A NOTE ON
REPRESENTING AND INTERPRETING MV-ALGEBRAS

COSTAS A. DROSSOS AND PANAGIS KARAZERIS

Abstract. We try to make a distinction between the idea of representing and that of interpreting a mathematical structure. We present a slight generalization of Di Nola's Representation Theorem as to incorporate this point of view. Furthermore, we examine some preservation and functorial aspects of the Boolean power construction.

1. Introduction

There are mathematical objects which are inherently and naturally 'nonstandard'. Examples include, the Itô integral, the Schwartz distributions, etc. We can add to this list the MV-algebras of many-valued logic, since their representation theorem refers to the nonstandard interval $^*[0,1]$.

Nonstandard methods (infinitesimal, Boolean-valued, topos-theoretic) rely on the existence of at least two levels of viewing mathematical objects, and the reduction of higher type standard objects to lower type non-standard ones. For example, equivalence classes of sequences are reduced to real numbers in a model of infinitesimal analysis, self-adjoint operators are reduced to real numbers in a Boolean model of set theory, continuous functions to a space are reduced to spaces in the topos of sheaves over that space.

The existence of a huge amount of nonstandard models implies that the structure of MV-algebras (alongside with many other first-order structures) can have many nonstandard interpretations. Although the aim of this note is not such a study, we try to make clear the distinction between representation and interpretation but also to indicate the interplay between the two.

The Boolean power construction is an important tool for representations of first-order structures. Its importance for the study of many-valued logic is even bigger as the construction constitutes a functorial passage from the algebras of classical logic to those of many-valued logic. We examine here some preservation property of that construction (preservation of hyperarchimedianess) and discuss possible adjunctions in which this functor participates. We close with some questions arising from our discussion.

2. Representations and Interpretations

First we would like to state that the main difference between representation and interpretation is that the former is usually seen within a specific model, whereas interpretations involve two, usually different, universes. The idea of interpretation

1991 *Mathematics Subject Classification.* 06D35,03B50,03c20, 03C90.

Key words and phrases. Representation, interpretation, MV-algebras, hypercubes, nonstandard thinking.

obtains its most general form in category theory, where theories are represented by small categories with suitable structure and interpretations of them become structure-preserving functors into categories with the same structure.

Thus one may interpret the notion of an MV-algebra (like any equationally defined structure) in an arbitrary category with finite products (in particular in an arbitrary topos). The interplay with familiar representations of MV-algebras begins when restricting to interpretations in universes of Boolean-valued or Heyting-valued sets. Then a representation of the MV-algebra usually involves an embedding of it in the global sections of such a "set". For example the familiar ([20]) representation of hyperarchimedian MV-algebras as Boolean products of subalebras of $[0,1]$ is an instance of such an approach. When it comes to arbitrary MV-algebras the most general representation known is Di Nola's embedding of an algebra in a power of an ultrapower of $[0,1]$. Of course Di Nola's representation constitutes a form of a Boolean ultraproduct representation over a special form of Boolean algebra (a powerset). We indicate how this representation can be extended to a more general one over a, possibly, more interesting Boolean algebra.

2.1. **Boolean Ultrapowers and Di Nola's Representation Theorem.** The next theorem is the Boolean analogue to Frayne's Lemma ([16, Thm. 2.16, p. 42]), and it is the result needed for such a generalization:

Theorem 2.1. *For structures $\mathscr{A} := \langle A, ...\rangle, \mathscr{B} := \langle B, ...\rangle$, then $\mathscr{A} \equiv \mathscr{B}$ if and only if $\mathscr{B}$ is elementary embeddable in some Boolean ultrapower $\mathscr{A}_U^{(\mathbb{B})}$ of $\mathscr{A}$. The Boolean $\mathbb{B}$ can be chosen as a minimal completion of the boolean algebra $F(\mathscr{A}, A \cup B)/\approx Th(\mathscr{A}, A)$. The latter denotes the Boolean algebra of formulae in the language of $\mathscr{A}$ with constants from A and B, modulo the theory of $\mathscr{A}$.*
The ultrafilter U can be chosen so that as including $Th(\mathscr{B}, B)/\approx Th(\mathscr{A}, A)$.

