion:

Synthetic	vs.	**Analytic**
– Direct manipulation of geometric objects		– Manipulation of analytic representation of geometric objects
– Logic at Bay. Use of naive logic		– Extensive use of classical logic
-Extensive use of infinitesimals		- Limits.

In page 72 of the same paper we find:

> "In three lectures at the University of Chicago in 1967, published in A Kock (Ed.1979), F. W. Lawvere proposed to use the theory of variable sets (= topos theory), developed by the Grothendieck school of Algebraic Geometry, as a foundation for synthetic reasoning."

"On the other hand in [12] Lawvere has expressed the following interesting view:

> "The program of investigating the connections between algebraic geometry and "intuitionistic" logic under the guidance of the form of objective dialectics known as category theory..."

We may summarize the above views as follows:

(i) F. W. Lawvere identifies Category and Topos Theory with a form of objective dialectics, where the dialectical processes take the form of adjoint functors. In [9] we find a diagrammatic presentation of a dialectical scheme:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"] & \mathscr{B} \arrow[l, bend left, "G"] \\
\mathscr{B}_0 \arrow[u, hook, "inclusion"] \arrow[r, leftrightarrow, "equivalence"'] & \mathscr{A}_0 \arrow[u, hook, "inclusion"']
\end{tikzcd}$$

along the following interpretation:

> "There are at least two ways in which we can interpret adjoint functors as exemplifying the dialectical process.
>
> **Interpretation I.** We may think of the pair (F, G) as establishing a contradiction between A and B. The equivalence between A_0 and B_0 is then the unity of opposites...
>
> **Interpretation II.** We may think of F as the thesis; its right adjoint G then becomes the antithesis. In this interpretation we should perhaps think of the adjunction itself as the synthesis. (This appears to be Lawvere's favorite interpretation.)".

For many examples taken from mathematics, see [9].

(ii) Synthetic reasoning and thought is based on a category over a base natural topos. Depending on the nature of the subject under consideration, the corresponding natural "geometric form" of the objects determine the natural base topos and its logic. The methodology of the *analytic-elementwise* vs. the *holistic- structural* remains always the same. For example, the objects of Physics, Chemistry, Biology, Psychology, Sociology, Arts, etc. have their own "*geometric form*" and corresponding logic. If the objects of the theory have a constant and crisp "geometric form" we may use classical logic, but if the "geometric form" is variable and fuzzy then we have to use a non-classical more flexible logic, like for example the intuitionistic one.

The dichotomic schemes that have been considered, on the basis of experimental evidence in split brain research, may be taken as empirical observations of the adjoint pair of functors resulting out of the basic dialectic system.

Thus from this point of view, all of our thinking is based on a dialectical game and on the strained relationships between the real and the concrete and the abstract and the imaginary, between the analytic-elementwise and the holistic-structural, between the algorithmic and the dialectic. The synthetic thought is the one, which in the framework of the natural base categories of the brain, composes and equilibrates all the above dialectical schemes, which penetrate and influence each other, giving the sense of a transcendental unity and of metaphysical whole: "the being of becoming".

