ALGEBRAIC STRUCTURES OF GENERALIZED MANY-VALUED LOGICS

(SEMINAR NOTES)

Costas A. Drossos
Department of Mathematics
University of Patras
GR-26110 Patras, GREECE

First Draft

(These are notes from a seminar given in the
Department of Mathematics, University of Patras 1996–97)

CONTENTS

1 POSETS **1**

1.1 Preordered Sets 1

1.2 Order Ideals and Filters 4

1.3 Mappings Between Posets 5

2 ADJOINTS FOR PREORDERS **11**

2.1 Adjoints 11

2.2 Triangle Inequalities
and Basic Properties of Adjunctions. 13

2.3 Adjoint Functor Theorem 14

2.4 Closure operators (monads) on preorders 18

2.5 The Adjoint Lifting Theorem. 24

3 cirl-MONOIDS **27**

3.1 Introduction 27

3.2 The Qualitative Case: Heyting Algebras. 29

3.3 The Quantitative Case:cirl-monoids 30

3.4 Monoid actions. 33

3.5 Integrality. 43

3.6 Algebraic strong de Morgan's law 44
3.7 Divisibility. 48
3.8 Involutive Negation and Girard monoids. 53
3.9 Square roots. 55

CHAPTER 1

POSETS

1.1 Preordered Sets

In these notes we shall need many implication symbols. To fix notation, we shall use

- '$\Rightarrow$' for 'implies';
- '$\Leftrightarrow$' for 'if and only if';
- '$=\!\wedge\!\Rightarrow$' for qualitative (lattice) implication;
- '$\Leftarrow\!\vee\!=$' for qualitative (lattice) co- implication;
- '$=\!\odot\!\Rightarrow$' for quantitative (monoidal) implication.
- '$\Leftarrow\!\oplus\!=$' for quantitative (monoidal) co-implication.
- '$\longrightarrow$', for function arrow.

1.1.1 Definition. A **preordered set** is a pair $\langle P, \leq \rangle$, consisting of a set P and a binary relation "$\leq$" on P such that:

(i) **Reflexivity:** $(\forall p \in P)[p \leq p]$

(ii) **Transitivity:** $(\forall p, q, r \in P)[(p \leq q) \wedge (q \leq r) \ \Rightarrow \ (p \leq r)]$

A preordered set it is not required to satisfy the antisymmetric property:

(iii) **Antisymmetric:** $(\forall p, q \in P)[(p \leq q) \wedge (q \leq p) \ \Rightarrow \ (p = q)]$

If (iii) is satisfied by P then it is called **a poset**.

Two elements $p, q \in P$ are said to be comparable if $p \leq q$ or $q \leq p$ (dichotomy property), otherwise we say that p and q are incomparable, denoted by $p \parallel q$. A **totally ordered set** or a **chain** is a poset in which every two elements are comparable.

Often we shall present the properties of a poset in a Gentzen-like style:

$$\frac{}{x \leq x} \quad \text{(reflexivity)}$$

$$\frac{x \leq y \quad y \leq z}{x \leq z} \quad \text{(transitivity)}$$

$$\frac{x \leq y \quad y \leq x}{x = y} \quad \text{(antisymmetric)}$$

One line means that the statements above the line imply the statement below, two lines mean that the above and below are logically equivalent.

1.1.2 Remark. Given any preordered set P, there is a naturally associated poset, denoted by $P/\equiv$ obtained from P, as a quotient structure, where "$\equiv$" is defined as

$$(\forall p, q \in P)[(p \equiv q) \quad \text{iff} \quad (p \leq q) \wedge (q \leq p)].$$

Similarly for the order on $P/\equiv$. Because of the above and the fact that in Category Theory "equality" means "isomorphic", preordered sets are more naturally adapted to a categorical treatment of order.

1.1.3 Definition. 1) Given a poset $\mathbb{P} = \langle P, \leq \rangle$ we may define the **dual** of **opposite** of $\mathbb{P}$, denoted by $\mathbb{P}^{op} = \langle P, \geq \rangle$.

The **duality principle** for posets allows us to dualize any theorem by interchanging $\leq$ with $\geq$.

2) The **direct product** of a family of posets $\langle P_i, \leq_i \rangle_{i \in I}$ is a poset $\langle \prod_{i \in I} P_i, \leq \rangle$, where "$\leq$" is defined componentwise, i.e. $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ iff $x_i \leq_i y_i$ for all $i \in I$.

If the index set I is also endowed with a total order, then the **lexicographic product** of the above family is a poset $\langle \prod_{i \in I} P_i, \leq_{\text{lex}} \rangle$ where "$\leq_{\text{lex}}$" is defined as

$$(x_i)_{i \in I} \leq_{\text{lex}} (y_i)_{i \in I} \quad \text{iff} \quad (\exists i \in I)[(x_i \leq y_i) \wedge (\forall k < i)[x_k = y_k]]$$

If in addition to the above, we have that $P_i \cap P_j = \emptyset$, whenever $i \neq j$, then we may define the **ordinal sum** of the above family, as: $\langle \bigcup_{i \in I} P_i, \leq \rangle$, where "$\leq$" is defined as

$$x \leq y \quad \text{iff} \quad (\exists i \in I)[x \leq_i y] \quad \text{or} \quad (\exists i, j \in I)[i < j \quad \textit{and} \quad x \in P_i, \quad \& \quad y \in P_j].$$

1.1.4 Definition. **(i)** Let $\langle P, \leq \rangle$ be a poset, and $S \subseteq P$. Then an element $u \in P$ is called an **upper bound of** S iff for every $p \in S$, we have $p \leq u$. Dually we may define lower bound of S.

(ii) A poset $\langle P, \leq \rangle$ is called **directed upwards** iff every pair of elements $p, q \in P$ has an upper bound. Similarly, $\langle P, \leq \rangle$ is called **directed downwards** iff every pair of elements $p, q \in P$ has a lower bound. If $\langle P, \leq \rangle$ is directed above and below it is called simply a **directed set**.

1.1.5 Definition. Let $\langle P, \leq \rangle$ be a poset. If for every pair of elements $p, q \in P$, the set of upper bounds of $\{p, q\}$ has a least element r, then r is called the **least upper bound** (join) of p and q and is denoted as $r = p \vee q$. The **greatest lower bound** (meet), denoted $p \wedge q$ is defined dually.

More precisely, let us denote by $S^u, S \subseteq P$, the set of all upper bounds of S (upper cone), and S^ℓ the set of all lower bounds (lower cone), i.e.

$$\begin{aligned} S^u &:= \{p \in P | (\forall s \in S)[s \leq p]\} \qquad \text{and} \\ S^\ell &:= \{p \in P | (\forall s \in S)[s \geq p]\} \end{aligned}$$

Now we define:

$$\begin{aligned} \sup S &\equiv \bigvee S \equiv \bigvee_{s \in S} s := \inf S^u \\ \inf S &\equiv \bigwedge S \equiv \bigwedge_{s \in S} s := \sup S^\ell. \end{aligned}$$

In particular we have:

$$\bigvee \emptyset \equiv \bigvee_{p \in \emptyset} p = 0 \qquad \text{and} \qquad \bigwedge \emptyset \equiv \bigwedge_{p \in \emptyset} p = 1$$

the bottom and top elements if they exist.

If for every pair of elements $p, q \in P$, the joint $p \vee q$ exists in P then $\langle P, \leq \rangle$ is called an **upper semilattice**. Dually, if $p \wedge q$ exists for every pair $p, q \in P$, then $\langle P, \leq \rangle$ is called a **lower semilattice**. If $\langle P, \leq \rangle$ is both an upper and a lower semilattice, then it is called a **lattice**. Compaire with the concept of 'directed set'.

1.1.6 Example. Posets.

1. Any set X with the **discrete** order: $x \leq y$ iff $x = y$;
2. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ with the usual order;
3. $\mathbf{2} \equiv \langle 2, \leq \rangle$ where $2 = \{0, 1\}$ and $0 \leq 1$;

4. if $\langle P, \leq \rangle$ is a poset and X an arbitrary set, then the set $P^X := \{f \mid f\colon X \longrightarrow P\}$, endowed with the **pointwise order**,

$$f \leq g \quad \text{iff} \quad (\forall x \in X)[f(x) \leq g(x)]$$

is a poset.

5. For any set X, $\langle P(X), \subseteq \rangle$ is a poset. If $\tau \subseteq P(X)$ is a topology on X, then $\langle \tau, \subseteq \rangle$ with the restriction order is also a poset.

6. The **specialization order** between points in a topological space,

$$x \leq y \quad \text{iff} \quad (\forall G \in \tau)\,[x \in G \Rightarrow y \in G]$$

is in ceneral a preorder. The specialization order is antisymmetric iff τ is a T_0 space. If τ is a T_1 then the specialization order is a discrete order.

7. Formulae of a propositional logic form a preorder under provability $\vdash$.

8. Let $G = \langle G_0, G_1 \rangle$ be an oriented multigraph with loops, see [1]. A subgraph $X = \langle X_0, X_1 \rangle$ is defined as: $X_0 \subseteq G_0, X_1 \subseteq G_1$ such that for every edge in X_1, its source and target are in X_0, i.e.

$$P(G) := \{X = \langle X_0, X_1 \rangle \mid X_0 \subseteq G_0, X_1 \subseteq G_1 \text{ such that}$$

$$(\forall f \in X_1)[\mathrm{dom(f)}, \mathrm{cod(f)} \in \mathrm{X_0}]\}.$$

We now define on $P(G)$ a partial order by: For $X, Y \in P(G)$,

$$X \subseteq Y \quad \text{iff} \quad X_0 \subseteq Y_0 \quad \text{and} \quad X_1 \subseteq Y_1.$$

Then $\langle P(G), \subseteq \rangle$ is a poset.

1.2 Order Ideals and Filters

Let $\langle P, \leq \rangle$ be a poset and $Q \subseteq P$.

1.2.1 Definition. (i) Q is called an **order ideal** (or o-ideal or a **down-set**) iff

$$(\forall p)(\forall q)[(p \in Q) \wedge (q \leq p) \Rightarrow (q \in Q)].$$

(ii) Dually, Q is called an **order filter** (or o-filter or an **up-set**) iff

$$(\forall p)(\forall q)[(p \in Q) \wedge (q \geq p) \Rightarrow (q \in Q)].$$

1.2.2 Proposition. *Q is a down-set iff $P \setminus Q$ is an up-set.*

Given an arbitrary subset Q of P and $x \in P$, we define:

$$\begin{array}{rcll} \downarrow Q & := & \{y \in P \mid (\exists x \in Q)[y \leq x]\} & \text{(down } Q\text{)} \\ \uparrow Q & := & \{y \in P \mid (\exists x \in Q)[y \geq x]\} & \text{(up } Q\text{)} \\ \downarrow x \equiv \downarrow \{x\} & := & \{y \in P \mid y \leq x\} & \text{(Principal order-ideal)} \\ \uparrow x \equiv \uparrow \{x\} & := & \{y \in P \mid y \geq x\} & \text{(Principal order-filter)} \end{array}$$

1.2.3 Remark. A preorder $\langle P, \leq \rangle$ can be considered as a category $C(P, \leq)$:

- The objects of $C(P, \leq)$ are the elements of P
- If $x, y \in P$ and $x \leq y$ then $C(P, \leq)$ has *exactly one* arrow from x to y denoted, $x \longrightarrow y$ of (x, y). Note that dom(x $\longrightarrow$ y) = x, cod(x $\longrightarrow$ y) = y.
- If $x \parallel y$ (x, y are not comparable) there is no arrow from x to y.
- The identity arrows of $C(P, \leq)$ are those of the form (x, x) or $x \longrightarrow x$. The transitive property of "$\leq$" is needed to ensure the existence of composition of two arrows, so that:

$$\frac{x \longrightarrow y \quad y \longrightarrow z}{x \longrightarrow z} \qquad \text{or} \qquad (x \longrightarrow y) \circ (y \longrightarrow z) = (x \longrightarrow z).$$

Slogan. *Think of a category as a generalized preordered set.*

We shall take the above slogan seriously and try to see what is the special form which category concepts take in the enviroment of preordered sets. Thus, a down-set, $\downarrow x = \{y \in P \mid y \longrightarrow x\}$ is all the arrows $y \longrightarrow x$, for $y \in P$, which is what is called a slice category $P{\downarrow}x$. Sometimes, when we would like to emphasize a categorical flavor we use the notation $P{\downarrow}x$ instead of $\downarrow x$. Similarly an upper set $\uparrow x$ can be regarded as the category $P{\uparrow}x$ of all arrows $x \longrightarrow y$, $y \in P$. The above notations are extended to the case of $\downarrow Q$ and $\uparrow Q$, i.e. $P{\downarrow}Q$ is the up set generated by Q, similarly for $P{\uparrow}Q$.

1.3 Mappings Between Posets

What we say for posets holds also for preorders but equality "=" should be changed to equivalence "$\approx$".

1.3.1 Definition. Let $\mathbb{P} = \langle P, \leq_P \rangle, \mathbb{Q} = \langle Q, \leq_Q \rangle$, be two posets and $f : P \longrightarrow Q$ be a function. Then we say that f **respects order** or it is **isotone** iff $p_1 \leq_P p_2$ implies $f(p_1) \leq_Q f(p_2)$.

In categorical terms, an isotone function is essentially a covariant functor.

An isotone function $f\colon P^{op} \longrightarrow Q$ is called an **antitone function** (a contravariant functor).

The function $f\colon P \longrightarrow Q$ preserves suprema (infima), when they exist iff for all $S \subseteq P$,

$$f(\bigvee S) = \bigvee f[S] \qquad (f(\bigwedge S) = \bigwedge f[S]).$$

1.3.2 Remark. Every isotone function $f\colon P \longrightarrow Q$ satisfies always the following inequalities:

$$f(\bigwedge S) \leq_Q \bigwedge_{s \in S} f(s) \leq_Q \bigvee_{s \in S} f(s) \leq_Q f(\bigvee S).$$

Thus, in order to prove that f preserves suprema, it is enough to show that:

$$f(\bigvee S) \leq \bigvee_{s \in S} f(s)$$

and similarly, to show that f preserves infima, it is enough to prove that:

$$\bigwedge_{s \in S} f(s) \leq f(\bigwedge S).$$

Let $\langle P, \leq_P \rangle, \langle Q, \leq_Q \rangle$ be two posets, then we denote by $Q^{\langle P \rangle}$ the set of all isotone functions $f\colon P \longrightarrow Q$ and $Q^{\langle P^{op} \rangle}$ the set of all antitone functions $f\colon P \longrightarrow Q$. In particular we consider $\langle Q, \leq_Q \rangle \equiv \langle 2, \leq \rangle$ the poset with $2 = \{0, 1\}$ and $0 \leq 1$. Every function $f\colon P \longrightarrow 2$ is a property or predicate and at the same time the indicator or characteristic function of a subset $S \subseteq P$, defined as: $S_f := \{x \in P \mid f(x) = 1\}$.

1.3.3 Proposition. **(i)** *$f \in 2^{\langle P \rangle}$ iff S_f is an up-set.*

(ii) *$f \in 2^{\langle P^{op} \rangle}$ iff S_f is a down-set.*

PROOF: **(i):** Let $f \in 2^{\langle P \rangle}$, i.e. $x \leq y \Rightarrow f(x) \leq f(y)$. Let also $S_f := \{x \in P \mid f(x) = 1\}$, $x \in S_f$ and $x \leq y$. We should prove that $y \in S_f$. Indeed, since $x \in S_f$ we have $f(x) = 1$, and since $x \leq y$ we have $f(x) \leq f(y)$ so that $f(y) = 1$ and thus $y \in S_f$, i.e. S_f is an up-set.

Conversely, let S_f be an up-set and let $x \leq y$. Consider $f(x)$ and $f(y)$. If $f(x) = 0$ then $f(x) \leq f(y)$. On the other hand if $f(x) = 1$ then $x \in S_f$ and since S_f is an up-set then $y \in S_f$, thus $f(x) \leq f(y)$. ⊣□

(ii): is proved similarly. ⊣■

1.3.4 Remark. 1) It is clear that each element $f \in 2^{\langle P^{op} \rangle}$ determines a unique down-set and conversely.

If we denote by $\mathcal{O}(P)$ the set of all down sets then we have:

$$\mathcal{O}(P) \cong 2^{\langle P^{op} \rangle}$$

The following Lemma is very helpfull in proving inequalities and equalities.

1.3.5 Lemma. *Let $\langle P, \leq \rangle$ be a poset and $x, y \in P$. Then the following are equivalent:*

(i) $x \leq y$

(ii) $\downarrow x \subseteq \downarrow y$ *or* $\uparrow y \subseteq \uparrow x$, *i.e.*

$$(\forall p \in P)\ [p \leq x \ \Rightarrow \ p \leq y] \qquad \text{or} \quad (\forall p \in P)\ [p \geq y \ \Rightarrow \ p \geq x]$$

(iii) $(\forall D \in \mathcal{O}(P))\ [y \in D \ \Rightarrow \ x \in D]$, *where* $\mathcal{O}(P)$ *is the set of all down sets of* P, *ordered by inclusion.*

PROOF: **(i) $\Rightarrow$ (ii):** Let $x \leq y$ and $a \in \downarrow x$ then $a \leq x$ by definition and so $a \leq y$, thus $a \in \downarrow y$. ⊣□

(ii) $\Rightarrow$ (i): Let $\downarrow x \subseteq \downarrow y$. Then for all a, $a \in \downarrow x \ \Rightarrow \ a \in \downarrow y$, i.e. $a \leq x \ \Rightarrow \ a \leq y$ and for $a = x$ we get $x \leq x \ \Rightarrow \ x \leq y$. ⊣□

(ii) $\Rightarrow$ (iii): Let $\downarrow x \subseteq \downarrow y$ and $D \in \mathcal{O}(P)$ with $y \in D$. Since D is a down-set and $x \leq y$ then $x \in D$. ⊣□

(iii) $\Rightarrow$ (ii): Let $D = \downarrow y$ then $y \in D$ and by (iii), $x \in D$, i.e. $x \in \downarrow y$ i.e. $x \leq y$ or $\downarrow x \subseteq \downarrow y$. ⊣∎

1.3.6 Corollary. *The following are equivalent:*

(i) $x = y$

(ii) $\downarrow x = \downarrow y$, *i.e.* $(\forall p \in P)\ [p \leq x \quad \text{iff} \quad p \leq y]$

(iii) $\uparrow x = \uparrow y$, *i.e.* $(\forall p \in P)\ [p \geq x \text{ iff } p \geq y]$

Thus in order to prove that $x = y$ it is enough to prove either (ii) or (iii). Lemma 1.3.5,Corollary 1.3.6 are very helpful in proving inequalities and equalities and especially in proving adjuctions in Chapter 2.

1.3.7 Theorem. *If* P, Q *are posets and* $f \colon P \longrightarrow Q$ *is any mapping, then the following conditions are equivalent:*

(i) f *is isotone.*

(ii) For all $D \in \mathcal{O}(Q)$, $f^{-1}[D] \in \mathcal{O}(P)$, *i.e. for all* $q \in Q$, $f^{-1}[\downarrow q]$ *is either empty or a non-empty down-set.*

(iii) For all $U \in \mathcal{O}(Q^{op})$, $f^{-1}[U] \in \mathcal{O}(P^{op})$, i.e. for all $q \in Q$, $f^{-1}[\uparrow q]$ is either empty or a non-empty up-set of P.

Proof: **(i) $\Rightarrow$ (ii):** Suppose (i). Then if $f^{-1}[\downarrow q] = \emptyset$ there is nothing to prove. So let $f^{-1}[\downarrow q] \neq \emptyset$, $y \in f^{-1}[\downarrow q]$ and $z \leq y$. We have from (i) and $z \leq y$, $f(z) \leq f(y) \leq q$, so that $z \in f^{-1}[\downarrow q]$. This shows that (i) $\Rightarrow$ (ii). ⊣□

Conversely, suppose that f satisfies (ii). For each $p \in P$, we have trivially, $f(p) \leq f(p)$ and so $p \in f^{-1}[\downarrow f(p)]$. Since by (ii), $f^{-1}[\downarrow f(y)]$ is a down-set of P, it follows that $x \leq y \Rightarrow x \in f^{-1}[\downarrow f(y)] \Rightarrow f(x) \leq f(y)$. ⊣□

A dual argument yields (i) $\Rightarrow$ (iii). ⊣▮

Define the functor:

$$P{\downarrow}(-)\colon P \longrightarrow \mathcal{O}(P) \cong 2^{\langle P^{op} \rangle} \quad /\!\!/ \quad x \mapsto P{\downarrow}x \equiv \ {\downarrow}x$$

then we have the *Yoneda Lemma for posets*:

1.3.8 Theorem. *(Yoneda Lemma). The embedding $P{\downarrow}(-)$ satisfies the following:*

$$(P{\downarrow}x) \subseteq (P{\downarrow}y) \quad \textit{iff} \quad x \leq y.$$

Proof: This is just Lemma 1.3.5. ⊣▮

The above theorem says that $P{\downarrow}(-)$ is isotone and full.