(Recall here that Boolean ultrapowers $\mathscr{A}_U^{(\mathbb{B})}$ are just isomorphic to direct limits of ordinary ultrapowers.)

Using the above Theorem we give a proof of the generalization by reproducing the steps of a proof given by Di Nola ([3]):

Theorem 2.2 (**Boolean generalization of Di Nola's Representation theorem**). *For any MV-algebra A there is a Boolean ultrapower $[0,1]_U^{(\mathbb{B})}$ of the MV-algebra $[0,1]$ such that A can be embedded into $\left([0,1]_U^{(\mathbb{B})}\right)^{Spec(A)}$, i.e. there exists:*

$$i : A \hookrightarrow \left([0,1]_U^{(\mathbb{B})}\right)^{Spec(A)}$$

Proof: Let A be an MV-algebra. By Chang's Representation Theorem, A can be embedded into the direct product

$$\prod\{A/P \mid P \in \mathrm{Spec}(A)\}.$$

If $P \in \mathrm{Spec}(A)$ then A/P is an MV-chain, so A/P can be embedded into a divisible MV-chain D_p. Next, we consider the set $\mathcal{F} := \{D_P \mid P \in \mathrm{Spec}(A)\}$. It is also known that any pair of MV-algebras from $\mathcal{F}$ are elementarily equivalent. But we know that there exists an MV-algebra D such that D_P can be elementarily embedded in D for any $P \in \mathrm{Spec}(A)$. It follows that D is also elementarily equivalent with the MV-algebras of $\mathcal{F}$. But $[0,1]$ is also elementarily equivalent with the MV-algebras of $\mathcal{F}$. It follows that D is also elementarily equivalent with $\mathcal{F}$, since $[0,1]$ is a divisible

MV-chain. Thus, by the Boolean analog of Frayne's Lemma 2.1, D is elementarily embeddable in some Boolean ultrapower $[0,1]_U^{(\mathbb{B})}$. For any $P \in \mathrm{Spec}(A)$ we get

$$A/P \hookrightarrow D_P \hookrightarrow D \hookrightarrow [0,1]_U^{(\mathbb{B})}$$

$$i_P : A/P \to [0,1]_U^{(\mathbb{B})}$$

Hence, if we define

$$i : A \hookrightarrow \left([0,1]_U^{(\mathbb{B})}\right)^{\mathrm{Spec}(A)} \quad /\!\!/ \quad a \mapsto i(a) : \{i_P(a) : P \in \mathrm{Spec}(A)\},$$

we get the desired embedding for the MV-algebra A. ⊣

2.2. **Some further remarks on interpretations.** The following theorem ([9, p. 212]) exemplifies the feature of reduction of type, mentioned above.

Gordon's Theorem: *Every universally complete vector lattice is an interpretation of the reals in an appropriate Boolean-valued model.*

The Boolean-valued interpretation of the reals have been introduced by Takeuti ([17]). Choosing as the Boolean algebra a Boolean algebra of projections in a Hilbert space, we get as interpreted reals self-adjoint operators, whereas if we choose a probability algebra we get correspondingly random variables as reals (an idea originally due to D. Scott). In general, appropriate Boolean-valued interpretations of the real and complex numbers give operator algebras (see e.g., [13, 14, 15]).

model of set theory. Let also $V^{(\mathbb{B})}$ be the Boolean valued model of V with respect to $\mathbb{B}$. We define the embedding

$$\check{(\cdot)} : V \hookrightarrow V^{(\mathbb{B})} \quad /\!\!/ \quad a \mapsto \check{a}$$

where $\check{a} : \mathrm{dom}(a) \to \mathbb{B}$ is defined as follows:

$$\mathrm{dom}(a) := \{\check{x} \mid x \in a\} \quad \text{and} \quad \check{a}(\check{x}) = 1, \quad x \in a.$$