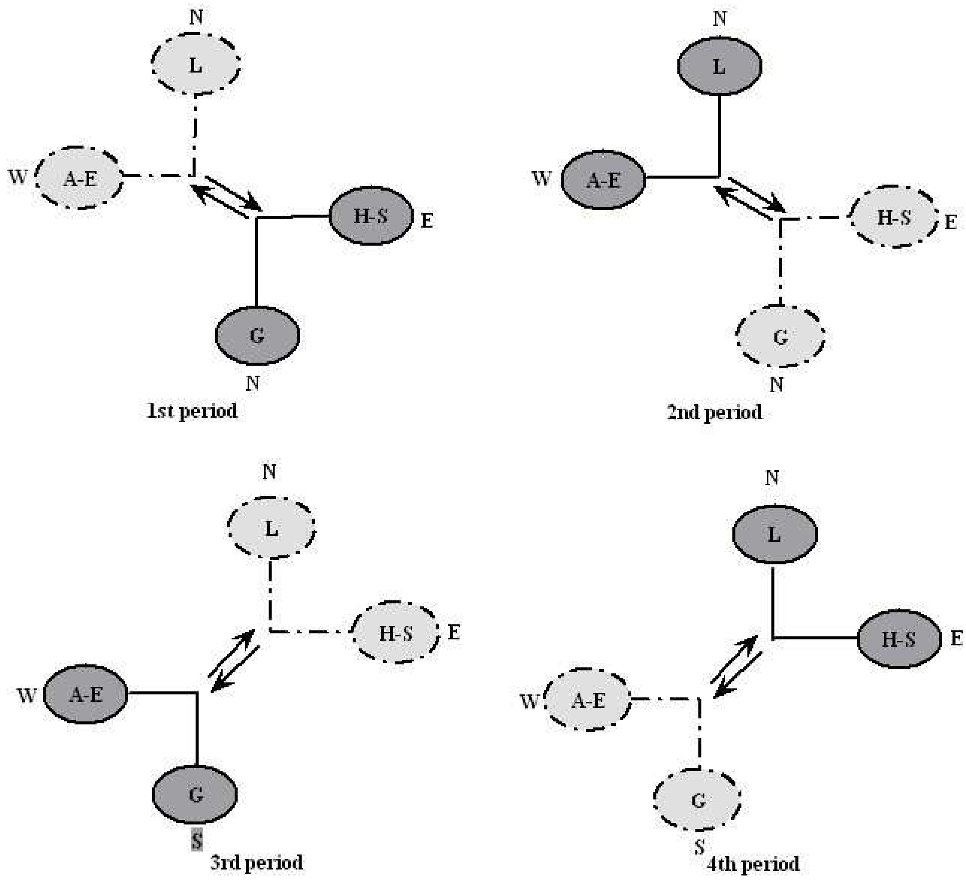
The synthetic human thinking avoids extreme use of analysis both in content and in form and tries to equilibrate the analytic-elementwise and the holistic-structural in content and in form. When the framework and the nature of the problem demands to work with some simple, but structured objects ,then forgetting the form and the structure and starting the study from a chaotic form and structure, makes the problem unnaturally treated and thus extremely difficult. Set theory is the classical example of using an extreme analysis, both in content and in form and this is the main reason that in many cases it disagrees with synthetic thought. The later uses simple but structured objects belonging to a natural base category, as units (urelements) of thinking. These units, are like the molecules of the matter or the cells of living matter, they retain the attributes of the whole to which are elements and thus they are the basic units of study.

The synthetic thought is strongly connected with the well known "Occam's razor": "Entia non sunt multiplicanda praeter necessitatem" that is "The entities must not be increased (analyzed), beyond that which is necessary." As an example, if we are interested to study the property that the water extinguish fire, then it is not appropriate to analyze water into oxygen and hydrogen, since both elements intensify fire. All these, from a philosophical point of view, are based on an amazing phenomenon, which characterizes all modern thinking: *The Non-Euclidean geometries are based upon the Euclidean one, the mathematics of variable objects upon the mathematics of constant objects, the mathematics of fuzzy objects upon the mathematics of sharp and crisp objects, the non-standard mathematics upon the standard ones, the irrational upon the rational, the metaphysical upon the concrete and physical.* We may add also, *the unconscious upon the conscious.* Through the above formalization and the flexibility of dialectics, there is a possibility for a mathematical understanding of previously unapproachable subjects.

7. PIAGET'S MODEL OF INTELLECTUAL DEVELOPMENT

In Section 1 at the beginning,we gave the basic concepts and framework of Piaget's theory.Next we are going to give a new description of the Piaget's model of intellectual development and the interconnections with the Van Hiele model for geometric learning.This new description will be based on the basic dialectical system, giving also to this models, a consistent with mathematics frame of foundation.