1.3.9 Proposition. *For any down-set $D \subseteq P$, we have:*

$$P{\downarrow}x \subseteq D \quad \textit{iff} \quad x \in D.$$

Proof: Easy. ⊣▮

We know that the Yoneda Lemma is extremely important in representing structures and objects in general, using **generalized elements** (*in going* to the object arrows) or **generalized properties** (*out going* to the object arrows). In the case of posets and preorders we have a similar situation.

(i) **Covariant or extensional representation:** Every element $x \in P$ can be represented as the down-set $P{\downarrow}x$, which is just the *in going* to x arrows. Similarly $x \leq y$ can be represented as an inclusion set operation: $(P{\downarrow}x) \subseteq (P{\downarrow}y)$.

(ii) **Contravariant or intensional representation:** Every element $x \in P$ can be represented as the up-set $P{\uparrow}x$, which is just the *out going* from x arrows. The order $x \leq y$ can be represented as the inclusion $(P{\uparrow}y) \subseteq (P{\uparrow}x)$.

$\mathcal{O}(P) \cong 2^{\langle P^{op} \rangle}$ can be considered as a completion of P in the following sense:

1.3.10 Theorem. *(i) For every ordered set $\langle P, \leq_P \rangle$, the poset $\langle \mathcal{O}(P), \subseteq \rangle$ is a complete lattice.*

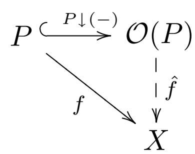
(ii) The embedding,

$$P{\downarrow}(-) \colon P \hookrightarrow \mathcal{O}(P) \;\; /\!\!/ \;\; p \mapsto P{\downarrow}p$$

has the following universal property: If $\langle X, \leq_X \rangle$ is another complete lattice and $f \colon P \longrightarrow X$ is any isotone function, then there exists a unique function, preserving arbitrary suprema,

$$\hat{f} \colon \mathcal{O}(P) \longrightarrow X$$

such that the following diagram commutes:



i.e. $\hat{f}(P{\downarrow}x) = f(x)$.

Proof: **(i):** $\mathcal{O}(P)$ is complete as a set lattice, since if $\{D_i\}_{i \in I} \subseteq \mathcal{O}(P)$ then $\bigcup\limits_{i \in I} D_i$ and $\bigcap\limits_{i \in I} D_i$ belong to $\mathcal{O}(P)$. ⊣⊓

(ii): Let, for every down-set $D \in \mathcal{O}(P)$, define:

$$\hat{f}(D) := \bigvee \{f(x) \in X \mid x \in D\}.$$

The function $\hat{f}$ preserves order since if $(P{\downarrow}x_1) \subseteq (P{\downarrow}x_2)$ then $\hat{f}(P{\downarrow}x_1) = \bigvee\{f(x) \in X \mid x \in (P{\downarrow}x_1)\} \leq \bigvee\{f(x) \in X \mid x \in (P{\downarrow}x_2)\} = \hat{f}(P{\downarrow}x_2)$ since $\{f(x) \mid x \in P{\downarrow}x_1\} = \{f(x) \mid x \leq x_1\} \subseteq \{f(x) \mid x \leq x_2\}$. The function $\hat{f}$, indeed makes the above diagram commutative:

$$\begin{aligned} \hat{f}(P{\downarrow}x) &= \bigvee\{f(y) \in X \mid y \in P{\downarrow}x\} \\ &= \bigvee\{f(y) \in X \mid y \leq x\}. \end{aligned}$$

Since f is isotone then $y \leq x$ implies that $f(y) \leq f(x)$, so that,

$$\bigvee\{f(y) \in X \mid y \leq x\} = f(x).$$

Furthermore $\hat{f}$ preserves arbitrary suprema:

Let $\{(P{\downarrow}x_i)\}_{i \in I} \subseteq \mathcal{O}(P)$, we should prove that:

$$\hat{f}\left[\bigcup_{i \in I}(P{\downarrow}x_i)\right] = \bigvee_{i \in I} \hat{f}[(P{\downarrow}x_i)].$$

Indeed,

$$\begin{aligned}
\hat{f}\left[\bigcup_{i\in I}(P{\downarrow}x_i)\right] &= \bigvee\left\{f(y)\in X \mid y\in\bigcup_{i\in I}(P{\downarrow}x_i)\right\}\\
&= \bigvee\{f(y)\in X \mid (\exists i\in I)[y\leq x_i]\}\\
&= \bigvee_{i\in I}\bigvee\{f(y)\in X \mid y\leq x_i\}\\
&= \bigvee_{i\in I} f(x_i) = \bigvee_{i\in I}\hat{f}[(P{\downarrow}x_i)]. \qquad \dashv
\end{aligned}$$

1.3.11 Remark. From the above Theorem it is clear that elements of the form $(P{\downarrow}x)$, freelly generate $\mathcal{O}(P)\cong 2^{\langle P^{op}\rangle}$, and thus every element $D\in\mathcal{O}(P)$ can be expressed as a union (supremum) of elements of the form $(P{\downarrow}x)$. Since $\hat{f}$ preserves suprema, it is enough to check only for elements $(P{\downarrow}x)$ in $\mathcal{O}(P)$.

CHAPTER 2

ADJOINTS FOR PREORDERS

2.1 Adjoints

Let $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ be two preorders.

2.1.1 Definition. An **adjunction** between P and Q, in symbols,

$$F : P \leftrightharpoons Q : G \quad \text{or} \quad P \underset{G}{\overset{F}{\rightleftarrows}} Q \quad \text{or even} \quad P \underset{G}{\overset{F}{\rightleftarrows}} Q$$

is a pair of functions (F, G) such that:

(i) Both $F : P \to Q$, $\quad G : Q \to P$ are isotone.

(ii) For all $p \in P, q \in Q$,

$$(AD) \qquad F(p) \leq_Q q \quad \text{iff} \quad p \leq_P G(q).$$

Sometimes we say that F is **left** or **lower adjoint** to G, and G is **right** or **upper adjoint** to F. This is also denoted by $F \dashv G$. On occations we shall also use the notation F^+ for the right adjoint G.

2.1.2 Remark. 1) In the case that we have two different preorders, i.e. $\langle P, \leq_P \rangle \neq \langle Q, \leq_Q \rangle$, then we may freely apply the above definition to anyone of the following pairs:

(i) $\langle P, \leq_P \rangle, \langle Q, \leq_Q \rangle$ (adjunction)

(ii) $\langle P, \leq_{P^{op}} \rangle, \langle Q, \leq_{Q^{op}} \rangle$ (dual adjunction)

(iii) $\langle P, \leq_{P^{op}} \rangle, \langle\, Q, \leq_{Q}\, \rangle$ (Galois connection)

Sometimes when we write P^{op} we mean $\langle P, \leq_{P^{op}} \rangle$. It is clear that:

$$F : P \leftrightharpoons Q : G \quad \text{iff} \quad G : Q^{op} \leftrightharpoons P^{op} : F$$

2) Adjunctions for preordered sets are also known with various other names:

(i) F is also called **residuated** and G **residual**.

(ii) An adjunction can be also called *a covariant Galois connection* and the usual Galois connection it is then called *contravariant Galois connection.*

2.1.3 Example. **1)** Let $X, Y \neq \emptyset$ and $R \subseteq X \times Y$. Let

$$P = \langle \mathcal{P}(X), \subseteq \rangle \quad \text{and} \quad Q = \langle \mathcal{P}(Y), \supseteq \rangle;$$

then,

$$F : \mathcal{P}(X) \leftrightharpoons \mathcal{P}(Y) : G$$

with

$$\begin{aligned} F(A) &:= \{y \in Y \mid (\forall x \in A)[(x, y] \in R]\}, \quad A \subseteq X \\ G(B) &:= \{x \in X \mid (\forall y \in B)[(x, y) \in R]\}, \quad B \subseteq Y \end{aligned}$$

defines an adjunction, called **polarity**.

2) Let $\mathbb{L}$ be a lattice, and

$$\Diamond : \mathbb{L} \longrightarrow \mathbb{L}$$

be a **possibility modal operator**, i.e. $\Diamond(0) = 0$ and $\Diamond(x \vee y) = \Diamond(x) \vee \Diamond(y)$, and

$$\Box : \mathbb{L} \longrightarrow \mathbb{L}$$

be a **necessity modal operator**, i.e. $\Box(1) = \perp$ and $\Box(x \wedge y) = \Box(x) \wedge \Box(y)$. Then,

$$\Diamond : \mathbb{L} \leftrightharpoons \mathbb{L} : \Box$$

is an adjunction. Usually when the elements of $\mathbb{L}$ are some kind of sets e.g. "open sets" then,

$x \in \Box(y)$ means that: "it is known (by a finite investigation) that x is in y", and

$x \in \Diamond(y)$ means that: "x cannot be finitely distinguished from y". As an example,

2.2 Triangle Inequalities and Basic Properties of Adjunctions.

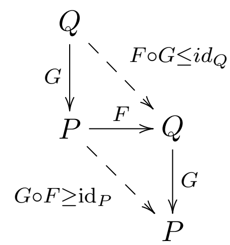
2.2.1 Theorem. *Let* $\langle\ P, \leq_P\ \rangle$ *and* $\langle\ Q, \leq_Q\ \rangle$ *be two preorders and* $F : P \longrightarrow Q$, $G : Q \longrightarrow P$ *are isotone maps. Then the following are equivalent:*

(i) $F \dashv G$

(ii) Triangle inequalities.

(α) $G \circ F \geq_P \mathrm{id}_P$

(β) $F \circ G \leq_Q \mathrm{id}_Q$



Proof: **(i) ⇒ (ii):** Let $F \dashv G$, then for all $p \in P, q \in Q$,

$$F(p) \leq_Q q \quad \text{iff} \quad p \leq_P F(q).$$

Choose now $q = F(p)$, then,

$$F(p) \leq_Q F(p) \quad \text{iff} \quad p \leq_P G(F(q))$$

i.e. $\mathrm{id}_P \leq_P G \circ F$.

Similarly it we take $p = Q(q)$ then we get $F \circ G \leq_Q \mathrm{id}_Q$. ⊣□

(ii) ⇒ (i): Let (ii) holds and assume $F(p) \leq_Q q$. Then since G is isotone (by hypothesis) we have $G \circ F(p) \leq_P G(q)$. But (ii) implies $\mathrm{id}_P(p) \leq_P G \circ F(p)$ and thus $p \leq_P G(q)$. Similarly, if $p \leq_P G(q)$ then,

$$F(p) \leq_Q (F \circ G)(q) \leq_Q q. \qquad ⊣■$$

2.2.2 Remark. 1) If we have $G \circ F = \mathrm{id}_P$ then G is called a **section** of F, whereas if $F \circ G = \mathrm{id}_Q$, then G is called a **retraction** of F. In the case that $G \circ F = \mathrm{id}_P$ and $F \circ G = \mathrm{id}_Q$ we say that $F : P \leftrightarrows Q : G$ forms a **duality** between P and Q.

2) The triangle inequalities have a geometric connotation. However it is instructive always to look for the basic duality of Lawvere: Geometry vs. Logic. Indeed an adjunction expresses just a dialectic system, see e.g. [28]. later when we apply adjunctions to monoids we will see that $G \circ F \geq \mathrm{id}_P$ expresses a quantitative (algebraic) property whereas $F \circ G \leq \mathrm{id}_Q$ expresses the logical principle of "modus ponens".

2.2.3 Theorem. *Left and right adjoints when exist there are unique, that is:*

If $F \dashv G$ *and* $F \dashv G'$ *then* $G = G'$ *and if* $F \dashv G$ *and* $F' \dashv G$ *then* $F = F'$.

Proof: Let $F \dashv G$ and $F \dashv G'$, then using the triangle inequalities we have:

$$\begin{aligned} G = \mathrm{id}_P \circ G &\leq (G' \circ F) \circ G, \quad \text{since} \quad F \dashv G' \\ &= G' \circ (F \circ G), \\ &\leq G' \circ \mathrm{id}_Q, \quad \text{since} \quad F \dashv G \\ &= G'. \end{aligned}$$

Similarly, $G \geq G'$, whence $G = G'$. The same method gives $F = F'$. ⊣▮

2.2.4 Lemma. *For every $p \in P, q \in Q$, $F \dashv G$ implies:*

$$\downarrow G(q) = F^{-1}[\downarrow q] \qquad \text{and} \qquad \uparrow F(p) = G^{-1}[\uparrow p].$$

Proof:

$$\begin{aligned} F^{-1}[\downarrow q] &= \{p \in P \mid F(p) \leq q\} \\ &= \{p \in P \mid p \leq G(q)\} \text{ since } F \dashv G \\ &= \downarrow G(q). \end{aligned}$$

The second relation is dual. ⊣▮

2.3 Adjoint Functor Theorem

2.3.1 Theorem. *(The Adjoint Functor Theorem) Let $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ be ordered sets and $F : P \longrightarrow Q$, $G : Q \longrightarrow P$ are isotone functions. Then the following are equivalent:*

(i) $F \dashv G$

(ii) *For each $q \in Q$,*

$$G(q) = \bigvee\{p \in P \mid F(p) \leq_Q q\}.$$

(iii) *For each $p \in P$,*

$$F(p) = \bigwedge\{q \in Q \mid p \leq G(q)\}.$$

Furthermore, if F has a right adjoint then F preserves all existing in P, suprema and dually if G has a left adjoint then G preserves all existing in Q infima.

Proof: **(i)** $\Rightarrow$ **(ii)**. First we will prove that if $F \dashv G$ then F preserves all suprema that exist in P. To this end let $p = \vee p_i$ exists in P, then since F is isotone we know that

$$\bigvee_{i \in I} F(p_i) \leq_Q F\left(\bigvee_{i \in I} p_i\right) \in Q$$

so that $\bigvee F(p_i) \neq \emptyset$ exists in Q. To prove now that $F\left(\bigvee_{i\in I} p_i\right) = \bigvee_{i\in I} F(p_i)$ we may prove the equivalent condition:

$$(\forall q \in Q)\left[F\left(\bigvee_{i\in I}\right) \leq_Q q \quad \text{iff} \quad \bigvee_{i\in I} F(p_i) \leq_Q q\right].$$

Indeed,

$$\begin{aligned} F\left(\bigvee_{i\in I} p_i\right) \leq_Q q \qquad & \text{iff} \quad \bigvee_{i\in I} p_i \leq_P G(q) \ \text{ since } \ F \dashv G \\ & \text{iff} \quad (\forall i \in I)[p_i \leq_P G(q)] \\ & \text{iff} \quad (\forall i \in I)[F(p_i) \leq_Q q] \\ & \text{iff} \quad \bigvee_{i\in I} F(p_i) \leq q. \qquad \dashv\Box \end{aligned}$$

Next we shall prove that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii): Let $F \dashv G$. Since $G(q) \in \downarrow G(q) = F^{-1}[\downarrow q]$ we conclude that $F^{-1}[\downarrow q] \neq \emptyset$. Finally we know that for all o-ideals,

$$\bigvee \downarrow G(q) = G(q).$$

Thus $\bigvee \downarrow G(q)$ always exists in P and since $\downarrow G(q) = F^{-1}[\downarrow q] = \{p \in P \mid F(p) \leq q\}$ we finally get

$$G(q) = \bigvee \{p \in P \mid F(p) \leq q\}. \qquad \dashv\Box$$

(ii) $\Rightarrow$ (i): Let $F^{-1}[\downarrow q] \neq \emptyset$ for all $q \in Q$ and that there exists $p \in P$ such that $F^{-1}[\downarrow q] = \downarrow p$. Suppose further that

$$G(q) = \bigvee \{p \in P \mid F(p) \leq q\}.$$

To prove (i) let $p \in G(q)$, i.e. $F(p) \leq q$ we should prove that $p \leq G(q)$. Indeed,

$$F(p) \leq q \ \text{ iff } \ F(p) \in \downarrow q \ \text{ iff } \ p \in F^{-1}[\downarrow q] = \downarrow G(q)$$

i.e. $p \leq G(q)$.

The proof that (i) $\Leftrightarrow$ (iii) is dual to (i) $\Leftrightarrow$ (ii). $\dashv\blacksquare$

2.3.2 Theorem. *(The Adjoint Functor Theorem: The Complete Case)*

Let $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ be complete lattices and $F : P \longrightarrow Q$, $G : Q \longrightarrow P$ are isotone functions. Then

(i) *$F : P \longrightarrow Q$ has a right adjoint, say $G : Q \longrightarrow P$, iff for all $\{p_i\}_{i\in I} \subseteq P$,*

$$F\left(\bigvee_{i\in I} p_i\right) = \bigvee_{i\in I} F(p_i)$$

and $G : Q \longrightarrow P$ *is uniquely determined by:*

$$G(q) = \bigvee\{p \in P \mid F(p) \leq_Q q\}, \quad q \in Q.$$

(ii) $G : Q \longrightarrow P$ *has a left adjoint, say* $F : P \longrightarrow Q$*, iff for all* $\{q_j\}_{j \in J} \subseteq Q$*,*

$$G\left(\bigwedge_{j \in J} q_j\right) = \bigwedge_{j \in J} G(q_j)$$

and $F : P \longrightarrow Q$ *is uniquely determined by:*

$$F(p) = \bigwedge\{q \in Q \mid p \leq G(q)\}.$$

Proof: Using the previous Theorem, it is enough to prove that if

$$F\left(\bigvee_{i \in I} p_i\right) = \bigvee F(p_i) \text{ for all } \{p_i\}_{i \in I} \subseteq P$$

then $G(q) := \bigvee\{p \in P \mid F(p) \leq_Q q\},\ q \in Q$, is isotone and $F \dashv G$. Indeed: If $F(p) \leq q$ then $p \in P$ is one member that makes up the supremum of $G(q)$ and so $p \leq G(q)$. Conversely, if $p \leq G(q) = \bigvee\{p \in P \mid F(p) \leq_Q q\}$, then since F preserves suprema we have:

$$\begin{aligned} F(p) &\leq F(G(q)) = F\left(\bigvee\{p \in P \mid F(p) \leq_Q q\}\right) \\ &= \bigvee\{F(p) \in Q \mid F(p) \leq_Q q\} \\ &\leq q. \end{aligned}$$

Thus $F \dashv G$. ⊣

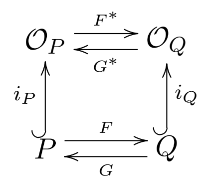
Let us go back to the functor $P{\downarrow}(-)$. We have seen that $P{\downarrow}(-)$ satisfies following properties:

1. $(\forall x, y \in P)[x \leq y \text{ iff } P{\downarrow}x \subseteq P{\downarrow}y]$
2. For any down-set $D \subseteq P$,
$$P{\downarrow}p \subseteq D \text{ iff } p \in D$$
3. If $F \dashv G$ then the adjunction condition
$$x \leq_P G(y) \text{ iff } F(x) \leq_Q y$$
is translated into
$$(P{\downarrow}x) \subseteq (P{\downarrow}G(y)) \quad \text{iff} \quad Q{\downarrow}F(x) \subseteq (Q{\downarrow}y)$$

4. If $F^* := G^{-1}$ and $G^* := F^{-1}$ then

$$F^*(P{\downarrow}p) = Q{\downarrow}F(p) \qquad \text{and} \qquad G^*(Q{\downarrow}q) = P{\downarrow}G(q)$$

Let $\mathcal{O}_P \equiv \langle \mathcal{O}(P), \subseteq \rangle$ and $\mathcal{O}_Q \equiv \langle \mathcal{O}(Q), \subseteq \rangle$. Consider the following diagram:



We shall prove the following theorem:

2.3.3 Theorem. *$F \dashv G$ iff $F^* \dashv G^*$.*

PROOF: ($\Rightarrow$) Let $F \dashv G$. Then,

$$x \leq_P G(y) \qquad \text{iff} \qquad F(x) \leq_Q y$$

which is equivalent with

$$(P{\downarrow}x) \subseteq (P{\downarrow}G(y)) \;\text{ iff }\; Q{\downarrow}F(x) \subseteq (Q{\downarrow}y)$$

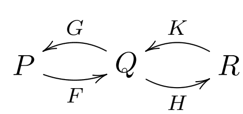
Using (4) above we get

$$F^*(P{\downarrow}x) \subseteq Q{\downarrow}y \quad \text{iff} \quad (P{\downarrow}x) \subseteq G^*(Q{\downarrow}y)$$

that is $F^* \dashv G^*$. Since $\mathcal{O}(P)$ is generated by elements of the form $(P{\downarrow}x), x \in P$, then the above proves that indeed $F^* \dashv G^*$. Simiarly for $\Leftarrow$. ⊣▌

Compositions of adjoints.