For $u \in V^{(\mathbb{B})}$ we define

$$\hat{u} := \{x \in V^{(\mathbb{B})} \mid [\![x \in u]\!] = 1\}.$$

The important property for Boolean powers with respect to a Boolean-valued model $V^{(\mathbb{B})}$ is that *the Boolean power $A^{(\mathbb{B})} \equiv A[\mathbb{B}]$ is isomorphic to* $(\check{A})\hat{}$. that "MV-

algebras are the non-commutative generalization of Boolean algebras". We would like to phrase this as: In the interpretation of Boolean and MV-algebras in the framework of C*-algebras, the later is a non-commutative generalization of the first.

Remark 2.3. The Theorem 2.2 holds true for *some* Boolean algebra. On the other hand the reals $\mathbb{R}$ can be interpreted in every Boolean-valued model as $\mathbb{R}^{(\mathbb{B})}$. More precisely, let $V^{(\mathbb{B})}$ be a Boolean-valued model such that the reals in it capture a complete vector lattice. Regarding the $\mathbb{B}$-reals as a $\mathbb{B}$-lattice order group we obtain a $\mathbb{B}$-interpretation of the MV-algebra $[0,1]$, by applying the gamma-functor inside $V^{(\mathbb{B})}$, i.e.

$$[\![\Gamma(\mathbb{R},1) = [0,1]]\!] = 1_{\mathbb{B}}.$$

Varying the Boolean algebra $\mathbb{B}$ appropriately, we essentially get Boolean valued interpretations-representations of the MV-algebra $[0,1]$. In each $V^{(\mathbb{B})}$ the interpreted MV-algebra $[0,1]^{(\mathbb{B})}$ $\mathbb{B}-$generates the variety of MV-algebras in $V^{(\mathbb{B})}$. This shows that by varying the Boolean algebra $\mathbb{B}$ we get various interpretations of the structure of MV-algebra. In this sense the above Boolean generalization of Di Nola's Representation theorem, holds for every Boolean algebra. Thus choosing appropriately the Boolean algebra $\mathbb{B}$ we may have various $\mathbb{B}$-representations of MV-algebras. For example choosing $\mathbb{B}$ to be a probability Boolean algebra we get $\mathbb{B}$-representations of MV-algebras in terms of stochastic processes $(f_t)_{t\in\mathrm{Spec}(A)}$, where f_t is a point free stochastic fuzzy set. Similarly, by choosing $\mathbb{B}$ as a projection Boolean algebra in a Hilbert space, we get a $\mathbb{B}$-representations of MV-algebras in terms of families of adjoint operators indexed by $\mathrm{Spec}(A)$, (see also [17, 18, 13, 14, 15]).

The above Boolean Representation Theorem initiates the study of the exact relationships between MV-algebras and C*-algebras, see also ([18, 7]). Another interesting result, connected with this, is the following, ([19, p. 18]):

Proposition. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If ${}^*(\cdot) : V(\mathbb{R}) \hookrightarrow V({}^*\mathbb{R})$ is the bounded ultrapower embedding over $\mathcal{U}$, then the C^*-algebras $S(\mathfrak{A})$ for $\mathfrak{A}$ hyperfinite dimensional internal C^*-algebra are exactly the ultraproducts over $\mathcal{U}$ of finite dimensional C^*-algebras. ($S(\mathfrak{A})$ is the infinitesimal hull of $\mathfrak{A}$.)*

We first state some known results for Boolean powers. By Boolean power we mean constantly "bounded Boolean power" (otherwise most of what follows holds for unbounded Boolean powers over complete seperable Boolean algebras).

