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Second-order, Third-order, and nth-order Ordinary Differential Equations: How the Exceptional Became Unexceptional

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Second- and third-order scalar ordinary differential equations of maximal symmetry in the traditional sense of point, respectively contact, symmetry are examined for the mappings they produce in fundamental first integrals. The properties of the "exceptional symmetries", *ie* those not considered to be generic to scalar equations of maximal symmetry, can be recast into a form which is applicable to all such equations of maximal symmetry. Thereby the exceptional symmetries are rendered unexceptional.

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1. Prologue

The core of the material presented here was initiated by two observations of Norbert Euler of the Department of Mathematics, Luleå University of Technology in northern Sweden [4]. The first observation was of the transformation of one first integral of the nonlinear ordinary differential equation under investigation to another and the second was the generation of the general solution of the source ordinary differential equation under a mapping of a particular solution of the equation.

These observations were combined with previous work on the symmetries and integrals of scalar ordinary differential equations of maximal symmetry [5, 6, 10, 13] in a final year class comprising Nicolette Caister, Veroshan Naicker and Ryan Warne to develop the basis of the ideas presented here.

The central purpose of this paper is to demonstrate that the additional Lie symmetries associated with scalar second- and third-order ordinary differential equations of maximal symmetry are not exceptional, but have precise parallels in all equations of maximal symmetry no matter the order.

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2. Some Background Material

A little knowledge of the concepts of symmetry and integral is necessary.

The infinitesimal transformation

$$\bar{x} = x + \varepsilon\xi \qquad \bar{y} = y + \varepsilon\eta,$$

where ε is the parameter of smallness, is generated by the differential operator

$$\Gamma = \xi\partial_x + \eta\partial_y$$

since we may write

$$\bar{x} = (1 + \varepsilon\Gamma)\, x \qquad \bar{y} = (1 + \varepsilon\Gamma)\, y.$$

Starting from this simple idea we may expand the application of the generator of the infinitesimal transformation, Γ, to functions of the variables x and y, to functions including derivatives of the dependent variable, y, and to differential equations. To include derivatives we need to determine a formula for the infinitesimal transformation induced in the derivative by the infinitesimal transformation generated in the original variables, x and y.

We illustrate the process of determination with the first derivative and present the formulæ for the higher derivatives.

$$\begin{aligned} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= \frac{\mathrm{d}\,(y + \varepsilon\eta)}{\mathrm{d}\,(x + \varepsilon\xi)} \\ &= \frac{y' + \varepsilon\eta'}{1 + \varepsilon\xi'} = (y' + \varepsilon\eta')\,(1 + \varepsilon\xi')^{-1} \\ &= (y' + \varepsilon\eta')\left(1 - \varepsilon\xi' + \varepsilon^2\xi'^2 - \ldots\right) \\ &= y' + \varepsilon\,(\eta' - y'\xi') \end{aligned}$$

to the first order in the infinitesimal parameter ε.

The formulæ for higher derivatives are calculated in a similar fashion. We have

$$\begin{aligned} \frac{d^2\bar{y}}{d\bar{x}^2} &= y'' + \varepsilon\left(\eta'' - 2y''\xi' - y'\xi''\right) \\ \frac{d^3\bar{y}}{d\bar{x}^3} &= y''' + \varepsilon\left(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi'''\right) \end{aligned}$$

for the second and third derivatives respectively. The obvious guess for the generalization to derivatives of higher order is the correct guess.

3. Symmetries of Differential Equations

A differential equation

$$E\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

possesses a symmetry, $\Gamma = \xi\partial_x + \eta\partial_y$, if

$$\Gamma^{[n]}E_{\big|E=0} = 0$$

has a nontrivial solution, where the differential operator has been
extended to apply to the nth derivative.