Piaget divides the human intellectual development into four periods: the sensorimotor period, the pre-operational period, the concrete-operational period, and the formal-operational period.We may also have some subdivisions of each period,into stages.It is claimed here that these four periods correspond to the four possible dialectical schemes, in which,continuous lines indicate the dominating part of the scheme,and dashed lines the secondary part, or even the one which exist at the beginning, in a seminal state:



(i) The sensorimotor period. This period encompasses infancy, approximately the first two years of life. This period is dominated by a holistic-structural perception, through direct interaction and experience with the objects. We have thus an activistic and "practical" intelligence. There is no analytical-logical abilities and no language and semiotic function. Therefore during this period we have a kind of pure categorical logic which corresponds to a logic of coordination of actions, and is based only on interactions with objects, that is on a kind of "empirical arrows".

The natural base category of the brain is characterized by the fluidness of objects and is more concentrated on processes, actions etc. that is on "empirical arrows". In [20] we find: "To know an object is not merely to register a copy of reality (its figurative aspect) but to transform it through assimilation to operative structures". "The inference that an object has permanence beyond our immediate perception of it comes even later, for the child must construct the very concept of "an object". The construction begins when the infant starts to coordinate various action schemes-for example, hearing and looking at the same object; reaching and grasping the same object; seeing, reaching, and grasping the same object; reaching, grasping, and sucking the same object. In order to construct the object scheme from those experiences, he must abstract from them the one feature that they all share-the one invariant among many experiences."

Thus the acquisition of the concept of the permanent and sharp object,is identical with defining the "extension" of the object, through all incoming arrows (actions, processes, etc.) which the child performs on the object. But this is an "empirical" realization of Yoneda's Lemma. With the permanence of objects we also have the opposing concept of variable and fuzzy object and through them a rudimentary conception of time and space.

(ii) The pre-operational period.(2-7 years) The main characteristics of this period are: Representational intelligence, the development of the semiotic function which pertains to sighs and symbols and the acquisition of language. Although Piagetians do not talk about the ability to analyze a whole to parts and the logic that this implies, we believe that all the main characteristics of this period may be subsumed by the development of the analytical-logical ability. Through the processes of accommodation which produces the figurative aspect of knowledge and assimilation, which produces the operative aspect, the child develops the semiotic function which is the analysis of a holistic "point" into signifier and significate. This essentially coincides with an analysis of whole into parts, see also [20]. This finally implies a representational stage for the intelligence. Thus the pure categorical, languageless logic of coordination of actions and processes (logic of arrows and diagrams) of the sensorimotor period, with the advent of representational intelligence, becomes the logic associated with the semiotic function (signifier-significate) in general, and with language in particular. As piaget puts it, language may be a necessary condition for the completion of logico-mathematical structures but it is not a sufficient condition for their formation.

There are also some limitations of preoperational thought. According to [20] we have the following limitations:

1. Concreteness
2. Irreversibility
3. Egocentrism

4. Centering
5. States vs. transformations
6. Transductive reasoning.

Concreteness here means that, "Much of his thinking takes the form of what Piaget calls mental experiment. Instead of the adult pattern of analyzing and synthesizing, the preoperational child simply runs through the symbols for the events as though he were actually participating in the events themselves. As compared to more advanced levels, his thinking at this stage tends to be dominated by its figural aspects by perceptions and images", see [20]. Irreversibility essentially means that his actions have rather a monoidal structure instead of a group structure. This monoidal structure seems to imply some variable and fuzzy aspects of intelligence. Egocentrism and centering, underline that the concepts and cognition have a rather local aspect. The last two limitations have to do with the inability to comprehend or to make a synthesis from parts (states) to a conscious structured whole, which may be of a higher hierarchical type.

(iii) The Concrete Operations Period.($\approx$ 7-11 years).

We have seen that the activistic intelligence of the sensorimotor period have changed to an intelligence which is dominated by perceptions and images,in the preoperational period. In the Concrete Operations Period we have real intellectual operations. An operation is an internalized transformation which has an inverse (group structures and Boolean Topoi). This essentially implies that in this period the child concentrates on rather sharp, concrete and constant space objects using a classical logic. During this period the child develops quantitative abilities and constructs a comprehensive, highly structured space and time.

(iv) The Formal Operations Period($\approx$ 11-15 Years).