Theorem 2.4. ([1])

(i) $\mathfrak{A}[\mathbb{2}] \cong \mathfrak{A}$;

(ii) *For any diagram $D\colon I \to$ **BAlg** in the category of Boolean algebras with limit $\lim_i B_i$, $\mathfrak{A}[\lim_i B_i] \cong \lim_{i\in I}\mathfrak{A}[\mathbb{B}_i]$. In particular,*

(iii) $\mathfrak{A}[\prod_{i\in I}\mathbb{B}_i] \cong \prod_{i\in I}\mathfrak{A}[\mathbb{B}_i]$.

Theorem 2.5. ([8, Th. 9.7.2 (c), p. 468]) *Let $\mathfrak{A}$ be a non-empty $\mathscr{L}$-structure. Then the Boolean power $\mathfrak{A}[-]$ defines a functor*

$$\mathfrak{A}[-] : \mathbf{BAlg} \to \mathscr{M},$$

where $\mathscr{M}$ is the category of $\mathscr{L}$-structures and homomorphisms; this functor preserves filtered colimits.

We also have:

Theorem 2.6. ([4])

(i) *For every MV-algebra A, the Boolean power $A[\mathbb{B}]$ is an MV-algebra;*

(ii) *Given a Boolean algebra $\mathbb{B}$, the center of the MV-algebra $[0,1][\mathbb{B}]$ is isomorphic to $\mathbb{B}$.*

(iii) *For every semisimple MV-algebra A, and every Boolean algebra $\mathbb{B}$, $A[\mathbb{B}]$ is semisimple.*

Recall that a hyperarchimedian MV-algebra is an MV-algebra in which every element x has a power x^n which is idempotent. In particular $[0,1]$ is a hyperarchimedian MV-algebra.

Theorem 2.7. *Let M be a hyperarchmedian MV-algebra. Then the Boolean power $M[B]$ is hyperarchimedian as well.*

Proof: For an $f \in M[B]$ one has

$$(f \odot f)(a) = \bigvee \{f(x) \wedge f(y) \mid x \odot y = a\}$$

and since f is a partition the above becomes

$$(f \odot f)(a) = \bigvee \{f(x) \mid x \odot x = a\},$$

so eventually

$$f^n(a) = \bigvee \{f(x) \wedge f(y) \mid x^n = a\}$$

Suppose that f takes m distinct non-zero values $f(x_1), ..., f(x_m)$ and (since M is hyperarchimedian) that $x_1^{n_1}, ..., x_m^{n_m}$ are idempotents.

Then

$$\begin{aligned} (f^{n_1 \cdots n_m} \odot f^{n_1 \cdots n_m})(a) &= \bigvee \{f(x) \mid x^{n_1 \cdots n_m} \odot x^{n_1 \cdots n_m} = a\} \\ &= \bigvee \{f(x) \mid x^{n_1 \cdots n_m} = a\} \\ &= f^{n_1 \cdots n_m}(a) \end{aligned}$$

so whenever the function $f^{n_1 \cdots n_m}$ takes non-zero values they have to be idempotent. This means that $f^{n_1 \cdots n_m}$ is itself an idempotent element of $M[B]$. ⊣

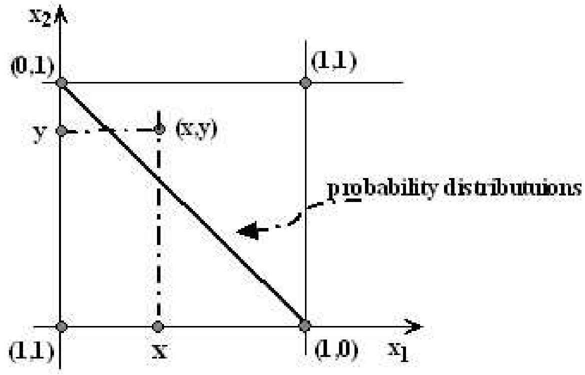
Corollary 2.8. *The Boolean power $[0,1][B]^*$ is a hypearchimedian MV-algebra.*