Let,



If $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.

PROOF: $GK(r) \leq p$ iff $G(K(r)) \leq P$, iff $F(p) \leq K(r)$ iff $r \leq HF(p)$. ⊣▌

2.3.4 Theorem. *Let A, B, C be posets and let,*

$$f : A \longrightarrow B, \qquad g : B \longrightarrow C$$

be residuated mappings. Then $g \circ f$ is residuated with

$$(g \circ g)^+ = f^+ \circ g^+.$$

Proof: Clearly $g \circ f$ and $f^+ \circ g^+$ are isotone. Moreover the isotonicity of f, g together with the triangle inequalities

$$f^+ \circ f \geq \mathrm{id}_A, \quad f \circ f^+ \leq \mathrm{id}_B, \quad g^+ \circ g \geq \mathrm{id}_B, \quad g \circ g^+ \leq \mathrm{id}_C$$

yields

$$(f^+ \circ g^+) \circ (g \circ f) \geq f^+ \circ \mathrm{id}_B \circ f = f^+ \circ f \geq \mathrm{id}_A; (g \circ f) \circ (f^+ \circ g^+) \leq g \circ \mathrm{id}_B \circ g^+ = g \circ g^+ \leq \mathrm{id}_C$$

from which we deduce, using the uniqueness of residuals, that $(g \circ f)^+$ exists and

$$(g \circ f)^+ = f^+ \circ g^+. \qquad \dashv$$

2.4 Closure operators (monads) on preorders

2.4.1 Definition. Let $\langle P, \leq \rangle$ be a poset. By a **closure operator (or a monad)** on P we mean an isotone mapping:

$$\mathrm{cl} : P \longrightarrow P$$

such that,

$$\mathrm{cl} = \mathrm{cl} \circ \mathrm{cl} \geq \mathrm{id}_P.$$

Alternatively, $\mathrm{cl}(\cdot)$ is a closure operator iff

(i) $p_1 \leq p_2 \Rightarrow \mathrm{cl}(p_1) \leq \mathrm{cl}(p_2)$ (isotonicity)

(ii) $(\forall p \in P)[p \leq \mathrm{cl}(p)]$ (increasing)

(iii) $(\forall p \in P)[\mathrm{cl}(p) = \mathrm{cl} \circ \mathrm{cl}(p)]$ (idempotency)

If instead of (ii) we have $(ii)'$ $(\forall p \in P)[p \geq \mathrm{int}(p)]$ then we have an **interior (or co-monad) operator**, that is $\mathrm{int} : P \longrightarrow P$ is an **interior operator** iff int is an isotone and

$$\mathrm{int} = \mathrm{int} \circ \mathrm{int} \leq \mathrm{id}_P.$$

2.4.2 Theorem. *If $\langle P, \leq \rangle$ is a poset and $f : P \longrightarrow P$ is a mapping, the following conditions are equivalent:*

(1) f is an interior operator

(2) $(\forall p \in P)[f^{-1}[\downarrow P] = f^{-1}[\downarrow f(p)]]$.

Likewise, the following conditions are equivalent:

(3) f is a closure operator

(4) $(\forall p \in P)[f^{-1}[\uparrow P] = f^{-1}[\uparrow f(p)]]$

Proof: We prove that (1) $\Leftrightarrow$ (2); the proof of (3) $\Leftrightarrow$ (4) is similar. Suppose that (1) holds. It is evident that for each $x \in P$,

(∗). $$x \in f^{-1}[\downarrow f(x)] \subseteq f^{-1}[\downarrow x]$$

Moreover,

(∗∗) $$y' \in f^{-1}[\downarrow x] f(y) \leq x f(y) \leq f(x) y \in f^{-1}[\downarrow f(x)]$$

Then,
$$f^{-1}[\downarrow p] = f^{-1}[\downarrow f(p)]. \qquad \dashv\square$$

Conversely if (2) holds, then we have:

$$\forall x \in P, \quad f^{-1}[\downarrow x] = f^{-1}[\downarrow f(x)] = f^{-1}[\downarrow f(f(x))].$$

Now x is an element of the second of these sets. It therefore belongs to the other two and so,
$$(\forall x \in P)\ [f(x) \leq x \quad \text{and} \quad f(x) \leq (f \circ f)(x)]$$
Giving $f \leq \mathrm{id}_P$ and $f \leq f \circ f$. Now f is isotone; for

$$y \leq x f(y) \leq y \leq x y \in f^{-1}[\downarrow x] = f^{-1}[\downarrow f(x)] f(y) \leq f(x).$$

We therefore deduce from $f \leq \mathrm{id}_P$ that $f \circ f \leq f$. Thus f is an interior operator. $\dashv\blacksquare$

2.4.3 Theorem. *If P is a poset then $f : P \longrightarrow P$ is a closure mapping iff there exists a poset Q and a left adjoint map (residuated map) $g : P \longrightarrow Q$ such that $f = g^+ \circ g$, where g^+ is the right adjoint to f (the residual).*

Proof: Suppose first that $g : A \longrightarrow B$ is residuated. Then since g is isotone and $g^+ \circ g \geq \mathrm{id}_A$, $g \circ g^+ \leq \mathrm{id}_B$ we have $g \circ g^+ \circ g \geq g \circ \mathrm{id}_A = g$ and $g \circ g^+ \circ g \leq \mathrm{id}_B \circ g = g$ so that

$$g \circ g^+ \circ g = g \quad \text{and similarly} \quad g^+ \circ g \circ g^+ = g^+.$$

From the above relation we get

$$g^+ \circ g = (g^+ \circ g) \circ (g^+ \circ g)$$

thus $g^+ \circ g$ is a closure mapping, it is also an isotone mapping for being the composition of two isotone mappings. $\dashv\square$

Conversely, suppose that A is a poset with $f : A \longrightarrow A$ a closure mapping. On A the function f induces an equivalence relation: for all $x, y \in A$,

$$x \sim_f y \qquad \text{iff} \qquad f(x) = f(y).$$

Let $A/\sim_f$ be the quotient set. We define on $A/\sim_f$

$$x/\sim_f \ \preceq\ y/\sim_f \quad :\Longleftrightarrow \quad f(x) \leq f(y).$$

It is readily seen that $\preceq$ is an ordering on $A/\sim_f$ and since f is isotone, the canonical surjection

$$k : A \longrightarrow A/\sim_f$$

is isotone. Now each equivalence class modulo $\sim_f$ has a greatest element, the greatest element in the class of x modulo $\sim_f$ being $f(x)$. We can therefore define a mapping

$$g : A/\sim_f \longrightarrow A \;/\!\!/\; x/\sim_f \mapsto g(x/\sim_f) := f(x).$$

We then have

$$\begin{aligned}(g \circ k)(x) &= g[k(x)] = g(x/\sim_f) = f(x) \geq x;\\ (k \circ g)(x/\sim_f) &= k[f(x)] = f(x)/\sim_f = x/\sim_f .\end{aligned}$$

from which it follows that k is residuated with $g = k^+$ and that $f = k^+ \circ k$. ⊣▮

An alternative treatment:

Question: Is every monad defined by means of a suitable pair of adjoint morphisms? The answer is YES!

2.4.4 Definition. Let T be a monad on the preorder P. Let

$$P_T := \{p \in P \mid T(p) \leq p\} \subseteq P.$$

Since T is a monad we have $(\forall p \in P)[p \leq T(p)]$. Thus

$$P_T := \{p \in P \mid T(p) = p\}.$$

That s P_T consists exactly by the "fixed points" of T.

Also since T is a monad we have

$$(\forall p \in P)[TT(p) \equiv T(p)]$$

we can therefore regard T as a morphism

$$F_T : P \longrightarrow P_T \;/\!\!/\; p \mapsto F_T(p) := T(p) = TT(p)$$

$T(p)$ is a fixed point of T and thus $T(p) \in P_T$.

Also we will denote the inclusion of P_T into P by

$$G_T : P_T \hookrightarrow P \;/\!\!/\; q \mapsto G_T(q) := q \equiv T(q).$$

Then we have:

2.4.5 Theorem. *For all $p \in P$ and all $q \in P_T$*

$$F_T(p) \leq q \quad \text{iff} \quad p \leq G_T(q)$$

i.e. $F_T \dashv G_T$.

PROOF:

$$\begin{aligned} F_T(p) \leq q \quad &\Leftrightarrow \quad T(p) \leq q \Leftrightarrow T(p) \leq T(q) \\ &\Leftrightarrow \quad p \leq T(p) \leq T(q) \\ &\Leftrightarrow \quad p \leq T(q) \equiv G_T(q). \end{aligned}$$ ⊣▮

2.4.6 Theorem. *Let T be a monad on the preorder P. Then there exists an adjunction $F_T : P \leftrightharpoons P_T : G_T$ such that, $T = G_T \circ F_T$.*

Comparison Morphism.

Let $F : P \leftrightharpoons Q : G$ be an adjunction and let the monad $T : P \longrightarrow P$ with $T := GF$. Let

$$P_T := \{p \in P \mid T(p) = p\}.$$

How do Q and P_T compare?

From the triangle inequalities, for any $q \in Q$

$$TG(q) = GFG(q) \leq G(q).$$

Thus $G(q) \in P_T$ and so we can regard G as a morphism,

$$G : Q \longrightarrow P_T \;/\!\!/\; q \mapsto G(q).$$

We will denote this morphism by

$$K : Q \longrightarrow P_T \;/\!\!/\; q \mapsto K(q) := G(q)$$

and call it **the comparison morphism** for the adjunction $F \dashv G$.

2.4.7 Definition. A morphism $F : P \longrightarrow Q$ is called **conservative** iff

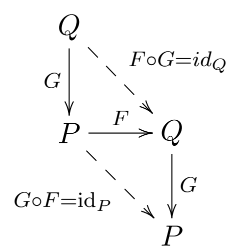
$$(\forall p, p' \in P)[F(p) \leq F(p')p \leq p']$$

A weaker than isomorphism association between preorders is "equivalence". Equivalence allows us to identify the preorders.

2.4.8 Definition. A morphism $G : Q \longrightarrow P$ is called a **quasi-inverse** for the morphism $F : P \longrightarrow Q$ iff

(i) $(\forall p \in P)[GF(p) \equiv p$, i.e. $GF \equiv \mathrm{id}_P]$

(ii) $(\forall q \in Q)[FG(q) \equiv q$, i.e. $FG \equiv \mathrm{id}_Q]$



Note. If G is a quasi-inverse for F, then G is both left-adjoint and right-adjoint to F, and moreover F is a quasi-inverse for G.

2.4.9 Definition. A morphism $F : P \longrightarrow Q$ is called **an equivalence** iff it has a quasi-inverse.

2.4.10 Theorem. *Let $F : P \leftrightarrows Q : G$ be an adjunction and let $T = GF$ be the associated monad on P. If G is conservative, then the comparison morphism*

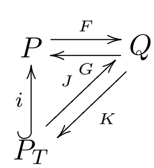
$$K : Q \longrightarrow P_T$$

is an equivalence.

Proof: Let

$$J : P_T \longrightarrow Q$$

be the the restriction of F to P_T,i.e. $F \upharpoonright P_T$.



Since G is conservative then,

$$(\forall q \in Q)[FG(q) \equiv q \equiv JK(q)]$$

since,

$$\begin{aligned} FG(q) &\leq q \quad \text{(Triangle ineq.)} \\ G(q) &= G(q) \\ F(G(q)) &= F(K(q)) \\ &= i \circ F(K(q)) \\ &= JK(q) \end{aligned}$$

Moreover, if $p \in P_T$, then

$$KJ(p) = GF(p) \equiv p.$$

Thus J is a quasi-inverse for K and so K is an equivalence. ⊣▌

Interior morphisms or comonads.

The dual concept of a monad is that of comonad.

2.4.11 Definition. A **comonad** or an **interior morphism** on a preorder Q is an endomorphism,

$$H : Q \longrightarrow Q.$$

Satisfying,

($\mathbf{M}_1^{op}$) $(\forall q \in Q)[H(q) \leq q]$;

($\mathbf{M}_2^{op}$) $(\forall q \in Q)[H(q) \leq HH(q)]$.

Dually we define, $Q_H := \{q \in Q \mid q \leq H(q)\}$ and

$$G_H : G \longrightarrow G_H \;\; /\!\!/ \;\; q \mapsto G_H(q) := H(q) = HH(q)$$

$$F_H : G_H \longrightarrow Q \;\; /\!\!/ \;\; q \mapsto F_H(q) := q \equiv H(q).$$

Then, $F_H : Q_H \leftrightharpoons Q : G_H$ is adjunction and $H = F_H G_H$.

2.4.12 Theorem. *If A is a poset and $f : A \longrightarrow A$ is a residuated mapping, then the following are equivalent:*

(a) f is a closure mapping;

(b) f^+ is an interior mapping;

(c) $f = f^+ \circ f$;

(d) $f^+ = f \circ f^+$.

Likewise, the following conditions are equivalent:

- *(α) f is an interior mapping;*
- *(β) f^+ is a closure mapping;*
- *(γ) $f = f \circ f^+$;*
- *(δ) $f^+ = f^+ \circ f$.*

Proof: **(a) $\Leftrightarrow$ (b):** Since f is residuated then,

$$f \leq \mathrm{id}_A \Leftrightarrow f^+ \geq \mathrm{id}_A$$
$$\text{and}$$
$$f \geq \mathrm{id}_A \Leftrightarrow f^+ \leq \mathrm{id}_A.$$

Indeed let $f \leq \mathrm{id}_A$, then by the triangle inequality f is residuated iff

$$\mathrm{id}_A \leq f^+ \circ f \leq f^+ \circ \mathrm{id}_A = f^+ \quad \text{so that} \quad f^+ \geq \mathrm{id}_A$$

Similarly for the second equivalence.

Now let (a), i.e. let f be a closure mapping then

$$f = f \circ f \geq \mathrm{id}_A, \quad \text{iff} \quad f^+ = f^+ \circ f^+ \leq \mathrm{id}_A, \ \text{ i.e. (a)} \Leftrightarrow \text{ (b).}$$

Now we shall establish (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b). Suppose that (a) holds; then (c) follows from the inequalities

$$\begin{array}{rcl} f^+ \circ f &=& f^+ \circ f \circ f \geq \mathrm{id}_A \circ f = f; \\ f &=& f \circ f^+ \circ f \geq \mathrm{id}_A \circ f^+ \circ f = f^+ \circ f. \end{array}$$

If now (c) holds then from $f = f^+ \circ f$ we deduce that $f \circ f^+ = f^+ \circ f \circ f^+ = f^+$ which is (d). Finally, if (d) holds, then $f^+ \circ f^+ = f \circ f^+ \circ f \circ f^+ = f \circ f^+ = f^+$ and $f^+ = f \circ f^+ \leq \mathrm{id}_A$ and hence (b) holds.

The equivalence of (α), (β), (γ), (δ) is proved similarly. ⊣

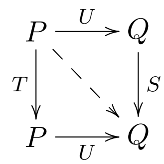
2.5 The Adjoint Lifting Theorem.

Suppose that,

$$T : P \longrightarrow P \quad \text{and} \quad S : Q \longrightarrow Q$$

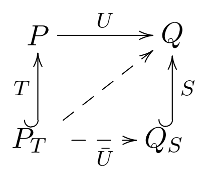
are monads on the preorders P and Q respectively.

2.5.1 Definition. A mapping $U : P \longrightarrow Q$ is **proper for** T **and** S iff



$$UT = SU$$

For any such $U : P \longrightarrow Q$ are obtained by restriction



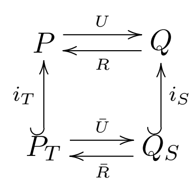
$$U \circ i_T = i_S \circ \bar{U}$$

2.5.2 Theorem. (Adjoint Lifting Theorem). *Let T and S be monads on P and Q respectively. Let $U : P \longrightarrow Q$ be a proper morphisms for T and S and let $\bar{U} : P_T \longrightarrow Q_S$ be the restriction of U. Then,*

(I) *If $U : P \longrightarrow Q$ has a right adjoint $R : Q \longrightarrow P$ then*

$$\bar{U} : P_T \longrightarrow Q_S \quad \text{has a right adjoint} \quad \bar{R} : Q_S \longrightarrow P_T$$

That is the following diagram is commutative:



given by the restriction: $\bar{R}(q) := R(q)$ for $q \in Q_S$.

(II) *If $U : P \longrightarrow Q$ has a left-adjoint $R : Q \longrightarrow P$ then*

$$\bar{U} : P_T \longrightarrow Q_S \quad \text{has a left-adjoint} \quad \bar{R} : Q_S \longrightarrow P_S, q \mapsto \bar{R}(q) := R(q),\ q \in Q_S.$$

CHAPTER 3

COMMUTATIVE, INTEGRAL, RESIDUATED, ℓ-MONOIDS (cirl-MONOIDS)

3.1 Introduction

With the revival of many-valued logics, fuzzy logics, linear logic etc. there is a strong interest for quantitative (not necessarily idempotent valuation structures (structures for truth values)).

To give a rationale for choosing **the commutative, residuated ℓ-monoids** as a kind of a generalized framework structure, it is helpful to begin with Bourbaki's mother-structures:

(i) Posets;

(ii) Topological spaces;

(iii) Algebraic structures.

To reach to a general structure which incorporates all the above mother structures, we find that posets are the primal basic structures and thus the starting point. This is clear since e.g. in set theory, '$\in$' essentially is an ordering. We would like to express (ii) and (iii) in a way which incorporates ordering. This means that we would like to see topology as the study of lattices of "opens" and algebra as the study of lattices of "ideals". Thus for topological spaces one may be forced to take **locales** and **frames** i.e. the lattice theoretic study of 'opens' (pointless topology) [26]. These structures are idempotent and thus they express qualitative concepts, associated with topology, in a lattice theoretic way. A **locale** is a complete lattice in which arbitrary joins distribute over meets, i.e. we have: For arbitrary I,

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

A locale is then equivalent with a complete Heyting algebra, i.e. with a complete lattice with top and bottom elements, in which for every element $b \in A$, the functor,

$$(\_) \wedge b : A \longrightarrow A \;/\!\!/\; a \mapsto a \wedge b$$

has a right adjoint, denoted by,

$$b \Rightarrow_{\wedge} (\_) : A \longrightarrow A \;/\!\!/\; c \mapsto b \Rightarrow_{\wedge} c$$

The above adjunction essentially expresses the Lawvere's duality:

Geometry vs. Logic

in its qualitative version. This is clear since the open sets express the geometric-extentional part, whereas the logical-intentional part is expressed by the Heyting algebra of intuitionistic propositional logic, where an open set is construe as "a finitely observable property"[33]. The element which connects the two interpretations is the concept of adjunction, which converts the extensional-geometric algebraic operation '$\wedge$' into a more sophisticated antithetical intentional-logical operation of implication. The 'synthesis' of the above two operations is a logic which incorporates the qualitative algebraic and logical operations. There are also negation operations, $\neg_{\wedge} a := (a \Rightarrow_{\wedge} 0)$ and $\neg_{\vee} a := (1 \Leftarrow_{\vee} a)$ where $\Leftarrow_{\vee}$ is the co-implication, which coincide with the corresponding pseudocomlements of a i.e.,

$$\neg_{\wedge} a = \bigvee \{x \mid a \wedge x = 0\} \quad \text{and} \quad \neg_{\vee} a = \bigwedge \{x \mid a \vee x = 1\}.$$

Passing now to the third element, i.e. **algebraic structures** we would like to generalize the Heyting algebra case by choosing a structure which is minimal, but at the same time rich enough to do logic and mathematics. This is the structure of **commutative, residuated ℓ-monoid** where the monoidal operation is not necessarily idempotent. If we want the idempotent (qualitative) case to be embedded into the non-idempotent one, we should chose:

1) The monoidal structure $\langle A; \leq, \odot, 1 \rangle$ to be lattice ordered which implies that it is also a po-monoid.