The last period , the formal operations, is characterized by a complete development of the abstract,the hypothetical and the possible. The child may operate on expressions in the form of propositions, properties and algebraic intentional functions. His reasoning and concept formations, is relatively independent of concrete reality and of visuo-spacial representations and thus holistic and structural. Element-free characterizations are possible.

8. THE VAN HIELE LEVELS

From our point of view, mathematics seems to be the unity of opposites of the basic dialectic system. The methodological aspect is reflected in the pair,

analytic-elementwise vs. holistic-structural

and the ontological (on, being+logos, logic) in the pair

Geometry vs. Logic

with "geometry the leading aspect". From this point of view, it seems natural that a model for the development of geometric thought and learning, should refer to all mathematics, since geometry constitutes the "content" and the leading aspect of mathematics. In fact, if one examines carefully the van Hiele model, can easily convince himself that the model essentially may coincides with the one implied by the basic dialectical system,and thus with the Piaget model as well.

Let us briefly describe the van Hiele model following [4]:

Level 0 (basic level): Visualization. At this level, students are aware

of space only as something that exist around them. Geometric concepts are viewed as total entities rather as having components or attributes. Geometric figures, for example,are recognized by their shape as a whole, that is, by their physical appearance,not by their parts or properties. A person functioning at this level can learn geometric vocabulary, can identify specified shapes, and given a figure, can reproduce it..."

Level 1 : Analysis. At level 1, an analysis of geometric concepts begins.

For example, through observation and experimentation students begin to discern the characteristics of figures. These emerging properties are then used to conceptualize classes of shapes. Thus figures are recognized as having parts and are recognized by their parts....Relationships between properties however, cannot yet be explained by students at this level, interrelationships between figures are still not seen, and definitions are not yet understood.

Level 2 : Informal Deduction. At this level, students can establish the interrelationships of properties both within figures... and among figures. Thus they can deduce properties of a figure and recognize classes of figures. Class inclusions is understood. Definitions are meaningful. Informal arguments can be followed and given. The student at this level, however, does not comprehend the significance of deduction as a whole or the role of axioms. Empirically obtained results are often used in conjunction with deduction techniques.Formal proofs can be followed, but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.

Level 3 : Deduction. At this level, the significance of deduction as a way of establishing geometric theory within an axiomatic system is understood. The interrelationships and the role of undefined terms, axioms, postulates, definitions, theorems and proof is seen. A person at this level can construct, not just memorize proofs; the possibility of developing a proof in more than one way is seen; the interaction of necessary and sufficient conditions is understood;distinctions between a statement and its converse can be made.

Level 4 : Rigor. At this stage the learner can work in a variety of axiomatic systems, that is, non-Euclidean geometries can be studied, and different systems can be compared.Geometry is seen in the abstract.

For more details about van Hiele levels see [4, 7, 24].

Next we shall make some remarks on the van Hiele levels: If one considers level 3 partly transitory partly belonging to level 4,then these levels can be incorporated into one level. Hence in this case we are left with four basic periods of developments. Therefore the van Hiele model, the Piaget's model and the model which is based on the basic dialectical system may be unified into one comprehensive model of development of mathematical thought and learning.We should like also to make the following remark: In the case of variable or fuzzy objects the attention should be concentrated more on the processes and actions rather than to objects. In this case, the categorical methods of the first Piaget's period, is more appropriate even for mathematically mature adults.

9. FINAL REMARKS

Every period of the development of mathematical thought is characterized by a dialectical scheme in which there is a primary or dominating part or a thesis. The other dialectical terms constitute the antithesis to this thesis. The unity of opposites or synthesis determines the state of the development. We may say that every period is characterized by a rough natural base topos which basically should be determined experimentally by measuring the different degrees with which each term participates into the basic dialectical system. We need to develop experimental techniques for measuring abilities like: analytic and logical processing holistic and structural determination of objects reasoning under fuzziness and variability, quantitative and qualitative reasoning etc. A mature person who have passed all the stages of intellectual development, may attain a balance of accommodation and assimilation to give a relatively stable intellectual equilibration.