Specializing Theorem 3.2, we know that for a fixed MV-algebra M the Boolean power construction is functorial, defining a functor $M[-]\colon \mathbf{BAlg} \to \mathbf{MVAlg}$, from the category of Boolean algebras to that of MV-algebras. On the other hand the center of an MV-algebra also depends functorially on the MV-algebra, thus defining a functor $C(-)\colon \mathbf{MVAlg} \to \mathbf{BAlg}$. As pointed out in Theorem 3.3(ii), for any Boolean algebra B, the center of the MV-algebra $[0,1][B]$ is isomorphic to B. Thus any MV-homomorphism $M \to [0,1][B]$ determines a Boolean homomorphism $C(M) \to B$ and this fact indicates a possibility to investigate an adjunction between the Boolean power functor $[0,1][-]$ and the center functor $C(-)$. The other direction of a correspondence though wouldn't work: One may not expect to extend a Boolean homomorphism $C(M) \to B$ to an MV-homomorphism $M \to [0,1][B]$. This can be made precise with the aid of Gluschankof's characterization of algebras $\mathscr{C}(X, [0,1]_d)$ of continuous functions from a Stone space to the unit interval equipped with its discrete topology (i.e Boolean powers of $[0,1]$). In particular ([6], Cor. 3.5) such an MV-algebra is injective iff X is finite. Now $C(-)$ preserves monomorphisms (as a matter of fact all limits, being right adjoint to the inclusion of Boolean algebras into MV-algebras) so if it were left adjoint to $[0,1][-]$, by a standard categorical argument, $[0,1][-]$ would have to preserve injective objects. Since there are more injective Boolean algebras than just the finite ones we conclude that there is no such adjunction:

Theorem 2.9. *The Boolean power construction $[0,1][-]\colon \mathbf{BAlg} \to \mathbf{MVAlg}$ is not a right adjoint to the center construction $C(-)\colon \mathbf{MVAlg} \to \mathbf{BAlg}$.*

3. Final Remarks

It is well known that there is a connection between fuzzy sets and semisimple MV-algebras. On the other hand cubes of the form $[0,1]^n$ play a vital role in both fuzzy sets and MV-algebras. We consider a simple example: If $X := \{x_1, x_2\}$ then the following square depicts classical and fuzzy sets:



The classical subsets of $\{x_1, x_2\}$ are $\{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$ and these are represented using indicator functions as:

$$(I_{\{x_2\}}(x_1), I_{\{x_2\}}(x_2)) = (0,1) \qquad (I_{\{x_1,x_2\}}(x_1) I_{\{x_1,x_2\}}(x_2)) = (1,1)$$

$$(I_{\emptyset}(x_1), I_{\emptyset}(x_2)) = (0,0) \qquad (I_{\{x_1\}}(x_1) I_{\{x_1\}}(x_2)) = (1,0)$$

Fuzzy sets on the diagonal $(0,1),(1,0)$ represent probability distributions. All other points represent fuzzy sets.

It appears that *classical sets are the extreme points of non classical sets, or that non classical sets constitute the convex hull of classical sets.*

It would be desirable to have a formal way that expresses this relationship, that is a concrete connection between the two worlds: Boolean algebras and classical logic and MV-algebras and many-valued logic. One such way is through Boolean powers.

First we notice that semisimple MV-algebras are represented using $[0,1]$-valued functions, whereas Boolean algebras are represented using $\mathbb{2} := \{0,1\}$-valued functions.

Using 2.4 we see that, $[0,1][\mathbb{2}] = [0,1]$ and $[0,1][\mathbb{2}^n] = [0,1]^n, \quad n \in \mathbb{N}$. In this way the Boolean power functor $[0,1][-]$ indeed gives the convex hull of the corresponding Cantor cubes. It would be interesting to investigate the passage from Boolean functions to McNaughton functions but unfortunately the Boolean power functor doesn't help in this direction.