2) The monoidal operation '$\odot$' has a right adjoined (or residual) operation '$\Rightarrow_{\odot}$', i.e. for all $a, b, c \in A$:

(AD) $\quad a \odot b \leq c \quad \text{iff} \quad b \leq a \Rightarrow_{\odot} c$

In the sequel we shall denote: $\neg_{\odot} a \equiv \neg a := a \Rightarrow_{\odot} 0$. To the adjoint pair $(\odot, \Rightarrow_{\odot})$ there is a dual adjoint pair (monoidal addition and co-implication) $(\oplus, \Leftarrow_{\oplus})$ with the co-implication $a \Leftarrow_{\oplus} b$ identical with difference $a \ominus b := a \odot \neg b$ satisfying the following dual adjunction: For all $a, b, c \in A$,

(DAD) $\quad a \oplus b \geq c \quad \text{iff} \quad b \geq c \Leftarrow_{\oplus} a$

There is here a corresponding co-negation oparation defined as:

$$\neg_{\oplus} a := 1 \Leftarrow_{\oplus} a.$$

3) When we restrict ourselves to the center (the idempotent elements) of the structure we would like to recover the Heyting algebra case in (ii).

From the above reasoning it is clear that the structures that constitute the basic framework are the "commutative residuated l-monoids" as studied e.g. in [24], with some additional conditions. In this Chapter we will follow closely [24].

3.2 The Qualitative Case: Heyting Algebras.

Heyting algebras have a dual interpetation:

(i) An extensional-geometric, and

(ii) An intentional-logical.

(i) Extensional-geometric. Heyting algebras are models of lattices of open sets. Depenting on the morphisms used Heyting algebras may be termed as locales or frames. These structures show that *a generalized topology is the study of lattices of opens*. In contradistiction, algebra can be conceived as *the study of lattices of ideals* and its main object concerns a quantization of geometry, based on either 'measurements', 'measures', or 'normalized metric spaces' which amounts to a kind of indistiguishability. As R. Street said in a forthcoming book [34]: "...commutative algebras are really *spaces* seen from the other side of your brain.' We would like to add that this other side essentially is the left hemisphere of your brain, space perception on the other hand, resides as we know in the right hemisphere.

(ii) Intentional-logical. Heyting algebras are also thought as models of systems of propositions in the first-order intuitionistic logic. Based on this dual interpretation one expects that the lattce order will be dialectically connected to logical implication operator through an adjunction, which is exactly the case.

Let $\langle \mathbb{L} \leq \rangle$ be a poset, in which every pair of elements a, b has a join $a \vee b$. Then $\langle \mathbb{L}, \leq, \vee, 0 \rangle$ where $0 := \bigvee \emptyset \equiv \bigvee_{p \in \emptyset}$ is a commutative, integral ℓ-monoid in which every element is idempotent. We term such monoids qualitative. Conversely,

3.2.1 Theorem. *Let $\langle \mathbb{L}, \vee, 0 \rangle$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on $\mathbb{L}$ such that $a \vee b$ is the join of a and b, and 0 is the least element.*

PROOF: See, [26, p.2]. ⊣

A **lattice** is a poset in which every finite subset has both a join and a meet. This means that $\langle \mathbb{L}, \vee, \wedge, 0, 1 \rangle$ is a structure such that both, $\langle \mathbb{L}, \vee, 0, \rangle$ and $\langle \mathbb{L}, \wedge, 1 \rangle$ are semilattices and the partial orders on $\mathbb{L}$ induced by the two semilattices are opposite to each other.

3.2.2 Proposition. *Suppose $\langle \mathbb{L}, \vee, 0\rangle$ and $\langle \mathbb{L}, \wedge, 1\rangle$ are semilattices. Then $\langle \mathbb{L}, \vee, \wedge, 0, 1\rangle$ is a lattice iff the absorption laws,*

$$a \wedge (a \vee b) = a \quad \text{and} \quad a \vee (a \wedge b) = a$$

are satisfied for all $a, b \in \mathbb{L}$.

PROOF: See, [26, p.3]. ⊣▮

If for every element, $a \in \mathbb{L}$ we require that the functor,

$$(-) \wedge a : \mathbb{L} \longrightarrow \mathbb{L} \;\; /\!\!/ \;\; x \mapsto (x \wedge a)$$

has a right adjoint,

$$a \mathrel{=\!\!\wedge\!\!\Rightarrow} (-) : \mathbb{L} \longrightarrow \mathbb{L} \;\; /\!\!/ \;\; x \mapsto (a \mathrel{=\!\!\wedge\!\!\Rightarrow} x)$$

then the resulting structure is called a **Heyting algebra**.

Peudocomplements and Negations

Bi-Heyting Algebras

Boolean Algebras ≡ Involutive bi-Heyting Algebras.

Structure of the Ideal Lattices

To be continued...

3.3 The Quantitative Case: Non indempotent generalizations

Let $\langle \mathbb{L} \leq\rangle$ be a lattice, i.e. $\langle \mathbb{L} \leq\rangle$ is a poset such that joins and meets of finite subsets of $\mathbb{L}$ exists. In particular,

$$\bigwedge \emptyset \equiv \bigwedge_{i \in \emptyset} x_i = \top \qquad (\text{resp. } \bigvee \emptyset \equiv \bigvee_{i \in \emptyset} x_i = \bot)$$

the universal upper (lower) bounds exists in $\mathbb{L}$.

We always suppose that $\mathbb{L}$ contains at least two elements, i.e. $\bot \neq \top$.

On $\mathbb{L}$ we consider an additional binary operation

$$\odot : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$$

such that, the structure $\langle \mathbb{L}, \odot, \top \equiv 1\rangle$ is a **commutative monoid** i.e.

(M1) $(\forall x, y, z \in \mathbb{L})[x \odot (y \odot z) = (x \odot y) \odot z]$

(M2) $(\forall x \in \mathbb{L})[x \odot 1 = x = 1 \odot x]$

(M3) $(\forall x, y \in \mathbb{L})[x \odot y = y \odot 1]$

We use here the integrality condition, where $\top \equiv 1$.

3.3.1 Remark. In a commutative monoid $\langle M, \odot, 1\rangle$ we may define the set

$$N := \{x, x \odot x, x \odot x \odot x, \ldots\}.$$

If the elements of N are all different, we may prove that N is isomorphic to the natural numbers $\mathbb{N}$. This means that having a commutative monoid we have at the same time a set N acting as the set of natural numbers. Thus when we construct various logics based on a commutative residuated ℓ-monoid we have always at our disposal a set of natural numbers. In this way commutative residuated ℓ-monoids are logico-mathematical in nature.

Commutative, residuated ℓ-monoids are also known as commutative autonomous posets, i.e. a **commutative autonomous poset** is a partially ordered set $\langle P, \leq\rangle$ together with an commutative and associative binary operation $\odot$ such that for all $a \in P$, $a \odot \_ : P \longrightarrow P$ is residuated.

- Clearly every quantale is an autonomous poset but not conversely (completeness may not be present).
- Usually the term **residuated monoid** for an autonomous poset is used, and the term **residuated lattice** is used in the presence of a certain amount of completeness.

3.3.2 Definition. The structure $\langle \mathbb{L}, \leq, \odot, 1\rangle$ is called **commutative ℓ-monoid** (short for: lattice ordered monoid) iff

(L01) $\langle \mathbb{L}, \leq\rangle$ is a lattice

(L02) $\langle \mathbb{L}, \odot, 1\rangle$ is a commutative monoid

(L03) $(\forall x, y, z \in \mathbb{L})[x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ and $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)]$

3.3.3 Remark. Property (L03) implies that $\langle \mathbb{L}, \leq, \odot 1\rangle$ is a **po-monoid**, i.e. : For all $a, b, x \in \mathbb{L}$,

$$a \leq b \Rightarrow x \odot a \leq x \odot b \quad \text{and} \quad a \odot x \leq b \odot x \qquad \text{(isotonicity)}$$

Indeed: Since $x \odot (a \vee b) = (x \odot a) \vee (x \odot b)$ and $a \leq b$

we have: $a \vee b = b$, so that,

$$x \odot (a \vee b) = x \odot b \quad \text{or} \quad (x \odot a) \vee (x \odot b) = x \odot b$$

i.e. $x \odot a \leq x \odot b$. Similarly for $a \odot x \leq b \odot x$.

Exercise. Let $\langle M, \odot, 1\rangle$ be a monoid and $m \in M$. Define a new operation on M by

$$\odot_m : M \times M \times \longrightarrow M \;\;/\!\!/\;\; (x,y) \mapsto x \odot_m y := x \odot m \odot y.$$

Show that this defines a semigroup. Under what conditions on m do we have a unit relative to $\odot_m$?

3.3.4 Definition. Let $f : X \times Y \longrightarrow Z$ be a function of two variables. Then we define:

(i) The x-**section**: For all $x \in X$, $\quad f_x : \longrightarrow Z \;/\!\!/\; y \mapsto f_x(y) := f(x,y)$

(ii) The y-**section**: For all $x \in X$, $\quad f^y : X \longrightarrow Z \;/\!\!/\; x \mapsto f^y(x) := f(x,y)$

Let $F : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$ be an associative operation on $\mathbb{L}$, and let:

$$F_a : \mathbb{L} \longrightarrow \mathbb{L} \;/\!\!/\; x \mapsto F_a(x) := F(a,x)$$

be the a-section, $a \in \mathbb{L}$, of F, and

$$F^b : \mathbb{L} \longrightarrow \mathbb{L} \;/\!\!/\; x \mapsto F^b(x) := F(x,b)$$

be the b-section, $b \in \mathbb{L}$, of F.

Then we have

3.3.5 Theorem. *The operation F is associative iff*

$$F_a \circ F^b = F^b \circ F_a \qquad \text{for all} \quad a,b \in \mathbb{L}.$$

PROOF: . For all $a,b,x \in \mathbb{L}$, the associativity of F means

$$F(a, F(x,b)) = F(F(a,x), b)$$

which is equivalent to

$$F_a(F^b(x)) = F^b(F_a(x)),$$

i.e.

$$(F_a \circ F^b)(x) = (F^b \circ F_a)(x).$$

Thus

$$F_a \circ F^b = F^b \circ F_a. \qquad ⊣$$

3.3.6 Theorem. *The operation F is commutative iff*

$$(\forall a \in \mathbb{L})[F_a = F^a].$$

PROOF: For all $a, x \in \mathbb{L}$, by commutativity of F we have: $F(a,x) = F(x,a)$ or $F_a(x) = F^a(x)$. ⊣▮

3.3.7 Corollary. *The operation F is associative and commutative iff for all $a, b \in \mathbb{L}$,*

$$F_a \circ F_b = F_b \circ F_a.$$

3.3.8 Theorem. *Let F be an associative operation on $\mathbb{L}$. If the element $a \in \mathbb{L}$ is idempotent under F then both F_a, F^a are idempotent functions,*

i.e. if $F(a,a) = a$ then $F_a \circ F_a = F_a$ and $F^a \circ F^a = F^a$.

PROOF:

$$\begin{aligned} F_a^2(x) = F_a(F_a(x)) = F_a(F(a,x)) &= F(a, F)a, x)) = \\ &= F(F(a,a),x) = F(a,x) = F_a(x). \end{aligned}$$ ⊣▮

3.4 Monoid actions.

Let $\mathbb{M} = \langle M, \odot, e \rangle$ be a monoid. Then for any $m \in M$ we define the **left multiplication** or **left translation**:

$$\lambda_m : M \longrightarrow M \;/\!\!/\; x \mapsto \lambda_m(x) := m * x$$

Consider the set $S_M = \{\lambda_m : m \in M\}$ of functions. Then we have:

(i) $\lambda_e = \mathrm{id}_M$ since $\lambda_e(x) = e \odot x = x$ and

(ii) $\lambda_m \circ \lambda_n = \lambda_{m \odot n}$ since $\lambda_m \circ \lambda_n(x) = \lambda_m(\lambda_n(x)) = m \odot (n \odot x) = (m \odot n) \odot x = \lambda_{m \odot n}(x)$.

So on S_M we have an operation:

$$\circ : S_M \times S_M \longrightarrow S_M \;/\!\!/\; (\lambda_m, \lambda_n) \mapsto \lambda_m \circ \lambda_n$$

such that $\langle S_M, \circ, \lambda_e \rangle$ forms a monoid with identity λ_e.

This can be generalized. Let X be a set and

$$\lambda_{(-)} : M \longrightarrow X^X \;/\!\!/\; m \mapsto \lambda_m$$

i.e. the family, $(\lambda_m)_{m \in M}$, where $\lambda_m : X \longrightarrow X$, and $m \in M$ (M is our original monoid), is a set of functions such that:

(i′) $\lambda_e = \mathrm{id}_X$;

(ii′) $\lambda_m \circ \lambda_m = \lambda_{m \odot p}$.

The collection of λ_m's is called **an action of M on the set X.**

$(\lambda_m)_{m\in M}$ can be replaced by a single function,

$$\lambda : M \times X \longrightarrow X. \;/\!\!/\; (m,x) \mapsto \lambda(m,x) := \lambda_m(x)$$

The above two conditions become:

(i′) $\lambda(e,x) = x$;

(ii′) $\lambda(m, \lambda(n,x)) = \lambda(m \odot n, x)$.

Let M be a monoid. An M**-set**, is defined to be a pair $\langle X, \lambda\rangle$ where $\lambda : M \times X \longrightarrow X$ is such an action of M on X.

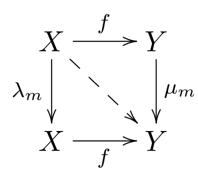
For a given monoid M, the M-sets are objects of a category M-**Set** which is a topos. An arrow

$$f : \langle X, \lambda\rangle \longrightarrow \langle Y, M\rangle$$

is an **equivariant** or **action-preserving function**

$$f : X \longrightarrow Y$$

i.e. such that,



$$f \circ \lambda_m = \mu_m \circ f$$

$$\left.\begin{array}{l}\text{Qualitative}\\ \text{Quantitative}\end{array}\right\} \quad \text{Geometry vs. Logic} \quad \left\{\begin{array}{l}\text{Qualitative (Boolean, Heyting,etc.)}\\ \text{Quantitative (MV-algebras, quantales, etc.)}\end{array}\right.$$

3.4.1 Definition. Let $\langle \mathbb{L}, \leq, \odot, 1\rangle$ be a commutative ℓ-monoid. We say that $\langle \mathbb{L}, \leq, \odot, 1\rangle$ is a **residuated, commutative ℓ-monoid** iff all translations

$$F_a \equiv a \odot (-) : \mathbb{L} \longrightarrow \mathbb{L} \;/\!\!/\; x \mapsto a \odot x$$

are residuated, i.e. have adjoints (or residuals)

$$G_a \equiv a \mathrel{-\!\!\odot\!\!\Rightarrow} (-) : \mathbb{L} \longrightarrow \mathbb{L} \;/\!\!/\; x \mapsto (a \mathrel{-\!\!\odot\!\!\Rightarrow} x)$$

Thus we have,

(AD)

$$\begin{array}{rl} & F_a(x) \leq b \quad \text{iff} \quad x \leq G_a(b) \\ \text{i.e.} & a \odot x \leq b \quad \text{iff} \quad x \leq (a \mathrel{-\!\!\odot\!\!\Rightarrow} b) \end{array}$$

or

$$(\forall a \in \mathbb{L})\ [F_a \dashv G_a].$$

3.4.2 Remark. The **logical operation of implication** is a more sophisticated notion than **the algabraic monoidal operation**, the adjointness here explains the emergence of one concept from the other, and at the same time the dialectical link of algebra and logic. Here we encounter again a logicomathematical reality, any separation of which into logic and algebra, results into unnutural mathematical settings.

3.4.3 Definition. A homomorphism between residuated, commutative ℓ-monoids is a structure preserving map, i.e.

$$h : \langle \mathbb{L}_1, \leq_1, \odot_1, 1_1\rangle \longrightarrow \langle \mathbb{L}_2, \leq_2, \odot_2, 1_2\rangle$$

such that:

(i) h is a lattice-homomorphism;

(ii) h is a monoid-homomorphism;

(iii) $h(a \Rightarrow\!\!\odot\!\!\Rightarrow_1 b) = h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 h(b)$.

3.4.4 Proposition. *Let $\langle P, \leq_1, \odot_1\rangle$, $\langle Q, \leq_2, \odot_2\rangle$ be two commutative residuated ℓ-monoids, and let $h : P \longrightarrow Q$ be a homomorphisms, then for all $a \in P, b \in Q$,*

$$h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) = (a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b))$$

Proof: To fix the notation let us consider what we have in a diagram: $\forall a \in P, b \in Q$

We have,

$$\begin{array}{rll} h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) \leq_1 a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b) & \text{iff} & a \odot_1 h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) \leq_1 h^+(b) \\ & \text{iff} & h[a \odot_1 h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b)] \leq_2 b \\ & \text{iff} & h(a) \odot_2 h(h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b)) \leq_2 b. \end{array}$$

By adjointness (triangle inequality), for all $c \in Q$,

$$h(h^+(c)) \leq_2 c$$

Thus, $h(a) \odot_2 h(h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b)) \leq h(a) \odot_2 (h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) \leq b$ (triangle).

Thus, the relation $h(a) \odot_2 h(h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b)) \leq_2 b$ holds true and so we proved that, $h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) \leq_1 a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b)$.

Conversely,

$$\begin{array}{rll} a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b) \leq_1 h^+(h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) & \text{iff} & h(a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b)) \leq_2 (h(a) \Rightarrow\!\!\odot\!\!\Rightarrow_2 b) \\ & \text{iff} & h(a) \odot_2 h(a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b)) \leq_2 b \\ & \text{iff} & h[a \odot_1 (a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b)] \leq_2 b. \end{array}$$

This holds iff $a \odot_1 (a \Rightarrow\!\!\odot\!\!\Rightarrow_1 h^+(b)) \leq_1 h^+(b)$. The last inequality always holds by the triangle inequality. ⊣

Residuated commutative ℓ-monoids and homomorphisms in the preceding sense form a category.

Interpreting the general characterizations for adjunction, we have the following basic properties:

3.4.5 Proposition. *The following are equivalent:*

(i) $(\forall a \in \mathbb{L})\ [F_a \dashv G_a]$ *(AD)*

(ii) **Triangle inequalities.**

1. $a \Rightarrow\!\!\odot\!\!\Rightarrow (a \odot x) \geq x$ *(fusing: from x we deduce 'a implies a and x))*
2. $a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow x) \leq x$ *(modus ponens: from a and $(a \Rightarrow\!\!\odot\!\!\Rightarrow x)$ we deduce x.)*

or

$$a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow x) \leq x \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow (a \odot x))$$

(iii) For each principal ideal $\downarrow x := \{y \in \mathbb{L} : y \leq x\}$ *of* $\mathbb{L}$*, and for all* $a \in \mathbb{L}$

$$F_a^{-1}[\downarrow x]$$

is a principal ideal of $\mathbb{L}$.