Given the hypothesis that each period of intellectual development is characterized by a natural base topos, a "theory of instruction" would be based on the concept of functor, connecting the base topos of the teacher to an "average base topos " of the students. The research in mathematical education should be directed to explicate the connection of this theory with a possible experimental determination of its concepts. The first to mention category theory in connection with the van Hiele model was A. Hoffer [7]. However his suggestion was not in the direction of using categories to characterize the natural base topos.

Bibliography

[1] Bell, J. L. [1986] "From absolute to Local Mathematics". *Synthese*, **69**, 409-26.

[2] Bradshaw J.L. and N.C. Nettleton [1981] "The nature of hemispheric specialization in man". *The Behavioral and Brain Sciences* **7**, 51-91.

[3] Carnap.R.[1947]. Logic and necessity.Univ. of Chicago press.

[4] Crowley H.L [1987] The van Hiele model of the development of Geometric Thought. In, Learning and teaching geometry k-12. NCTM 1987, Year book, pp 1-11.

[5] Drossos C.A.[1988] Foundations of Nonstandard Mathematics: Infinitesimals, Boolean analysis and Categories. (Notes in Greek).

[6] Drossos,C.A. and G.Markakis [1987] Boolean Powers and Stochastic Spaces. (submitted for Publication).

[7] Hoffer,A. [1983] Van-Hiele Based Research. In: *Acquisition of Mathematics Concepts and Processes.* Academic press, 205 -227.

[8] Kock, A. [1981] *Synthetic Differential Geometry.* Cambridge.

[9] Lambek, J. [1981]. The influence of Heraclitus on modern Mathematics. In J.Agassi and R.S.Cohen (Ed). *Scientific philosophy today.* Reidel, 111-114

[10] Lawvere, F.W. [1969] Adjointness in foundations. *Dialectica* **23**, pp. 281-296.

[11] Lawvere, F.W. [1970] Quantifiers and sheaves. *Actes du Congres Internat., des mathematiciens*, Nice, pp.329-334.

[12] Lawvere, F.W. [1972]. "Introduction" to *Toposes, Algebraic Geometry and Logic*, SLMN # 274, pp 1-12.

[13] Lawvere, F.W. [1976] Variable quantities and variable structures in topoi. In: *Algebra, topology and category theory:a collection of papers in honor of S. Eilenberg*. Academic press, 101-131.

[14] Lawvere, F.W. [1987] *Algebraic concepts in the foundations of physics and engineering*. Lecture Notes (Buffalo).

[15] Lazerson, A.(Ed.) [1975] *Psychology Today. An Introduction*, 3rd Ed. Random House.

[16] Lewis, A. C. [1977] H. Grassmann's 1844 Ausdehnungslehre and Schleiermacher's Dialectic. *Annals of Science*, **34**, 103-62.

[17] Piaget,J. [1968] *Structuralism*, (trans. 1977) Basic Books, N.Y.

[18] Piaget,J. [1970] *Psychology and Epistemology*, (trans.1971) Grossman, N.Y.

[19] Piaget,J. [1975] *The Development of Thought: Equilibration of Cognitive Structures*, (trans.1977), Viking N.Y.

[20] Phillips, J. L. [1981] *Piaget's Theory: A Primer*. Freeman.

[21] Reyes, G. [1986] Synthetic Reasoning and Variable Sets. In *Categories in Continuum Physics*, edited by Lawvere and Schanuel. Springer LNM # 1174, 69-82.

[22] Skemp, R. [1971] *The Psychology of Learning Mathematics*, Penguin Books.

[23] Springer, S. P. and G. Deutch [1981] *Left Brain, Right Brain*, Freeman.

[24] Wirszup, I. [1976] Breakthroughts in the Psychology of learning and teaching geometry. In: Martin(Ed.) "*Space and Geometry: Papers from a research workshop*, Colombus Ohio ERIC/SMEAC.

[25] Wraith,G. [1975] *Lectures on Elementary Topoi*, Springer L.N.M. # 445, 114-206.

Department of Mathematics
University of Patras
GR 265 10 Patras, Greece