In addition one can prove that the sets 2^ω and $[0,1]$ are isomorphic by choosing in $[0,1]$ the non-terminating expansions. For a set with the power of continuum we may represent its subsets by using hyperfinite cubes 2^H, where H is a hyperfinite natural number. The nonstandard study of hypercubes and MV-algebras seems to be promising in finding an equivalent formulation for MV-algebra problems, avoiding the heavy algebraic geometric formulations.

References

1. S. Burris, Boolean powers, *Algebra Univ.*, **5** (1975), No. 2, 341-360.
2. R. O. I. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., (2000), Dordrecht, Boston, London.
3. A. Di Nola, Representation and Reticulation by quotients of MV-algebras. *Ricerche di Matematica*, 40, (1991), pp. 291-297, and personal communication.
4. C. A. Drossos and P. Karazeris, Coupling an MV-algbera with a Boolean Algebra, *Internat. J. Approx. Reason.* 18 (1998), 231-238.
5. Joan Gispert and Daniele Mundici, "MV-algebras: a variety for magnitudes with archimedean units", *Algebra univers.* 53 (2005), 743
6. D. Gluschankof, Prime deductive systems and injective objects in the algebras of Lukasiewicz infinite valued logic, *Algebra Universalis*, 29 (1992), 354–377.
7. C. W. Henson and J. Iovino, Ultraproducts in Analysis. In: C. Ward Henson, José Iovino, Alexander S. Kechris, Edward Odell (Eds) *Analysis and Logic* Cambridge University Press, 2003, pp. 1-114.
8. W. Hodges, *Model Theory.* Cambridge, Univ. Press, 1993.
9. A. G. Kusraev and S. S. Kutateladze, *Boolean Valued Analysis*, Kluwer Acad. Publ., (1999) Dordrecht, Boston, London.
10. Lawvere, F.W., Categories of Space and of Quantity.In: J. Echeveria, et al (Eds) *The Space of Mathematics: Philosophical, Epistemological and Historical Explorations.* (International Symposium on Structures in Mathematical Theories. San Sebastian, Spain 1990), De- Gruyter, Berlin 1992, 14-30.
11. R. Mansfield, The theory of Boolean ultrapowers, *Ann. Math. Logic* **2** (1971) 297-323.
12. D. Mundici. Interpretation of AF C*-algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, 65:15-63, 1986.
13. M. Ozawa, Boolean valued interpretation of Hilbert space theory, *J. Math. Soc. Japan*, 35 (1983), 609-627.
14. M. Ozawa, Boolean valued analysis and type I AW*-algebras, *Proc. Japan Acad.*, 59A (1983), 368-371.
15. M. Ozawa, Boolean valued interpretation of Banach space theory and module structures of von Neumann algebras, *Nagoya Math. J.* **117** (1990) 1-36.
16. K. Potthoff, "Boolean Ultrapowers", *Arch. math. Logik* 16 (1974), 37-48.
17. G. Takeuti, *Two Applications of Logic to Mathematics,* Princeton University Press, Princeton, 1978. (MR0505474)
18. G. Takeuti, C*-Algebras and Boolean valued analysis, *Japan, J. Math.* **9** (1983) 207-245.
19. FJ Thayer, "Quasidiagonal C^*-algebras and Nonstandard Analysis"-*Arxiv preprint math.OA/0209292*, 2002 - arxiv.org
20. A. Torrens, W-algebras which are Boolean products of members of $SR[1]$ and CW-algebras, *Studia Logica* 46 (1987), 265–274.

Department of Mathematics, University of Patras, GR-26500 Patras, Greece
E-mail address: cdrossos@{math.upatras.gr,upatras.gr,otenet.gr}

Department of Mathematics, University of Patras, GR-26500 Patras, Greece
E-mail address: pkarazer@upatras.gr