(iv) For all $a \in \mathbb{L}$ F_a *preserves arbitrary suprema, i.e. if* $x = \bigvee_{i \in I} x_i$ *exists then for all* $a \in \mathbb{L}$,

$$F_a(P) = \bigvee_{i \in I} F_a(x_i)$$

i.e.

$$a \odot \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (a \odot x_i)$$

In particular,

$$a \odot (x_1 \vee x_2) = (a \odot x_1) \vee (a \odot x_2)$$

Furthermore,

$$G_a(x) \equiv (a \Rightarrow\!\!\odot\!\!\Rightarrow x) = \bigvee \{y \in \mathbb{L} : a \odot y \leq x\}$$

(v) For all $a \in \mathbb{L}$, G_a *preserves arbitrary infima, i.e. if* $y = \bigwedge_{j \in J} y_j$ *exists in* $\mathbb{L}$. *then*

$$G_a(y) = \bigwedge_{j \in J} G_a(y_j)$$

i.e.

$$a \Rightarrow\!\!\odot\!\!\Rightarrow \left(\bigwedge_{j \in J} y_j \right) = \bigwedge_{j \in J} (a \Rightarrow\!\!\odot\!\!\Rightarrow y_j)$$

In particular,

$$a \Rightarrow\!\!\odot\!\!\Rightarrow (y_1 \wedge y_2) = (a \Rightarrow\!\!\odot\!\!\Rightarrow y_1) \wedge (a \Rightarrow\!\!\odot\!\!\Rightarrow y_2)$$

Furthermore,

$$a \odot x \equiv F_a(x) = \bigwedge\{y \in \mathbb{L} : x \leq a \Rightarrow\!\!\odot\!\!\Rightarrow y\}.$$

Note that since $F_a \dashv G_a$ *then for all* $a \in \mathbb{L}$,

$$G_a F_a G_a = G_a \qquad \text{and} \qquad F_a G_a F_a = F_a$$

or

$$\begin{aligned} a \Rightarrow\!\!\odot\!\!\Rightarrow [a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow x)] &= (a \Rightarrow\!\!\odot\!\!\Rightarrow x) \\ a \odot [a \Rightarrow\!\!\odot\!\!\Rightarrow (a \odot x)] &= a \odot x \end{aligned}$$

Now if $F^a : \mathbb{L} \longrightarrow \mathbb{L} \ /\!\!/ \ x \mapsto F^a(x) := F(x, a) = x \odot a$ then since $\mathbb{L}$ is commutative we have,

$$F_a = F^a \quad \text{and} \quad a \leq b \Rightarrow F_a \leq F_b \quad \& \quad F^a \leq F^b.$$

However if $G^a : \mathbb{L} \longrightarrow \mathbb{L} \ /\!\!/ \ x \mapsto G^a(x) := (x \Rightarrow\!\!\odot\!\!\Rightarrow a)$ then,

$$a \leq b \quad \text{implies} \quad G_a \geq G_b \ \& \ G^a \leq G^b$$

i.e.

$$(a \Rightarrow\!\!\odot\!\!\Rightarrow x) \geq (b \Rightarrow\!\!\odot\!\!\Rightarrow x) \quad \text{and} \quad (a \Rightarrow\!\!\odot\!\!\Rightarrow a) \leq (x \Rightarrow\!\!\odot\!\!\Rightarrow b)$$

That is,

$(*)$ $\quad a \leq b \quad$ implies $\quad (\forall x \in \mathbb{L})[(x \Rightarrow\!\!\odot\!\!\Rightarrow a) \leq (x \Rightarrow\!\!\odot\!\!\Rightarrow b)$ and $(a \Rightarrow\!\!\odot\!\!\Rightarrow x) \geq (b \Rightarrow\!\!\odot\!\!\Rightarrow x)]$

This means that $G(.,.)$ is antitone in the first and isotone in the second variable or equivalently

$$G_a \text{ is antitone} \quad \text{and} \quad G^a \text{ is isotone}$$

We can see the second relation in $(*)$ as follows: Let $a \leq b$ then,

$(**)$ $$F_a \leq F_b$$

This in turn implies:

$$G_b \leq G_a \circ F_a \circ G_b \leq G_a \circ F_b \circ G_b \leq G_a.$$

Thus $G_b \leq G_a$ or $(b \Rightarrow\!\!\odot\!\!\Rightarrow x) \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow x)$.

More precisely, all functions F_a, G_a, F_b, G_b are isotone functions, thus taking the composite from the left and right of $(**)$ gives again an inequality, i.e.

$$\begin{aligned} F_a \leq F_b \quad &\Rightarrow \quad G_a \circ F_a \leq G_a \circ F_b \\ &\Rightarrow \quad G_a \circ F_a \circ G_b \leq G_a \circ F_b \circ G_b \end{aligned}$$

but due to the triangle inequalities we have

$$G_a \circ F_a \geq \mathrm{id}_P \qquad \text{and} \qquad F_b \circ G_b \leq \mathrm{id}_Q.$$

So that,

$$G_b \leq G_a \circ F_a \circ G_b \leq G_a \circ F_b \circ G_b \leq G_a.$$

Thus,

$$G_b \leq G_a \qquad \dashv\!\!\blacksquare.$$

SUMMARY OF THE BASIC PROPERTIES

We summarize results which have beeen already prooved or will be proved for easy refference. If something reffers to e.g. (i) etc., it will reffers to these properties:

(AD) $a \odot x \leq b$ iff $x \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow b)$ for all $a \in \mathbb{L}$,
i.e. $(\forall a \in \mathbb{L})[F_a \dashv G_a]$

(i) po-monoid: $\langle \mathbb{L}; \leq, \odot, 1\rangle$ is a po-monoid, i.e.

$$(\forall a, b, x \in \mathbb{L})[a \leq b \ \Rightarrow \ a \odot x \leq b \odot x]$$

(ii) Triangle Inequalities: For all $a \in \mathbb{L}$,

$$\left.\begin{array}{lll} a \Rightarrow\!\!\odot\!\!\Rightarrow (a \odot x) \geq x & i.e. & G_a \circ F_a \geq \mathrm{id}_P \\ a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow x) \leq x & i.e. & F_a \circ G_a \leq \mathrm{id}_Q \end{array}\right\}$$

(iii) For all $a, b, c \in \mathbb{L}$

$$(a \Rightarrow\!\!\odot\!\!\Rightarrow (b \Rightarrow\!\!\odot\!\!\Rightarrow c)) = ((a \odot b) \Rightarrow\!\!\odot\!\!\Rightarrow c) = (b \Rightarrow\!\!\odot\!\!\Rightarrow (a \Rightarrow\!\!\odot\!\!\Rightarrow c))$$

(iv) $a \odot (\cdot)$ *preserves finite joins and infinite suprema when exist*, i.e., when the sups involed exist,

$$a \odot \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (a \odot x_i)$$

(v) $a \Rightarrow\!\!\odot\!\!\Rightarrow (\cdot)$ *preserves finite meets and infinite infima when exist*, i.e., when the infs involed exist,

$$a \Rightarrow\!\!\odot\!\!\Rightarrow \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I} (a \Rightarrow\!\!\odot\!\!\Rightarrow x_i)$$

In addition we have: $(a \vee b) \Rightarrow\!\!\odot\!\!\Rightarrow c = (a \Rightarrow\!\!\odot\!\!\Rightarrow c) \wedge (b \Rightarrow\!\!\odot\!\!\Rightarrow c)$
where, $a \odot x := \bigwedge\{u \in \mathbb{L} : x \leq a \Rightarrow\!\!\odot\!\!\Rightarrow y\}$ and $(a \Rightarrow\!\!\odot\!\!\Rightarrow x) := \bigvee\{y \in \mathbb{L} : a \odot y = x\}$.

(vi) For all $x, y \in \mathbb{L}$,

$$x \odot (x \Rightarrow\!\!\odot\!\!\Rightarrow y) = y \quad \text{iff} \quad (\exists z \in \mathbb{L})[y = x \odot z]$$

(vii) For all $x, y \in \mathbb{L}$,

$$x \Rightarrow\!\!\odot\!\!\Rightarrow (x \odot y) = y \quad \text{iff} \quad (\exists z \in \mathbb{L})[y = (x \Rightarrow\!\!\odot\!\!\Rightarrow z)]$$

Integrality implies:

(iix) $a \odot b \leq a \wedge b$.

(ix) $a \leq b$ iff $a \mathrel{=\!\odot\!\!\Rightarrow} b = 1$.

(x) The universal bound $\perp$ is the zero element with respect to $\odot$, i.e.: $\perp = 0$.

Divisibility implies:

(xi) $a \odot (a \mathrel{=\!\odot\!\!\Rightarrow} b) = a \wedge b$.

(xii) $a \odot a = a$ then $(\forall b \in \mathbb{L})[a \wedge b = a \odot b]$

(xiii) $a_1 \leq a_2$ implies $a_1 \odot b = a_1 \odot (a_2 \mathrel{=\!\odot\!\!\Rightarrow} (a_2 \odot b))$

(xiv) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$

(xv) $a \mathrel{=\!\odot\!\!\Rightarrow} (b \wedge c) = (a \mathrel{=\!\odot\!\!\Rightarrow} b) \odot ((a \wedge b) \mathrel{=\!\odot\!\!\Rightarrow} c)$

(MV) $((a \mathrel{=\!\odot\!\!\Rightarrow} b) \mathrel{=\!\odot\!\!\Rightarrow} b) = a \vee b$.

Finally if $a \leq b$ then $(a \Rightarrow x) \geq (b \Rightarrow x)$, i.e. $G_a \geq G_b$ and $(x \Rightarrow a) \leq (x \Rightarrow b)$ i.e. . $G^a \leq G^b$.

3.4.6 Theorem. *Let $\langle \mathbb{L}, \leq, \odot, 1 \rangle$ be an commutative, residuated $\ell - monoid$ and $x, y \in \mathbb{L}$, then:*

(i) $x \odot (x \mathrel{=\!\odot\!\!\Rightarrow} y) = y$ *iff* $(\exists z \in \mathbb{L})[y = x \odot z]$
(ii) $x \mathrel{=\!\odot\!\!\Rightarrow} (x \odot y) = y$ *iff* $(\exists z \in \mathbb{L})[y = (x \mathrel{=\!\odot\!\!\Rightarrow} z)]$
(iii) $(y \mathrel{=\!\odot\!\!\Rightarrow} x) \mathrel{=\!\odot\!\!\Rightarrow} x = y$ *iff* $(\exists z \in \mathbb{L})[y = (z \mathrel{=\!\odot\!\!\Rightarrow} x)]$

Proof: If $f : A \longrightarrow B$ is a residuated map then by Theorem 2.6 in Blyth & Janowitz, the following $(i), (ii), (iii)$ as well as $(i'), (ii'), (iii')$ are equivalent:

(i) $f^+ \circ f = \mathrm{id}_A$		**(i′)**	$f \circ f^+ = \mathrm{id}_B$
(ii) f is injective	&	**(ii′)**	f is surjective
(iii) f^+ is surjective		**(iii′)**	f^+ is injective.

Thus,

$$f^+ \circ f = \mathrm{id}_A \quad \text{iff} \quad f^+ \text{ is surjective iff } (\forall x \in A)(\exists y \in B)[x = f^+(y)]$$
$$f \circ f^+ = \mathrm{id}_B \quad \text{iff} \quad f \text{ is surjective iff } (\forall y \in B)(\exists x \in A)[y = f(x)]$$

Applying these observations in turn to $F_x, F_x^+ \equiv G_x$ we have

(i) $(\forall y \in \mathbb{L})\ [F_x(G_x(y)) = y]$ iff $(\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = F_x(z)]$ or
$(\forall y \in \mathbb{L})\ [x \odot (x \mathrel{=\!\odot\!\!\Rightarrow} y) = y]$ iff $(\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = x \odot z]$ $\dashv\Box$

(ii) Similarly, $(\forall y \in \mathbb{L})\ [G_x(F_x(y)) = y]$ iff $(\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = G_x(z)]$ or
$(\forall y \in \mathbb{L})\ [x \Rightarrow (x \odot y) = y]$ iff $(\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = x \Rightarrow z]$ ⊣▯

(iii) Since,

$$x \odot b \leq a \quad \text{iff } b \leq x \Rightarrow a \quad \text{iff } x \leq (b \Rightarrow a)$$

If we define,

$$g_a : \mathbb{L}^{op} \longrightarrow \mathbb{L}\ /\!\!/\ x \mapsto g_a(x) := (x \Rightarrow a)$$

and,

$$g_a^+ : \mathbb{L} \longrightarrow \mathbb{L}^{op}\ /\!\!/\ x \mapsto g_a^+(x) := (x \Rightarrow a)$$

then, $g_a(x) \leq^{op} b$ iff $x \leq g_a^+(b)$ *i.e.* $g_a \dashv g_a^+$.

Applying the above to g_a, g_a^+ we have

$$(\forall y \in \mathbb{L})\ [g_x(g_x^+(y)) = y] \qquad \text{iff} \qquad (\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = g_x(z)]$$

or

$$((y \Rightarrow x) \Rightarrow x) = y \qquad \text{iff} \qquad (\forall y \in \mathbb{L})(\exists z \in \mathbb{L})\ [y = (z \Rightarrow x)]. \quad ⊣▮$$

3.4.7 Theorem. *The following conditions are equivalent:*

(i) $\langle \mathbb{L}, \leq, \odot, 1\rangle$ *is a commutative residuated ℓ-monoid;*

(ii) $(\forall x, y, z \in \mathbb{L})\ [z \Rightarrow (y \Rightarrow x) = ((y \odot z) \Rightarrow x)]$*;*

(iii) $(\forall x, y, z \in \mathbb{L})\ [z \Rightarrow (y \Rightarrow x) = (y \Rightarrow (z \Rightarrow x))]$*;*

(iv) $(\forall x, y, z \in \mathbb{L})\ [z \odot (y \Rightarrow x) \leq (y \odot (z \Rightarrow x)) \leq (z \Rightarrow (y \odot x))]$

Proof: **(i) ⇒ (ii):** (First proof) Due to commutativity and associativity we have (Th. 6.6)

$$(\forall y, z \in \mathbb{L})\ [F_y \circ F_z = F_z \circ F_y] = F_{y \odot z}$$

Since F_y, F_z are residuated maps then $F_y \circ F_z = F_z \circ F_y$ is also residuated with residual,

$$(*) \qquad G_z \circ G_y = G_y \circ G_z = G_{y \odot z}$$

But this says that, for all $x, y, z \in \mathbb{L}$,

$$z \Rightarrow (y \Rightarrow x) = (y \odot z) \Rightarrow x. \quad ⊣▯$$

(i) ⇒ (iii): By $(*)$ we get also the equivalence of (i) ⇔ (iii), i.e. from $G_z \circ G_y = G_y \circ G_z$ we get,

$$z \Rightarrow (y \Rightarrow x) = y \Rightarrow (z \Rightarrow x).$$

(Second proof of (i) ⇒ (ii): We can see that: For all $a, b \in \mathbb{L}$

$$a = b \quad \text{iff} \quad (\forall x \in \mathbb{L})\ [x \leq a \ \text{ iff } \ x \leq b] \qquad (\text{ see Remark 3.1})$$

Thus in order to prove that $(z \Rightarrow\!\!\!\odot\!\!\Rightarrow (y \Rightarrow\!\!\!\odot\!\!\Rightarrow x)) = ((y \odot z) \Rightarrow\!\!\!\odot\!\!\Rightarrow x)$ we have to prove that for all $\omega \in \mathbb{L}$,

$$\omega \leq (z \Rightarrow\!\!\!\odot\!\!\Rightarrow (y \Rightarrow\!\!\!\odot\!\!\Rightarrow x)) \quad \text{iff} \quad \omega \leq ((y \odot z) \Rightarrow\!\!\!\odot\!\!\Rightarrow x)$$

But $\omega \leq (z \Rightarrow\!\!\!\odot\!\!\Rightarrow (y \Rightarrow\!\!\!\odot\!\!\Rightarrow x)) \overset{\text{(AD)}}{\Leftrightarrow} \omega \odot z \leq (y \Rightarrow\!\!\!\odot\!\!\Rightarrow x) \overset{\text{(AD)}}{\Leftrightarrow} \omega \odot y \odot z \leq x \overset{\text{(AD)}}{\Leftrightarrow} \omega \leq ((y \odot z) \Rightarrow\!\!\!\odot\!\!\Rightarrow x)$.

We note that we could easily prove that (ii) $\Leftrightarrow$ (iii). Indeed from (ii) we get

$$(z \Rightarrow\!\!\!\odot\!\!\Rightarrow (y \Rightarrow\!\!\!\odot\!\!\Rightarrow x)) = ((z \odot y) \Rightarrow\!\!\!\odot\!\!\Rightarrow x).$$

But $z \odot y = y \odot z$ since $\mathbb{L}$ is commutative. Thus:

$$(z \odot y) \Rightarrow\!\!\!\odot\!\!\Rightarrow x = (y \odot z) \Rightarrow\!\!\!\odot\!\!\Rightarrow x \overset{\text{(i)}}{=} y \Rightarrow\!\!\!\odot\!\!\Rightarrow (z \Rightarrow\!\!\!\odot\!\!\Rightarrow x) \equiv \text{(iii)}$$

and conversely. Or alternatively use $(\odot)$ to prove (ii) $\Leftrightarrow$ (iii). ⊣▯

Before we prove (i) $\Leftrightarrow$ (iv), we shall prove a lemma.

3.4.8 Lemma. *If $\langle \mathbb{L}, \leq \rangle$ is a poset and $F_1, F_2 \in \mathbb{L}^{\mathbb{L}}$ are residuated maps with residuals $G_1 \equiv F_1^+, G_2 \equiv F_2^+$ respectively, then*

$$F_1 \circ F_2 \leq F_2 \circ F_1 \quad \textit{iff} \quad F_2 \circ G_1 \leq G_1 \circ F_2$$

Proof: Let $F_1 \circ F_2 \leq F_2 \circ F_1$ then

$$\begin{aligned} id_{\mathbb{L}} \circ F_2 \circ G_1 &\leq G_1 \circ F_1 \circ F_2 \circ G_1 \quad \text{(triangle inequality)} \\ &= G_1 \circ (f_1 \circ F_2) \circ G_1 \leq G_1 \circ (F_2 \circ F_1) \circ G_1 \quad \text{by hypothesis} \\ &= (G_1 \circ F_2) \circ (F_1 \circ G_1) \leq G_1 \circ F_2 \quad \text{since } F_1 \circ G_1 \leq id_{\mathbb{L}} \end{aligned}$$

and conversely, if $F_2 \circ G \leq G_1 \circ F_2$ then,

$$\begin{aligned} F_1 \circ F_2 \leq F_1 \circ F_2 \circ (id_{\mathbb{L}}) &\leq F_1 \circ F_2 \circ (G_1 \circ F_1) \\ &\leq F_1 \circ G_1 \circ F_2 \circ F_1 \\ &\leq F_2 \circ F_1. \end{aligned}$$ ⊣▮

(i) $\Leftrightarrow$ (iv):(continuation of Th.Theorem 3.4.7) The commutativity and associativity is equivalent to $(\forall y, z)\ [F_y \circ F_z = F_z \circ F_y]$. But this is equivalent with

$$F_y \circ F_z \leq F_z \circ F_y \quad \text{and} \quad F_z \circ F_y \leq F_y \circ F_z.$$

Using the Lemma, these are equivalent to

$$F_z \circ G_y \leq G_y \circ F_z \quad \text{and} \quad F_y \circ G_z \leq G_z \circ F_y$$

which are exactly relation (iv). ⊣▮

3.4.9 Remark. If $\langle \mathbb{L}, \leq, \odot \rangle$ is a commutative residuated ℓ-monoid then it is also a po-monoid, see also,Remark 3.3.3 i.e. :

$$a \leq b \;\Rightarrow\; a \odot x \leq b \odot x \qquad \text{for all } \; x \in \mathbb{L}.$$

Indeed, from triangle inequalities we have:

$$(\forall b, x \in \mathbb{L})\ [b \leq (x \mathrel{=\!\odot\!\Rightarrow} (b \odot x))]$$

By the transitivity of $\leq$, we get:

$$a \leq [x \mathrel{=\!\odot\!\Rightarrow} (b \odot x)] \;\overset{\text{(AD)}}{\Longleftrightarrow}\; a \odot x \leq b \odot x,$$

hence, $\langle \mathbb{L}, \leq, \odot \rangle$ is a po-monoid.