Usually one limits the nature of the variable dependence in the
coefficient functions, ξ and η. For example, if one seeks the point symmetries, *ie* ξ and η depend only upon the variables x and y, of the differential equation

$$y'' = \frac{1}{y^3},$$

one obtains the three symmetries

$$\begin{aligned} \Gamma_1 &= \partial_x \\ \Gamma_2 &= 2x\partial_x + y\partial_y \\ \Gamma_3 &= x^2\partial_x + xy\partial_y. \end{aligned}$$

A feature of these differential operators, the symmetries, is that they form an algebra under the operation of taking what is called the Lie Bracket. Thus we have

$$\begin{aligned} [\Gamma_1, \Gamma_2]_{LB} &= \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1 = 2\Gamma_1 \\ [\Gamma_1, \Gamma_3]_{LB} &= \Gamma_1\Gamma_2 - \Gamma_3\Gamma_1 = \Gamma_2 \\ [\Gamma_2, \Gamma_3]_{LB} &= \Gamma_2\Gamma_3 - \Gamma_3\Gamma_2 = 2\Gamma_3. \end{aligned}$$

The algebra is called $sl(2, R)$, which is a representation of the special linear group in two dimensions and which plays an important role in the group theoretical study of differential equations.

One can calculate various types of symmetry. In particular one may vary the nature of the variable dependence in the coefficient functions. This is an important aspect of this talk. The actual calculation can be quite tedious and lengthy, especially as the order and number of differential equations increases. Not surprisingly a number of programs have been developed to enable the computations to be performed using a computer. The carrier code reflects the sophistication of the times. Thus LIE, developed by Alan Head almost thirty years ago [7, 19], is embedded in MUMATH, the interactive package of Clara Nucci, which is about twenty years old [17, 18], is based on REDUCE and SYM, the baby of Stelios Dimas and Dimitris Tsoubelis [2, 3], is an add-on to Mathematica.

The number of Lie point symmetries depends upon the equation. The minimum number is zero. For a scalar nth-order ordinary differential equation the maximum number of Lie point symmetries is $n+4$ for $n \geq 3$ and $n+6$ for $n = 2$ [9, 11]. The two additional symmetries in the case $n = 2$ are called exceptional symmetries. If one includes contact symmetries, there are three additional symmetries in the case of $n = 3$.

Why the cases $n = 2$ and $n = 3$ should be exceptional has been a matter of some interest to me for many years.

Eventually it came to me that the cause of the exceptionality was the lack of audacity in my thinking.

In what follows we see how to remove the exception. The process is illustrated with extremely simple ordinary differential equations, not because the answer is limited to them but to enable the obvious to be seen clearly.

4. The First Simple Equation

The Symmetries:
The Lie point symmetries of

$$y'' = 0 \tag{1}$$

are

$$\begin{array}{lll} \Gamma_h = y\partial_y & \big] & \text{homogeneity; } 1A_1 \\ \left.\begin{array}{l}\Gamma_{s1} = 1\partial_y \\ \Gamma_{s2} = x\partial_y\end{array}\right] & & \text{solution; } 2A_1 \\ \left.\begin{array}{l}\Gamma_{sl1} = \partial_x \\ \Gamma_{sl2} = x\partial_x + \frac{1}{2}y\partial_y \\ \Gamma_{sl3} = x^2\partial_x + xy\partial_y\end{array}\right] & & \text{special linear; } A_{3,8}\,(sl(2,R)) \\ \left.\begin{array}{l}\Gamma_{nc1} = y\partial_x \\ \Gamma_{nc2} = xy\partial_x + y^2\partial_y\end{array}\right] & & \text{noncartan; } 2A_1. \end{array} \tag{2}$$

A little explanation of this table is required! The eight Lie point symmetries are a representation of the Lie algebra $sl(3, R)$. However, our interest is more in the detail of the different subalgebras comprising this representation of $sl(3, R)$. The first symmetry, Γ_h, represents the homogeneity of the equation. The second and third symmetries, Γ_{si}, $i = 1, 2$, are called solution symmetries since the coefficient functions, $\eta_1 = 1$ and $\eta_2 = x$, are solutions of (1). The next three symmetries, Γ_{sli}, $i = 1, 3$, constitute a representation of the algebra, $sl(2, R)$. The remaining two symmetries, Γ_{nci}, $i = 1, 2$, are called noncartan symmetries. The transformations which they induce are not fibre-preserving [8]. The labels given to the various subalgebras based upon the Mubarakzyanov classification scheme [12, 14–16]. Apart from $sl(2, R)$ the other subalgebras are abelian. In the case of Γ_h this is trivial.