3.4.10 Example. Any complete abelian po-group $\langle G, \cdot \rangle$ is a commutative quantale with

$$F_a(x) := a \cdot x \qquad \text{and} \qquad G_a(x) \equiv (a \mathrel{=\!\odot\!\Rightarrow} x) := a^{-1} \cdot x.$$

3.4.11 Proposition. *For all $a \in \mathbb{L}$, let $C_a(\cdot) : \mathbb{L} \longrightarrow \mathbb{L} \;/\!\!/\; x \mapsto C_a(x) := ((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a)$, then $C_a(\cdot)$ is a closure operator.*

Proof: **(i):** $C_a(\cdot)$ preserves order, since if $x_1 \leq x_2$ then

$$(x_1 \mathrel{=\!\odot\!\Rightarrow} a) \geq (x_2 \mathrel{=\!\odot\!\Rightarrow} a) \quad \text{and so} \quad a \mathrel{=\!\odot\!\Rightarrow} (x_1 \mathrel{=\!\odot\!\Rightarrow} a) \leq a \mathrel{=\!\odot\!\Rightarrow} (x_2 \mathrel{=\!\odot\!\Rightarrow} a).$$

but $(a \mathrel{=\!\odot\!\Rightarrow} (x_1 \mathrel{=\!\odot\!\Rightarrow} a)) = ((x_1 \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a)$ and $(a \mathrel{=\!\odot\!\Rightarrow} (x_2 \mathrel{=\!\odot\!\Rightarrow} a)) = ((x_2 \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a)$, thus $C_a(x_1) \leq C_a(x_2)$ ⊣▯

(ii): $(\forall x \in \mathbb{L})\ [x \leq C_a(x)]$. This is clear since from triangle inequality we have

$$x \odot (x \mathrel{=\!\odot\!\Rightarrow} a) \leq a \;\overset{\text{(AD)}}{\Longleftrightarrow}\; x \leq ((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a).$$

(iii): Idempotency. We have to prove that: $C_a \circ C_a = C_a$ or

$$(x \mathrel{=\!\odot\!\Rightarrow} a) = (((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a).$$

From (ii) since $x \leq ((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a)$ it follows that

$$((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a) \leq (x \mathrel{=\!\odot\!\Rightarrow} a) \quad \text{since } (-) \mathrel{=\!\odot\!\Rightarrow} a \;\text{ is antitone.}$$

The reverse inequality follows, since from triangle inequality we have:

$$(x \mathrel{=\!\odot\!\Rightarrow} a) \odot ((x \mathrel{=\!\odot\!\Rightarrow} a) \mathrel{=\!\odot\!\Rightarrow} a) \leq a. \qquad ⊣▮$$

3.5 Integrality.

Let $\langle \mathbb{L}, \leq, \odot, 1\rangle$ be a commutative residuated ℓ-monoid. Let also $\top$ be the top element in the lattice $\langle \mathbb{L}, \leq\rangle$. The interrelationships of the top element and the unit element 1 of the monoid $\langle \mathbb{L}, \odot, 1\rangle$ is interesting.

We may say that our element a of $\langle \mathbb{L}, \odot, 1\rangle$ is **integral** iff $a \leq 1$. If all elements of $\langle \mathbb{L}, \odot, 1\rangle$ are integral, i.e. $(\forall a \in \mathbb{L})\ [a \leq 1]$ then $\langle \mathbb{L}, \odot, 1\rangle$ is called **integral ℓ-monoid**.

If $\top \equiv 1$ then an ℓ-monoid is integral. More precisely we have:

3.5.1 Definition. A residuated, commutative ℓ-monoid $\langle \mathbb{L}, \leq, \odot\rangle$ is called **integral** iff $\top \equiv 1$, i.e. the universal upper bound $\top$ acts as the unit element w.r.t. $\odot$.

3.5.2 Lemma. *Let $M = \langle \mathbb{L}, \leq, \odot\rangle$ be a commutative residuated ℓ-monoid, and 1 be the monoidal unit element, whereas $\top$ is the universal upper bound of the lattice $\langle \mathbb{L}, \leq\rangle$. For every element $x \in \mathbb{L}$,*

(a) The following assertions are equivalent:

(a1) $x = 1$;

(a2) $x \leq (a \Rightarrow_{\odot} b)$ *iff* $a \leq b$;

(a3) $a = (x \Rightarrow_{\odot} a)$ *for all* $a \in \mathbb{L}$.

(b) Likewise the following assertions are equivalent:

(b1) $\langle \mathbb{L}, \leq, \odot\rangle$ *is integral;*

(b2) $1 = (a \Rightarrow_{\odot} b)$ *iff* $a \leq b$;

(b3) $a = (1 \Rightarrow_{\odot} a)$ *for all* $a \in \mathbb{L}$.

(c) If there exists $x \in \mathbb{L}$, with $\top \odot x = 1$ then $\top = 1$.

PROOF: Assertion (b) is an immediate consequence of assertion (a). In order to verify (a) we proceed as follows:

(a1) $\Rightarrow$ (a2). This is obvious, since if $x = 1$ then by (AD) we have

$$a \odot 1 \leq b \qquad \text{iff} \qquad 1 \leq a \Rightarrow_{\odot} b.$$

(a2) $\Rightarrow$ (a3). Let (a2), then since $a \leq a$ iff $x \leq a \Rightarrow_{\odot} a$ iff $a \leq x \Rightarrow_{\odot} a$.

To prove now that $(x \Rightarrow_{\odot} a) \leq a$ we use the triangle inequality

$$x \odot (x \Rightarrow_{\odot} a) \leq a \ \text{ iff } \ x \leq ((x \Rightarrow_{\odot} a) \Rightarrow_{\odot} a) = (a \Rightarrow_{\odot} (x \Rightarrow_{\odot} a)).$$

(a3) $\Rightarrow$ (a1). Since $a = (x \Rightarrow a)$ for all $a \in \mathbb{L}$ then by Th. 6.12 (iii) with $y = a$, $z = x,\ x = a$ we have

$$((a \Rightarrow a) \Rightarrow a) = a$$

By Theorem 3.4.6 (ii), we have with $y = a, x = x, z = a$,

$$x \Rightarrow (x \odot a) = a$$

But $x \odot a = (x \Rightarrow (x * a))$, then $x \odot a = a$ or $x = 1$.

(c) Because $\langle \mathbb{L}, \leq, \odot \rangle$ is a po-monoid we have

$$\top = \top \odot 1 \leq \top \odot \top \leq \top.$$

On the other hand we infer from the hypothesis of assertion (c): $\top \odot (\top \Rightarrow 1) = 1$. Multiplying both sides by $\top$ we have

$$\top = 1 \odot \top = \top \odot \top \odot (\top \Rightarrow 1) = \top \odot (\top \Rightarrow 1) = 1. \qquad \dashv$$

- In any integral, residuated, commutative ℓ-monoid the following relation holds:

  $(*)$ $$a \odot b \leq a \wedge b$$

- Due to integrality $\forall a, b \in \mathbb{L},\ b \leq 1$ iff $b \leq a \Rightarrow a$ iff $a \odot b \leq a$ or since $a \odot b \leq a \wedge b$ then

  $$a \odot b \leq a, \quad \& a \odot b \leq b$$

  From $(*)$ we have:

  $$(a \wedge b) \Rightarrow x \leq (a \odot b) \Rightarrow x \quad \text{and} \quad x \Rightarrow (a \odot b) \leq x \Rightarrow (a \wedge b)$$

  So that:

  $$x \odot (x \Rightarrow (a \odot b) \leq a \wedge b \leq k, \quad \text{where} \quad k \in \{a, b\}$$

  and

  $$(a \odot b) \odot [(a \wedge b) \Rightarrow x] \leq x.$$

3.6 Algebraic strong de Morgan's law: Distributivity of the underline lattice structure

The logical principle:

$$(p \Rightarrow q) \vee (q \Rightarrow p) = 1$$

is clearly true in Boolean algebras. However it need not hold in a frame or in general in an integral commutative, residuated ℓ-monoid. This principle was termed by Johnstone [SLNM # 753 (1979), 479–491] Strong de Morgan's law (SDML).

A related property to SDML is the property:

$$\neg(a \wedge b) = \neg a \vee \neg b \qquad (DML).$$

(The first de Morgan's law $\neg(a \vee b) = \neg a \wedge \neg b$ automatically holds in $\mathbb{L}$ by the adjointness of $\wedge$ and $\Rightarrow\!\!\odot\!\!\Rightarrow$.)

P. Johnstone proved that:

"Let X be a topological space. The frame $\Omega(X)$ of open sets of X satisfies DML iff X is extremally disconnected" (disjoint opens have disjoint closures).

3.6.1 Definition. An integral, commutative, residuated ℓ-monoid is said to satisfy the **algebraic strong de Morgan's law (ASDML)** iff for all $a, b \in \mathbb{L}$,

$$(a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (b \Rightarrow\!\!\odot\!\!\Rightarrow a) = 1.$$

3.6.2 Proposition. *In any integral, commutative, residuated ℓ-monoid the following assertions are equivalent:*

(i) $(a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (b \Rightarrow\!\!\odot\!\!\Rightarrow a) = 1$ *for all* $a, b \in \mathbb{L}$.

(ii) $a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c) = (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow c)$ *for all* $a, b, c \in \mathbb{L}$.

(iii) $a \wedge b) \Rightarrow\!\!\odot\!\!\Rightarrow c = (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow c)$ *for all* $a, b, c \in \mathbb{L}$.

Proof: **(i)** $\Rightarrow$ **(ii):** Since $a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \leq b \leq b \vee c$ and $a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow c) \leq c \leq b \vee c$

we have,

$$\left.\begin{array}{lcl}(a \Rightarrow\!\!\odot\!\!\Rightarrow b) & \leq & a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c) \\ (a \Rightarrow\!\!\odot\!\!\Rightarrow c) & \leq & a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)\end{array}\right\} \Rightarrow (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow c) \leq q \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)$$

i.e. holds always.

Thus we have to prove that,

$$a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c) \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow c).$$

Since $a \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)) \leq b \vee c$ (triang. ineq.) then,

$$a \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)) \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c).$$

So that

$$1 = a \Rightarrow\!\!\odot\!\!\Rightarrow a \quad \leq \quad a \Rightarrow\!\!\odot\!\!\Rightarrow [\,(a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)) \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c)\,]$$

since $b \vee c \leq (c \Rightarrow\!\!\odot\!\!\Rightarrow b) \Rightarrow\!\!\odot\!\!\Rightarrow b)$

$$\leq \quad a \mathrel{=\!\odot\!\!\Rightarrow} [\,(a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} ((c \mathrel{=\!\odot\!\!\Rightarrow} b) \mathrel{=\!\odot\!\!\Rightarrow} b)\,]$$

since $a \mathrel{=\!\odot\!\!\Rightarrow} (y \mathrel{=\!\odot\!\!\Rightarrow} z) = y \mathrel{=\!\odot\!\!\Rightarrow} (x \mathrel{=\!\odot\!\!\Rightarrow} z)$

$$= \quad a \mathrel{=\!\odot\!\!\Rightarrow} [\,(c \mathrel{=\!\odot\!\!\Rightarrow} b) \mathrel{=\!\odot\!\!\Rightarrow} ((a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} b\,]$$

since $a \mathrel{=\!\odot\!\!\Rightarrow} (y \mathrel{=\!\odot\!\!\Rightarrow} z) = y \mathrel{=\!\odot\!\!\Rightarrow} (x \mathrel{=\!\odot\!\!\Rightarrow} z)$

$$= \quad (c \mathrel{=\!\odot\!\!\Rightarrow} b) \mathrel{=\!\odot\!\!\Rightarrow} [\,a \mathrel{=\!\odot\!\!\Rightarrow} ((a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} b\,]$$

Thus,

$$(*) \qquad 1 \odot (c \mathrel{=\!\odot\!\!\Rightarrow} b) = c \mathrel{=\!\odot\!\!\Rightarrow} b \leq a \mathrel{=\!\odot\!\!\Rightarrow} [\,(a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} b]$$

and by reversing the rôles of b and c we get

$$(**) \qquad (b \mathrel{=\!\odot\!\!\Rightarrow} c) \leq a \mathrel{=\!\odot\!\!\Rightarrow} [\,(a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} c\,]$$

Thus,

$$\begin{aligned}
&(a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} ((a \mathrel{=\!\odot\!\!\Rightarrow} b) \vee (a \mathrel{=\!\odot\!\!\Rightarrow} c)) \\
&\quad \geq (a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} (a \mathrel{=\!\odot\!\!\Rightarrow} b) \vee (a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} (a \mathrel{=\!\odot\!\!\Rightarrow} c) \\
&\quad = a \mathrel{=\!\odot\!\!\Rightarrow} ((a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} b) \vee a \mathrel{=\!\odot\!\!\Rightarrow} ((a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c)) \mathrel{=\!\odot\!\!\Rightarrow} c) \\
&\quad \geq (c \mathrel{=\!\odot\!\!\Rightarrow} b) \vee (b \mathrel{=\!\odot\!\!\Rightarrow} c) = 1, \qquad \text{by ASDML .}
\end{aligned}$$

It thus follows that

$$a \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c) \leq (a \mathrel{=\!\odot\!\!\Rightarrow} b) \vee (a \mathrel{=\!\odot\!\!\Rightarrow} c).$$

(ii) $\Rightarrow$ (iii):

$$\begin{aligned}
((b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a) \quad &\mathrel{=\!\odot\!\!\Rightarrow} \quad ((b \mathrel{=\!\odot\!\!\Rightarrow} a) \vee (c \mathrel{=\!\odot\!\!\Rightarrow} a) \\
&= \quad ((b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a) \mathrel{=\!\odot\!\!\Rightarrow} (b \mathrel{=\!\odot\!\!\Rightarrow} a) \vee ((b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a) \mathrel{=\!\odot\!\!\Rightarrow} (c \mathrel{=\!\odot\!\!\Rightarrow} a) = \\
&= \quad b \mathrel{=\!\odot\!\!\Rightarrow} (((b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a) \mathrel{=\!\odot\!\!\Rightarrow} a) \vee c \mathrel{=\!\odot\!\!\Rightarrow} (((b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a) \mathrel{=\!\odot\!\!\Rightarrow} a) \\
&\geq \quad (b \mathrel{=\!\odot\!\!\Rightarrow} (b \wedge c)) \vee (c \mathrel{=\!\odot\!\!\Rightarrow} (b \wedge c)) \\
&= \quad (b \mathrel{=\!\odot\!\!\Rightarrow} c) \vee (c \mathrel{=\!\odot\!\!\Rightarrow} b) = (b \vee c) \mathrel{=\!\odot\!\!\Rightarrow} c \vee (b \vee c) \mathrel{=\!\odot\!\!\Rightarrow} b \\
&= \quad (b \vee c) \mathrel{=\!\odot\!\!\Rightarrow} (b \vee c) = 1
\end{aligned}$$

So, $(b \wedge c) \mathrel{=\!\odot\!\!\Rightarrow} a \leq (b \mathrel{=\!\odot\!\!\Rightarrow} a) \vee (c \mathrel{=\!\odot\!\!\Rightarrow} a)$. The opposite inequality is always true.

(iii) $\Rightarrow$ (i): $(a \mathrel{=\!\odot\!\!\Rightarrow} b) \vee (b \mathrel{=\!\odot\!\!\Rightarrow} a) \geq (a \mathrel{=\!\odot\!\!\Rightarrow} (a \wedge b)) \vee (b \mathrel{=\!\odot\!\!\Rightarrow} a \wedge b)) = (a \wedge b) \mathrel{=\!\odot\!\!\Rightarrow} (a \wedge b) = 1.$ ⊣▌

3.6.3 Lemma. *Let $\langle \mathbb{L}, \leq \odot \rangle$ be an integral, commutative residuated ℓ-monoid, satisfying the algebraic strong de Morgan's law. Then the following assertions are valid:*

(i) $a \odot b \leq (a \odot a) \vee (b \odot b),\ (a \odot a) \wedge (b \odot b) \leq a \odot b$;

(ii) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$;

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *i.e. the lattice* $(\mathbb{L}, \leq)$ *is distributive.*

(iv) For all $x, y \in \mathbb{L}$,

$$x \vee y = [\,(x \Rightarrow_{\odot} y) \Rightarrow_{\odot} y\,] \wedge [\,(y \Rightarrow_{\odot} x) \Rightarrow_{\odot} x\,]$$

Proof: **(i):**

$$\begin{aligned} a \odot b &= (a \odot b) \odot 1 = (a \odot b)[(a \Rightarrow_{\odot} b) \vee (b \Rightarrow_{\odot} a)] \\ &= [\,(a \odot b) \odot (a \Rightarrow_{\odot} b)] \vee [(a \odot b) \odot (b \Rightarrow_{\odot} a)\,] \\ &\leq [\,b \odot (a \odot (a \Rightarrow_{\odot} b)))\,] \vee [\,a \odot (b \odot (b \Rightarrow_{\odot} a))\,] \\ &\leq (b \odot b) \vee (a \odot a) \end{aligned}$$

$$\begin{aligned} (a \odot a) \wedge (b \odot b) &= [(a \odot a) \wedge (b \odot b) \odot 1 = [(a \odot a) \wedge (b \odot b)] \odot (a \Rightarrow_{\odot} b) \vee (b \Rightarrow_{\odot} a)] \\ &= [[(a \odot a) \wedge (b \odot b)] \odot (a \Rightarrow_{\odot} b)] \vee [[\,(a \odot a) \wedge (b \odot b)\,] \odot (b \Rightarrow_{\odot} a)] \\ &\leq [\,(a \odot a) \odot (a \Rightarrow_{\odot} b)\,] \vee [\,(b \odot b) \odot (b \Rightarrow_{\odot} a)\,] \\ &= [\,a \odot (a \odot (a \Rightarrow_{\odot} b))\,] \vee [\,b \odot (b \odot (b \Rightarrow_{\odot} a))\,] \\ &\leq [\,a \odot b\,] \vee [\,b \odot a\,] = a \odot b \end{aligned}$$

$$(a \odot a) \wedge (b \odot b) \leq a \odot b. \qquad \dashv\!\square$$

(ii):

$$\begin{aligned} (a \odot b) \wedge (a \odot c) &= [\,(a \odot b) \wedge (a \odot c)\,] \odot \overbrace{[\,(b \Rightarrow_{\odot} c) \vee (c \Rightarrow_{\odot} b)\,]}^{=1} \\ &= [\,((a \odot b) \wedge (a \odot c)) \odot (b \Rightarrow_{\odot} c)\,] \vee [\,((a \odot b) \wedge (a \odot c)) \odot (c \Rightarrow_{\odot} b)\,] \\ &\leq [\,(a \odot b) \odot (b \Rightarrow_{\odot} c)] \vee [(a \odot c) \odot (c \Rightarrow_{\odot} b)\,] \\ &\leq (a \odot c) \vee (a \odot b) = a \odot (b \odot c) \leq a \odot (b \wedge c). \end{aligned}$$

Since $\odot$ is isotone in both variables,

$$(a \odot b) \wedge (a \odot c) \leq a \odot (b \wedge c).$$

The other way is trivial. $\dashv\!\square$

Other proof: Since $\odot$ is isotone and $b \geq b \wedge c$ and $c \geq b \wedge c$ then,

$$\left.\begin{array}{lcl} a \odot b & \geq & a \odot (b \wedge c) \\ a \odot c & \geq & a \odot (b \wedge c) \end{array}\right\} \Rightarrow_{\odot} (a \odot b) \wedge (a \odot c) \geq a \odot (b \wedge c) \qquad \dashv\!\square$$

(iii): It is well known that to prove distributivity, it is sufficient to show that if $b \wedge c \leq a$ then $(a \vee b) \wedge (a \vee c) \leq a$. However,

$$\begin{aligned} (a \wedge (b \wedge c)) \Rightarrow_{\odot} ((a \wedge b) \vee (a \wedge c)) &= [(a \wedge (b \vee c)) \Rightarrow_{\odot} (a \wedge b)] \vee [(a \wedge (b \wedge c) \Rightarrow_{\odot} (a \wedge c)] \\ &\geq [(b \wedge c) \Rightarrow_{\odot} (a \wedge b)] \vee [(b \wedge c) \Rightarrow_{\odot} (a \wedge c)] \end{aligned}$$

since $(-) \Rightarrow_{\odot} x$ is antitone

$$\geq \;\; [(b \wedge c) \Rightarrow\!\!\odot\!\!\Rightarrow b] \vee [(b \wedge c) \Rightarrow\!\!\odot\!\!\Rightarrow c] \;\; \text{since } x \Rightarrow\!\!\odot\!\!\Rightarrow (-) \;\text{ is isotone}$$
$$\geq \;\; (c \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (b \Rightarrow\!\!\odot\!\!\Rightarrow c) = 1 \qquad \text{since } x \Rightarrow\!\!\odot\!\!\Rightarrow (-) \;\text{ is isotone}$$

$$a \wedge (b \wedge c) \leq (a \wedge b) \vee (a \wedge c). \qquad \dashv\Box$$

(iv): First we shall prove the inequality:

$$x \vee y \leq [\,(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y\,] \wedge [\,(y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x\,]$$

Due to integrality we have for all $z, y \in \mathbb{M}$,

$$z \odot y \leq z \wedge y \leq y$$

From adjunction (AD) we have,

$$z \odot y \leq y \quad \text{iff} \quad y \leq z \Rightarrow\!\!\odot\!\!\Rightarrow y$$

Choosing $z = (x \Rightarrow\!\!\odot\!\!\Rightarrow y)$ we finally get,

$$y \leq (x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y \tag{$*$}$$

On the other hand, adjunction (triangle inequality) implies that,

$$x \odot (x \Rightarrow\!\!\odot\!\!\Rightarrow y) \leq y \quad \overset{(AD)}{\Longleftrightarrow} \quad x \leq (x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y \tag{$**$}$$

So by $(*)$ and $(**)$ we get,

$$x \vee y \leq (x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y$$

Interchanging the roles of x, y we also get:

$$x \vee y \leq (y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x$$

hence we obtain,

$$x \vee y \leq [(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y] \wedge [(y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x]$$

For the converse inequality, we have:

$$\begin{aligned}
&[\,(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y\,] \wedge [\,(y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x\,] = \\
&\quad [\,(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \vee (y \Rightarrow\!\!\odot\!\!\Rightarrow x)\,] \odot \{[\,(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y\,] \wedge [\,(y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x\,]\} \\
&\quad \leq [\,(x \Rightarrow\!\!\odot\!\!\Rightarrow y) \odot ((x \Rightarrow\!\!\odot\!\!\Rightarrow y) \Rightarrow\!\!\odot\!\!\Rightarrow y)\,] \vee [\,(y \Rightarrow\!\!\odot\!\!\Rightarrow x) \odot ((y \Rightarrow\!\!\odot\!\!\Rightarrow x) \Rightarrow\!\!\odot\!\!\Rightarrow x)\,] \\
&\quad \leq y \vee x = x \vee y. \qquad \dashv\blacksquare
\end{aligned}$$

3.7 Divisibility.

Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative, residuated ℓ-monoid and $\langle \mathbb{L}, \leq^{op} \rangle \equiv \langle \mathbb{L}, \geq \rangle$ the opposite poset.

3.7.1 Definition. $\langle \mathbb{L}, \leq, \odot \rangle$ is called **divisible** iff

$$(*) \qquad (\forall a, b \in \mathbb{L}) \big[a \leq b \Longleftrightarrow (\exists c \in \mathbb{L})[b = a \odot c] \big]$$

and it is called **dual-divisible** iff $\langle \mathbb{L}, \leq^{op}, \odot \rangle$ is divisible, i.e.

$$(**) \qquad (\forall a, b \in \mathbb{L}) \big[b \leq a \Longleftrightarrow (\exists c \in \mathbb{L})[b = a \odot c] \big]$$

In the framework of integral commutative, residuated ℓ-monoid we have that $\top = 1$, $\bot = 0$ and $(\forall x \in \mathbb{L})[x \leq 1]$ so that if $\mathbb{L}$ is divisible then $(\forall x \in \mathbb{L})(\exists c \in \mathbb{L})[c \odot x = 1]$ which is not possible. For example let us take $\mathbb{L} - [0,1]$, $\quad x \odot y := \max\{x + y - 1, 0\}$, then $\frac{1}{4} < \frac{1}{2}$ but for $x < \frac{3}{4}$, $\quad \frac{1}{4} \odot x = 0 \neq \frac{1}{2}$ for $x \geq \frac{3}{4}$, then $\frac{1}{4} \odot x = \frac{1}{2}$ does not have a solution in $[0,1]$, since $x = \frac{10}{8} > 1$. So for integral ℓ-monoids $(*)$ is not the appropriate concept of divisibility. From now on when we say divisible we shall always mean dual-divisible.

A characterization of divisibility is given by:

3.7.2 Lemma. *Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative residuated ℓ-monoid. Then the following assertions are equivalent:*

(i) $\langle \mathbb{L}, \leq, \odot \rangle$ *is divisible;*

(ii) $a \wedge b = a \odot (a \Longrightarrow b)$;

(iii) $a \Longrightarrow (b \wedge c) = (a \Longrightarrow b) \odot ((a \wedge b) \Longrightarrow c)$.

Proof: **(i) $\Rightarrow$ (ii):** Let $\langle \mathbb{L}, \leq, \odot \rangle$ be divisible. Then by definition:

$$(\forall a, b \in \mathbb{L}) \big[b \leq a \Longleftrightarrow (\exists c \in \mathbb{L})[b = a \odot c] \big].$$

But since $b \leq a$ iff $a \wedge b = b$ we get

$$(\exists c \in \mathbb{L}) \, [a \wedge b = a \odot c].$$

Since $b = a \odot c$ we have in particular,

$$a \odot c \leq b \quad \overset{(AD)}{\Leftrightarrow} \quad c \leq (a \Longrightarrow b)$$

Thus,

$$a \wedge b = a \odot c \leq a \odot (a \Longrightarrow b) \leq b = a \wedge b \qquad \text{(triangle inequality)}$$

i.e.

$$a \wedge b = a \odot (a \Longrightarrow b).$$

In general we have by triangle inequality,

$$a \odot (a \Longrightarrow b) \leq b.$$

However in the presence of divisibility we have that:

$$\begin{aligned} a \odot (a \Rrightarrow b) &= a \wedge b \leq b \\ \text{and} \quad a \odot (a \Rrightarrow b) &= a \wedge b \leq a \qquad \dashv\Box \end{aligned}$$

(ii) $\Rightarrow$ (iii): First we note that:

$$\begin{aligned} (a \wedge b) \Rrightarrow c &= a \odot (a \Rrightarrow b) \Rrightarrow c \quad \text{by (ii)} \\ &= a \Rrightarrow (a \Rrightarrow b) \Rrightarrow c \end{aligned}$$

since $(a \odot b) \Rrightarrow c = a \Rrightarrow (b \Rrightarrow c)$ with $a \leftarrow a, b \leftarrow (a \Rrightarrow b)$ and $c \leftarrow c$

$$= (a \Rrightarrow b) \Rrightarrow (a \Rrightarrow c)$$

since $a \Rrightarrow (b \Rrightarrow c) = b \Rrightarrow (a \Rrightarrow c)$

Thus,

$$\begin{aligned} (a \Rrightarrow b) \odot ((a \wedge b) \Rrightarrow c) &= (a \Rrightarrow b) \odot ((a \Rrightarrow b) \Rrightarrow (a \Rrightarrow c)) \\ &= (a \Rrightarrow b) \wedge (a \Rrightarrow c) \end{aligned}$$

since $x \odot (x \Rrightarrow y) = x \wedge y$, with $x = (a \Rrightarrow b), y = (a \Rrightarrow c)$

$$= a \Rrightarrow (b \wedge c). \qquad \dashv\Box$$

(iii) $\Rightarrow$ (i): We know that integrality is equivalent to:

$$(\forall x \in \mathbb{L})\ [x = (1 \Rightarrow x)].$$

Suppose now that $b \leq a$. In order to prove (i) we must prove that there exists $c \in \mathbb{L}$ so that $b = c \odot a$. Indeed:

$$\begin{aligned} a \odot (a \Rrightarrow b) &= (1 \Rrightarrow a) \odot ((1 \wedge a) \Rrightarrow b); \quad (1 \wedge a = a) \\ &= 1 \Rrightarrow a \wedge b \quad \text{by (iii)} \\ &= 1 \Rrightarrow b \qquad \text{since } a \wedge b = b \\ &= b. \end{aligned}$$

$c = (a \Rrightarrow b)$. $\dashv\blacksquare$

3.7.3 Proposition. *Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, divisible, commutative residuated ℓ-monoid, then we have:*

(i) If a is idempotent w.r.t. $\odot$, i.e. if

$$a \odot a = a \quad \textit{then} \quad a \wedge b = a \odot b \quad \textit{for all} \quad b \in \mathbb{L}.$$

(ii) $a_1 \leq a_2 \Rrightarrow a_1 \odot b = a_1 \odot (a_2 \Rrightarrow (a_2 \odot b))$

(iii) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$

(iv) $a \odot b \leq (a \odot a) \vee (b \odot b)$

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

PROOF: **(i):** Since for every $a, b \in \mathbb{L}$ we have $a \odot b \leq a \wedge b$ then from the previous Lemma (ii) we have:

$$\begin{aligned} a \odot b \leq a \wedge b &= a \odot (a \Rightarrow b) \\ &= a \odot a \odot (a \Rightarrow b) \quad \text{since } a = a \odot a \\ &= a \odot (a \odot (a \Rightarrow b)) \\ &\leq a \odot b \quad \text{since } a \odot (a \Rightarrow b) \leq b \ \text{ (triangle inequality).} \end{aligned}$$

Thus $a \wedge b = a \odot b$ for all $b \in \mathbb{L}$ ⊣⊓

(ii): Let $a_1 \leq a_2$. Divisibility implies that $[a_2 \odot (a_2 \Rightarrow a_1) = a_2 \wedge a_1 = a_1]$

$$a_1 \odot (a_2 \Rightarrow (a_2 \odot b)) = a_2 \odot (a_2 \Rightarrow a_1) \odot (a_2 \Rightarrow (a_2 \odot b))$$

since $a_2 \odot (a_2 \Rightarrow a_1) = a_1 \wedge a_2 = a_1$

$$\begin{aligned} &= (a_2 \Rightarrow a_1) \odot a_2 \ (a_2 \Rightarrow (a_2 \odot b)) \\ &= (a_2 \Rightarrow a_1) \odot [a_2 \wedge (a_2 \odot b)] \end{aligned}$$

But

$$\begin{aligned} a_2 \wedge (a_2 \odot b) &= (a_2 \odot 1) \wedge (a_2 \odot b) \\ &= a_2 \odot (1 \wedge b) \\ &= a_2 \odot b \end{aligned}$$

Thus

$$\begin{aligned} a_1 \odot (a_2 \Rightarrow (a_2 \odot b)) &= (a_2 \Rightarrow a_1) \odot a_2 \odot b \\ &= [a_2 \odot (a_2 \Rightarrow a_1)] \odot b \\ &= (a_1 \wedge a_2) \odot b = a_1 \odot b. \end{aligned}$$ ⊣⊓

(iii): In the relation $a \wedge b = a \odot (a \Rightarrow b)$ we put for a, $\leftarrow (a \odot b)$, and for b, $\leftarrow (a \odot c)$ and then we have:

$$\begin{aligned} (a \odot b) \wedge (a \odot b) &= (a \odot b) \odot [(a \odot b) \Rightarrow (a \odot c)] \\ &= a \odot (b \odot [(a \odot b) \Rightarrow (a \odot c)]) \\ &= a \odot (b \odot [a \Rightarrow (b \Rightarrow (a \odot c))]) \\ &\qquad \text{in } a \Rightarrow (b \Rightarrow c) = (a \odot b) = (a \odot b) \Rightarrow c \quad \text{put} \quad a \leftarrow a, b \leftarrow b, c \leftarrow a \odot c \\ &= a \odot (b \odot (b \Rightarrow (a \Rightarrow (a \odot c)))) \\ &\qquad \text{since } x \Rightarrow (y \Rightarrow z) = y \Rightarrow (a \Rightarrow c) \\ &= a \odot [b \wedge (a \Rightarrow (a \odot c))] \qquad \text{since } x \odot (x \Rightarrow y) = x \wedge y \\ &= a \odot (b \wedge c). \end{aligned}$$

This is so because $a \Rightarrow_{\odot} (a \odot c) = c$ and this is clear since $(\forall x \in \mathbb{L})\ x \leq a \Rightarrow_{\odot} (a \odot c)$ iff $x \leq c$, this in turn is so since

$$\begin{aligned} x \leq a \Rightarrow_{\odot} (a \odot c) &\overset{(AD)}{\Longleftrightarrow} \quad a \odot x \leq a \odot c \\ &\Leftrightarrow \quad x \leq c. \end{aligned}$$

The last equivalence follows since

$$x \leq c \Rightarrow_{\odot} a \odot x \leq a \odot c,\ a \in \mathbb{L} \quad \text{is po-monoid.}$$

Suppose now that $(\forall x \in \mathbb{L})\ [a \odot x \leq a \odot c]$. Put $a = 1$ then $x \leq c$. ⊣▯

(iv): Let us first consider an arbitrary pair $(a, b) \in \mathbb{L} \times \mathbb{L}$. Since $\langle \mathbb{L}, \leq, \odot \rangle$ is divisible and $a \leq a \vee b$ and $b \leq a \vee b$ there exist elements $c_1, c_2 \in \mathbb{L}$ such that

$$a = (a \vee b) \odot c_1 \quad \text{and} \quad b = (a \vee b) \odot c_2$$

Since $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$ we have

$$a \odot b = [(a \vee b) \odot c_1] \odot [(a \vee b) \odot c_2] = c_1 \odot c_2 \odot (a \vee b) \odot (a \vee b)$$

But $a \vee b = [(a \vee b) \odot c_1] \vee [(a \vee b \odot c_2] = (a \vee b) \odot (c_1 \vee c_2)$ since $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

Thus

$$\begin{aligned} c_1 \odot c_2 \odot (a \vee b) \odot (a \vee b) &= c_1 \odot c_2 \odot (c_1 \vee c_2) \odot (a \vee b) \odot (a \vee b) \\ &\leq [(c_1 \odot c_1) \vee (c_2 \odot c_2)] \odot (a \vee b) \odot (a \vee b) \\ &= [[(a \vee b) \odot (a \vee b)] \odot (c_1 \odot c_1)] \vee [[(a \vee b) \odot (a \vee b)] \odot (c_2 \odot c_2)] \\ &= (a \odot a) \vee (b \odot b). \quad ⊣▯ \end{aligned}$$

(v). Since $a \wedge b = a \odot (a \Rightarrow_{\odot} b)$ then $a \wedge (b \vee c) = (b \vee c) \odot ((b \vee c) \Rightarrow_{\odot} a)$ $\leq (b \odot (b \Rightarrow_{\odot} a)) \vee (c \odot (c \Rightarrow_{\odot} a)) = (a \wedge b) \vee (a \wedge c)$. ⊣▮

3.7.4 Corollary. *Let $M = \langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative, residuated ℓ-monoid. If M is divisible and satisfies the algebraic strong de Morgan law, then the subset H_M of all idempotent elements w.r.t. $\odot$ form a Heyting algebra, and the implication in H_M coincides with the implication based on $\odot$.*

Proof: Let

$$H_M := \{e \in M \mid e \odot e = e\}.$$

Now for all $a, b \in H_M$ idempotency implies that

$$a \odot b = a \wedge b.$$

Thus (AD) becomes in this case,

$$(*) \qquad (\forall a, b, c \in H_M)\ [\, a \wedge b \leq c \quad \text{iff} \quad a \leq b \Rightarrow_{\odot} c \,]$$

But this is the definition of a Heyting algebra. (A Heyting algebra is a lattice with 0 such that for all $a \in \mathbb{H}$, the endomorphism,

$$F_a : \mathbb{H} \longrightarrow \mathbb{H} \;/\!\!/\; x \mapsto F_a(x) = a \wedge x$$

has a right-adjoint, i.e. (∗).) It is also clear by the way we have defined H_M that implication in H_M coincides with implication based on $\odot$. This is so since $a \odot b = a \wedge b$. ⊣

3.7.5 Remark. From the above Corollary, we see that the structure of cirl-monoid which satisfy in addition, the divisibility and the algebraic Strong De Morgan's law, is a direct quantitative generalization of Heyting algebras.

The next characteristic which refines the classification of our structures is negation. Like in the Heyting algebras, negation is not necessarily involutive, and there is no bi-duality. Thus we may have bi-Heyting algebras and Heyting algebras with an involutive negation. Similarly in the quantitative case we have corresponding classifications.

3.8 Involutive Negation and Girard monoids.

In an integral, commutative, residuated ℓ-monoid $M = \langle \mathbb{L}, \leq, \odot \rangle$ we have that $\bot = 0$ and $\top = 1$.

3.8.1 Definition. (i) For all $a \in M$ we define **the negation of** a:

$$\neg a := (a \Rightarrow_{\odot} 0)$$

(ii) The negation is an **involution** iff for all $a \in \mathbb{L}$,

$$\neg(\neg a) = a$$

or

$$((a \Rightarrow_{\odot} 0) \Rightarrow_{\odot} 0) = a$$

(iii) An integral, commutative, residuated ℓ-monoid is called an integral, commutative Girard monoid iff the negation is an involution.

3.8.2 Proposition. *$\neg a$ is the greatest element such that*

$$a \odot \neg a = 0$$

i.e.

$$\neg a = \bigvee \{x \mid x \odot a = 0\}$$

$\neg a$ is the greatest element such that

PROOF: Since $\neg a := (a \Rightarrow_\odot 0)$ and we know that for all a, b

$$a \Rightarrow_\odot b := \bigvee\{x \in \mathbb{L} \mid a \odot x \leq b\}$$

then putting $b = 0$ we get:

$$\neg a = \bigvee\{x \in \mathbb{L} \mid a \odot x \leq 0\}$$

Due to integrality we have $0 \leq a \odot x$, so that

$$\neg a = \bigvee\{x \in \mathbb{L} \mid a \odot x = 0\}. \qquad \dashv$$

Note. Thus $\neg a$ is a pseudocomplement-like operation w.r.t. the general monoidal operation $\odot$, and $\forall b \in \mathbb{L}$,

$$\neg a := a \Rightarrow_\odot 0 \leq a \Rightarrow_\odot b,$$

$(a \Rightarrow_\odot -)$ is isotone and $0 \leq b$.

3.8.3 Proposition. *Let $M = \langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative Girard-monoid. Then the following relations are valid:*

(1) $a \Rightarrow_\odot b = [a \odot (b \Rightarrow_\odot 0)] \Rightarrow_\odot 0$, i.e. $a \Rightarrow_\odot b = \neg(a \odot \neg b)$ $\quad [\equiv (a \Rightarrow_\odot b) = (\neg b \Rightarrow_\odot \neg a)]$

(2). $(a \wedge b) \Rightarrow_\odot 0 = (a \Rightarrow_\odot 0) \vee (b \Rightarrow_\odot 0)$, i.e. $\neg(a \wedge b) = \neg a \vee \neg b$.

PROOF: **(1):** Since M is Girard-monoid, the negation is an involution, thus:

$$\begin{aligned} a \Rightarrow_\odot b &= a \Rightarrow_\odot ((b \Rightarrow_\odot 0) \Rightarrow_\odot 0) \\ &= (a \odot (b \Rightarrow_\odot 0)) \Rightarrow_\odot 0 \qquad \text{by (iii)} \end{aligned}$$

Note also that $(a \odot (b \Rightarrow_\odot 0) \Rightarrow_\odot 0 = (b \Rightarrow_\odot 0) \Rightarrow_\odot (a \Rightarrow_\odot 0) = \neg b \Rightarrow_\odot \neg a$, i.e.

$$a \Rightarrow_\odot b = (a \odot (b \Rightarrow_\odot 0)) \Rightarrow_\odot 0 = \neg b \Rightarrow_\odot \neg a$$

(2): We have:

$$\begin{aligned} ((a \Rightarrow_\odot 0) \vee (b \Rightarrow_\odot 0)) \Rightarrow_\odot 0 &= [(a \Rightarrow_\odot 0) \Rightarrow_\odot 0] \wedge [(b \Rightarrow_\odot 0) \Rightarrow_\odot 0] \qquad \text{by (v)} \\ &= a \wedge b \qquad \text{by (xvi)} \end{aligned}$$

$$[((a \Rightarrow_\odot 0) \vee (b \Rightarrow_\odot 0)) \Rightarrow_\odot 0] \Rightarrow_\odot 0 = (a \wedge b) \Rightarrow_\odot 0$$

or

$$(a \Rightarrow_\odot 0) \vee (b \Rightarrow_\odot 0) = (a \wedge b) \Rightarrow_\odot 0. \qquad \dashv$$

3.8.4 Lemma. *Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative Girard monoid. Then the following assertions are equivalent:*

(1) $\langle \mathbb{L}, \leq, \odot \rangle$ *satisfies the algebraic strong de Morgan law;*

(2) $(\forall a, b, c \in \mathbb{L})\ [a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)]$

PROOF: **(1) ⇒ (2):** This is just the previous Lemma (2) ⊣▯

(2) ⇒ (1): To prove that (2) ⇒ (1) we shall prove instead of (1) the equivalent condition (Prop 2.3 (a)),

$$a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c) = (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow c).$$

Indeed,

$$\begin{aligned} a \Rightarrow\!\!\odot\!\!\Rightarrow (b \vee c) &= [a \odot ((b \vee c) \Rightarrow\!\!\odot\!\!\Rightarrow 0)] \Rightarrow\!\!\odot\!\!\Rightarrow 0 \\ &= [a \odot ((b \Rightarrow\!\!\odot\!\!\Rightarrow 0) \wedge (c \Rightarrow\!\!\odot\!\!\Rightarrow 0))] \Rightarrow\!\!\odot\!\!\Rightarrow 0 \\ &= [(a \odot (b \Rightarrow\!\!\odot\!\!\Rightarrow 0)) \wedge (a \odot (c \Rightarrow\!\!\odot\!\!\Rightarrow 0))] \Rightarrow\!\!\odot\!\!\Rightarrow 0 \quad \text{by (2)} \\ &= [(a \odot (b \Rightarrow\!\!\odot\!\!\Rightarrow 0)) \Rightarrow\!\!\odot\!\!\Rightarrow 0] \vee [(a \odot (c \Rightarrow\!\!\odot\!\!\Rightarrow 0)) \Rightarrow\!\!\odot\!\!\Rightarrow 0] \quad \text{by (xviii)} \\ &= (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \qquad \text{by (xvii).} \end{aligned}$$
⊣▮

3.8.5 Lemma. *Let* $\langle \mathbb{L}, \leq, \odot \rangle$ *be an integral, commutative Girard monoid satisfying the ASDML. Then the "negation" has at most one fixpoint, i.e. :*

$$a = a \Rightarrow\!\!\odot\!\!\Rightarrow 0 \quad \textit{and} \quad b = b \Rightarrow\!\!\odot\!\!\Rightarrow 0 \quad \textit{implies} \quad a = b.$$

PROOF:

$$\begin{aligned} a &= a \odot 1 \\ &= a \odot [(a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (b \Rightarrow\!\!\odot\!\!\Rightarrow a)] \\ &= a \odot [(a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee ((a \Rightarrow\!\!\odot\!\!\Rightarrow 0) \Rightarrow\!\!\odot\!\!\Rightarrow (b \Rightarrow\!\!\odot\!\!\Rightarrow 0))] \quad \text{since } b \Rightarrow\!\!\odot\!\!\Rightarrow a = (a \Rightarrow\!\!\odot\!\!\Rightarrow 0) \Rightarrow\!\!\odot\!\!\Rightarrow (b \Rightarrow\!\!\odot\!\!\Rightarrow 0) \\ &= a \odot [(a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow b)] \quad \text{by hypothesis } a \Rightarrow\!\!\odot\!\!\Rightarrow 0 = a \text{ and } b \Rightarrow\!\!\odot\!\!\Rightarrow 0 = b \\ &= a \odot a \Rightarrow\!\!\odot\!\!\Rightarrow b \qquad \text{since } (a \Rightarrow\!\!\odot\!\!\Rightarrow b) \vee (a \Rightarrow\!\!\odot\!\!\Rightarrow b) = a \Rightarrow\!\!\odot\!\!\Rightarrow b \\ &\leq b \qquad \text{interchanging the rôle of } a \text{ and } b \text{ we have } a = b. \end{aligned}$$
⊣▮

3.9 Square roots.

An integral, commutative, residuated ℓ-monoid $\langle \mathbb{L}, \leq, \odot \rangle$ has **square roots** iff there exists a unary operation,

$$S : \mathbb{L} \longrightarrow \mathbb{L} \ /\!\!/ \ a \mapsto S(a)$$

having the following properties:

(S1) $(\forall a \in \mathbb{L})\ [S(a) \odot S(a) = a]$.

(S2) $(\forall a, b \in \mathbb{L})\ [b \odot b \leq a \Rightarrow\!\!\odot\!\!\Rightarrow b \leq S(a)]$

Properties (S1) and (S2) uniquely determined the operation S. Indeed, let S' be another operation satisfying (S1), (S2), then,

$$S(a) \odot S(a) = a = S'(a) \odot S'(a).$$

Since S is uniquely defined we denote $S(a)$ also $a^{1/2}$ or $\sqrt{a}$.

In the following proposition we collect some useful inequalities beyond the basic,

$$x \odot (y \Rrightarrow z) \leq y \Rrightarrow (x \odot z).$$

3.9.1 Proposition. *(i)* $(a \Rrightarrow b) \odot (c \Rrightarrow \delta) \leq (a \odot c) \Rrightarrow (b \odot \delta)$

(ii) $(a^{1/2} \Rrightarrow b^{1/2}) \odot (a^{1/2} \Rrightarrow b^{1/2}) \leq a \Rrightarrow b$

(iii) $(a^{1/2} \odot (a \Rrightarrow b)^{1/2}) \odot (a^{1/2} \odot (a \Rrightarrow b)^{1/2}) \leq b$

Proof: **(i):**

$$\begin{array}{rcll} (a \Rrightarrow b) \odot (c \Rrightarrow \delta) & \leq & c \Rrightarrow [(a \Rrightarrow b) \odot \delta] & \text{since } x \odot (y \Rrightarrow z) \leq y \Rrightarrow (x \odot z) \\ & \leq & c \Rrightarrow [a \Rrightarrow (a \odot \delta)] & \quad - \text{''} - \\ & = & (a \odot c) \Rrightarrow (a \odot \delta). & \dashv\square \end{array}$$

(ii): Using (i) with $a = \delta \leftarrow a^{1/2}, b = c \leftarrow b^{1/2}$ we get the result. $\dashv\square$

(iii): $(a^{1/2} \odot (a \Rrightarrow b)^{1/2}) \odot (a^{1/2} \odot (a \Rrightarrow b)^{1/2}) = (a^{1/2} \odot a^{1/2}) \odot ((a \Rrightarrow b)^{1/2} \odot (a \Rrightarrow b)^{1/2}) = a \odot (a \Rrightarrow b) \leq b.$ $\dashv\blacksquare$

3.9.2 Proposition. *(General properties of square roots.) Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative, residuated ℓ-monoid with square roots. Then we have:*

(i) $a \leq a^{1/2},\ a \leq b \Rrightarrow a^{1/2} \leq b^{1/2}$.

In a complete Boolean algebra we have $x = 1$ and the relation $a^{1/2} = a$ holds for all $a \in \mathbb{B}$ (prove it!). Thus a square root in a Boolean algebra is a closure operator.

(ii) $a^{1/2} \odot b^{1/2} \leq (a \odot b)^{1/2}$

(iii) $(a \wedge b)^{1/2} = a^{1/2} \wedge b^{1/2}$

(iv) $(a \wedge b)^{1/2} = a^{1/2} \wedge b^{1/2}$

(v) $a \odot b \leq (a \odot a) \vee (b \odot b)$

(vi) $a \wedge b \leq a^{1/2} \odot b^{1/2} \leq a \vee b$.

PROOF: **(i):** Due to integrality we have: $a \odot a \leq a \wedge a = a \overset{(S2)}{\Rightarrow} a \leq a^{1/2}$. We have $a^{1/2} \odot a^{1/2} = a \leq b$. Thus $a^{1/2} \odot a^{1/2} \leq b \overset{(S2)}{\Rightarrow} a^{1/2} \leq b^{1/2}$. ⊣⊓
(ii): $(a^{1/2} \odot b^{1/2}) \odot (a^{1/2} \odot b^{1/2}) = (a^{1/2} \odot a^{1/2}) \odot (b^{1/2} \odot b^{1/2}) = a \odot b$. In particular $(a^{1/2} \odot b^{1/2}) \odot (a^{1/2} \odot b^{1/2}) \leq a \odot b \overset{(k)}{\Rightarrow} a^{1/2} \odot b^{1/2} \leq (a \odot b)^{1/2}$. ⊣⊓
(iii): By the previous proposition we have

$$(a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2}) \odot (a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2}) \leq a \Rightarrow\!\!\odot\!\!\Rightarrow b \text{ and } (a^{1/2} \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2}) \odot (a^{1/2} \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2}) \leq b.$$

Using the axiom (S2) we infer:

$$a^{1.2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2} \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2} \text{ and } a^{1/2} \odot (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2} \leq b^{1/2} \text{ iff } (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2} \leq a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2}$$

$$a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2} \leq (a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2} \leq a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2}$$

i.e.

$$(a \Rightarrow\!\!\odot\!\!\Rightarrow b)^{1/2} = a^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow b^{1/2}. \quad ⊣⊓$$

(iv): Since the square root function is isotone we have always

$$S(a \wedge b) \leq S(a) \wedge S(b) \text{ or } (a \wedge b)^{1/2} \leq a^{1/2} \wedge b^{1/2}.$$

For the converse, we observe that:

$$(a^{1/2} \wedge b^{1/2}) \odot (a^{1/2} \wedge b^{1/2}) \leq a \wedge b \qquad \text{(why?)}$$

so that

$$(a^{1/2} \wedge b^{1/2}) \leq (a \wedge b)^{1/2}. \quad ⊣⊓$$

(v): We know that $a \odot (b \odot c) = (a \odot b) \vee (a \odot c)$ (iv), so that

$$\begin{aligned}(a \vee b) \odot (c \vee \delta) &= (a \odot c) \vee (a \odot \delta) \vee (b \odot c) \vee (b \odot \delta) \\ &\leq (a \odot c) \vee (b \vee \delta)\end{aligned}$$

Thus,

$$(a \vee b) \odot (a \vee b) \leq (a \odot a) \vee (b \odot b)$$

and by (S2) we get

$$a \vee b \leq [(a \odot a) \vee (b \odot b)]^{1/2}$$

Squaring both sides and applying (iv) we get:

$$(a \odot a) \vee (a \odot b) \vee (b \odot b) \leq (a \odot a) \vee (b \odot b)$$

since $(a \odot b) \vee (b \odot a) = (a \odot b)$ so that

$$(a \odot b) \leq (a \odot a) \vee (b \odot b). \quad ⊣⊓$$

(vi): By (S1)

$$\begin{aligned}a \wedge b &= (a \wedge b)^{1/2} \odot (a \wedge b)^{1/2} \\ &\leq a^{1/2} \odot b^{1/2}\end{aligned}$$

since $a \wedge b \leq a$ and $a \wedge b \leq b$ implies $(a \wedge b)^{1/2} \leq a^{1/2}$ and $(a \wedge b)^{1/2} \leq b^{1/2}$ and this in turn implies $(a \wedge b)^{1/2} \odot (a \wedge b)^{1/2} \leq a^{1/2} \odot b^{1/2}$.

Using now $a \odot b \leq (a \odot a) \vee (b \odot b)$ (xxiii), we get,

$$a^{1/2} \odot b^{1/2} \leq (a^{1/2} \odot a^{1/2}) \vee (b^{1/2} \odot b^{1/2}) = a \vee b. \qquad \dashv$$

3.9.3 Corollary. *Let $\langle \mathbb{L}, \leq, \odot \rangle$ be an integral, divisible, commutative, residuated ℓ-monoid with square roots. Then,*

$$b \leq a^{1/2} \odot b^{1/2} \Longrightarrow b \leq a.$$

PROOF: Since $b \leq a^{1/2} \odot b^{1/2}$ we have that $b \wedge (a^{1/2} \odot b^{1/2}) = b$. Thus

$$\begin{aligned}
b &= (a^{1/2} \odot b^{1/2}) \wedge b = (a^{1/2} \odot b^{1/2}) \wedge (b^{1/2} \odot b^{1/2}) \\
&= (a^{1/2} \wedge b^{1/2}) \odot b^{1/2} \quad \text{by (xiv)} \\
&= b^{1/2} \odot b^{1/2} \odot (b^{1/2} \Longrightarrow a^{1/2}) \quad \text{since } a^{1/2} \wedge b^{1/2} = b^{1/2} \odot (b^{1/2} \Longrightarrow a^{1/2}) \text{ (xi)} \\
&= b \odot (b^{1/2} \Longrightarrow a^{1/2}) \\
&\leq a^{1/2} \odot b^{1/2} \odot (b^{1/2} \Longrightarrow a^{1/2}) \quad \text{by hypothesis)} \\
&\leq a^{1/2} \odot a^{1/2} = a \quad \text{since } b^{1/2}(b^{1/2} \Longrightarrow a^{1/2}) \leq a^{1/2} \text{ (triangle)}
\end{aligned}$$

$$b \leq a. \qquad \dashv$$

3.9.4 Corollary. *Let $M = \langle \mathbb{L}, \leq, \odot \rangle$ be an integral, commutative Girard-monoid with square roots. Then*

(i) $(a \odot b)^{1/2} = ((a^{1/2} \odot b^{1/2}) \Longrightarrow 0^{1/2}) \Longrightarrow 0^{1/2}$

(ii) If M satisfies the algebraic strong de Morgan's law, then

$$(a \vee b)^{1/2} = a^{1/2} \vee b^{1/2}$$

holds.

PROOF: **(i):**

$$\begin{aligned}
((a^{1/2} \odot b^{1/2}) \Longrightarrow 0^{1/2}) \Longrightarrow 0^{1/2} &\overset{\text{(iii)}}{=} (a^{1/2} \Longrightarrow (b^{1/2} \Longrightarrow 0^{1/2})) \Longrightarrow 0^{1/2} \\
&\overset{\text{(xxi)}}{=} (a^{1/2} \Longrightarrow (b \Longrightarrow 0)^{1/2}) \Longrightarrow 0^{1/2} \\
&\overset{\text{(xxi)}}{=} ((a \Longrightarrow (b \Longrightarrow 0)) \Longrightarrow 0)^{1/2} \\
&\overset{\text{(iii)}}{=} ((a \odot b) \Longrightarrow 0) \Longrightarrow 0)^{1/2} \\
&\overset{\text{(xvi)}}{=} (a \odot b)^{1/2}
\end{aligned}$$

(ii):

$$
\begin{aligned}
(a \vee b)^{1/2} &\overset{\text{(xvi)}}{=} [((a \vee b) \Rightarrow\!\!\odot\!\!\Rightarrow 0) \Rightarrow\!\!\odot\!\!\Rightarrow 0]^{1/2} \\
&\overset{\text{(v)}}{=} [((a \Rightarrow\!\!\odot\!\!\Rightarrow 0) \wedge (b \Rightarrow\!\!\odot\!\!\Rightarrow 0)) \Rightarrow\!\!\odot\!\!\Rightarrow 0]^{1/2} \\
&\overset{\text{(xxii)}}{=} ((a \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2} \wedge (b \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2}) \Rightarrow\!\!\odot\!\!\Rightarrow 0^{1/2} \\
&\overset{\text{SANDML}}{=} \Big(((a \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow 0^{1/2}) \vee ((b \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2} \Rightarrow\!\!\odot\!\!\Rightarrow 0^{1/2})\Big) \\
&\qquad \text{(by Algeb. Strong de Morgan's law)} \\
&= \Big[((a \Rightarrow\!\!\odot\!\!\Rightarrow 0) \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2} \vee ((b \Rightarrow\!\!\odot\!\!\Rightarrow 0) \Rightarrow\!\!\odot\!\!\Rightarrow 0)^{1/2}\Big] \\
&= a^{1/2} \vee b^{1/2}. \qquad \dashv
\end{aligned}
$$

BIBLIOGRAPHY

[1] M. Barr and C. Wells. *Category Theory for Computing Science*. 2nd Ed. Prentice Hall, 1995.

[2] T. S. Blyth and M. F. Janowitz. *Residuation Theory.* Pergamon press 1972.

[3] C. C. Chang. Algebraic analysis of many valued logics. *Trans. of AMS*, **88**, 1957, 467-490.

[4] C. C. Chang. A new proof of the completeness of Łukasiewicz axioms. *Trans. of AMS*, **93**, 1958, 74-80.

[5] M. L. Dalla Chiara and Roberto Giuntini. Partial and Unsharp Quantum Logics. Priprint.

[6] C. A. Drossos. Foundations of fuzzy sets: A nonstandard approach. *Fuzzy Sets and Systems*, 37:287–307, 1990.

[7] C. A. Drossos and P. Karazeris. Coupling an MV-algebra with a Boolean Algebra. (To appear in Jounal of Approximate Peasoning) 1998.

[8] C. A. Drossos and G. Markakis. Boolean fuzzy sets. *Fuzzy Sets and Systems*, 46:81–95, 1992.

[9] C. A. Drossos and G. Markakis. Boolean representations of fuzzy sets. *Kybernetes*, 22(3):35–40, 1993.

[10] C. A. Drossos and G. Markakis. Boolean powers and stochastic spaces. *Math. Slovaca* **44** # 1 (1994), 1-19.

[11] C. A. Drossos and M. Navara. Generalized t-conorms and closure operators. EUFIT 96, Aachen, Germany, September 2-5, 1996.

[12] C. A. Drossos and P. Theodoropoulos. $I\!B$-fuzzy probability *Fuzzy Sets and Systems,* **78** (1996), 355-369.

[13] J. Michel Dunn. *Gaggle Theory:An Absraction of Galois Connections and Residuation, with Applications to Negation, and Various Logical Operators.* Indiana Univ. Logic Group, Preprint No. IULG-91-3, May,1991.

[14] J. Michel Dunn. *Star and Perp: Tow Treatments of Negation* Indiana Univ. Logic Group, Preprint No. IULG-93-21, April,1993.

[15] J. Michel Dunn. *Partial-Gaggles Applied to Logics with Restricted Structural Rules.* Indiana Univ. Logic Group, Preprint No. IULG-93-22, April,1993.

[16] J. Michel Dunn. Gaggle Theory Applied to Intuitionistic, Modal, and Relevance Logics. In: I. Max and W. Stelzner. *Logic und Mathematik*, Walter de Gruyter, Berlin 1995, pp.335-368.

[17] M. Erné, J. Kolowski, A. Melton, G. E. Stecker. A primer on Galois conections. In: *Papers on General Topology and Applications* (Madison Wisc. 1991), *Ann. New York Acad. Sci.*, # 704, N.Y. Acad. Sci. 1993, pp. 103-125.

[18] R. C. Flagg. Church's thesis with epistemic arithmetic. In: S Shapiro (Ed) *Intentional Mathematics* Norh-Holland, 1985

[19] Roberto Giuntini. Quantum MV Algebras. *Studia Logica* **56:** 393-417, 1996

[20] Roberto Giuntini. Quantum MV Algebras and Commutativity. Preprint.

[21] Petr Hájek. Metamathematics of Fuzzy Logic: Part I. Book in preparation, Prague 1995.

[22] C. Hartonas and J. Michel Dunn. *Duality Theorems for Partial Orders, Semilattices, Galois Connections and Lattices* Indiana Univ. Logic Group, Preprint No. IULG-93-26, August,1993.

[23] U. Höhle: M-valued Sets and Sheaves over Integral Commutative CL-monoids. In: S. E. Rodabaugh et all. *Applications of Category Theory to Fuzzy Subsets*, 33-72, Kluwer (1992).

[24] U. Höhle: Commutative, residuated l-monoids. In U. Höhle and E. P. Klement *Non-Classical Logics and Their Applications to Fuzzy Subsets: A Handbook on the Math. Foundations of Fuzzy Set Theory.* Kluwer 1995.

[25] U. Höhle: On the Fundamentals of Fuzzy Set Theory. Priprint.

[26] Peter Johnstone. *Stone Spaces* Cambridge Univ. Press, 1982.

[27] Kappos, D.: *Probability Algebras and Stochastic spaces.* Academic Press, (1969).

[28] J. Lambek. The influence of Heraclitus on modern mathematics, In, J. Agassi and R.S. Cohen *Scientific Philosophy Today*, Reidel, 1981.

[29] D. Mundici. Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *J. of Funct. Anal.,* **65:**15-63, 1986.

[30] G. Panti. *Multi-valued Logics.* In: D. Gabbay and P. Smets (Eds) *Qunatified Representation of Uncertainty and Impresision: A Handbook of Defensible Reasoning and Uncertainty Management Systems.* Kluwer ?.

[31] B. Riečan and T. Neubrunn. *Integral, Measure and Ordering.* Kluwer, 1997.

[32] K. I. Rosenthal. *Quantales and Their Applications.* Longman Scientific & Technical, Longman Burnt Mill, Harlow 1990.

[33] M. B. Smith. Topology. In S. Abramsky D. Gabbay and T. S. E. Maibaum: *Handbook of Logic in Computer Science, vol. 1.:Background: Mathematical Structures.* pp. 641-761, Oxford Uviv. Press, 1992.

[34] Ross Street. *Quantum Groups.* Univ. of Macquarie, Sydney, Australia. Available by ftp: `ftp.mpce.mq.edu.au` in `pub/maths/TeX/Quantum`.

[35] P. Zlatoš. Two-levelled logic and model theory. In: Colloquia Mathematica Societatis Janos Bolyai, No. 28: Finite Algebra and Multiple-valued logic, Szeged (Hungary), 1979, pp.825-871.

[36] P. Zlatoš. Unitary Conguence Adjunctions. In: Colloquia Mathematica Societatis Janos Bolyai, No. 43. Lectures in Universal Algebra. Szeged (Hungary), 1983, pp. 587-647.