The Fundamental Integrals:

As a second-order ordinary differential equation (1) possesses two functionally independent first integrals which can be expressed in a variety of ways. We choose the particular representation, termed fundamental [10],

$$I_1 = y - xy' \quad \text{and} \quad I_2 = y' \tag{3}$$

which correspond to the integrating factors, $-x$ and 1, of (1) respectively.

The actions of the eight Lie point symmetries, once extended, of (1) on the first integrals given in (3) are

$$\begin{array}{ll}
\Gamma_h I_1 = I_1 & \Gamma_h I_2 = I_2 \\
\hline
\Gamma_{s1} I_1 = 1 & \Gamma_{s1} I_2 = 0 \\
\Gamma_{s2} I_1 = 0 & \Gamma_{s2} I_2 = 1 \\
\hline
\Gamma_{sl1} I_1 = -I_1 & \Gamma_{sl1} I_2 = 0 \\
\Gamma_{sl2} I_1 = \frac{1}{2} I_1 & \Gamma_{sl2} I_2 = -\frac{1}{2} I_2 \\
\Gamma_{sl3} I_1 = 0 & \Gamma_{sl3} I_2 = I_2 \\
\hline
\Gamma_{nc1} I_1 = -I_1 I_2 & \Gamma_{nc1} I_2 = -I_2^2 \\
\Gamma_{nc2} I_1 = I_1^2 & \Gamma_{nc2} I_2 = I_1 I_2
\end{array} \tag{4}$$

in which we see that the actions of the different classes of symmetry have noticeably different effects on the fundamental first integrals.

5. The Second Simple Equation

The Symmetries:
The Lie point symmetries of

$$y''' = 0 \tag{5}$$

are

$$\begin{array}{rcll}
\Gamma_h & = & y\partial_y & \text{homogeneity; } 1A_1 \\
\Gamma_{s1} & = & 1\partial_y & \\
\Gamma_{s2} & = & x\partial_y & \text{solution; } 3A_1 \\
\Gamma_{s3} & = & \frac{1}{2}x^2\partial_y & \\
\Gamma_{sl1} & = & \partial_x & \\
\Gamma_{sl2} & = & x\partial_x + y\partial_y & \text{special linear; } A_{3,8}\,(sl(2,R))I \\
\Gamma_{sl3} & = & x^2\partial_x + 2xy\partial_y & \\
\Gamma_{ic1} & = & y'\partial_x + \frac{1}{2}y'^2\partial_y & \\
\Gamma_{ic2} & = & (xy' - y)\,\partial_x + \frac{1}{2}xy'^2\partial_y & \text{intrinsically contact; } 3A_1, \\
\Gamma_{ic3} & = & \left(\frac{1}{2}xy' - y\right)\left[x\partial_x + \left(\frac{1}{2}xy' + y\right)\partial_y\right] &
\end{array} \tag{6}$$

with the intrinsically contact symmetries, Γ_{icj}, $j = 1, 3$, replacing the noncartan point symmetries of (1). The ten symmetries are a representation of the algebra $sp(4)$ [1].

Note that the major difference is that the two-element representations of the abelian algebra, $2A_1$, of (1) have been replaced by three-element representations of the abelian algebra, $3A_1$.

The Fundamental Integrals:

The fundamental first integrals of (5) are

$$\begin{aligned}
I_1 &= y - xy' + \tfrac{1}{2}x^2 y'' \\
I_2 &= y' - xy'' \\
I_3 &= y''
\end{aligned} \tag{7}$$

and the actions of the symmetries on the integrals are given as

$$\begin{array}{lll}
\Gamma_h I_1 = I_1 & \Gamma_h I_2 = I_2 & \Gamma_h I_3 = I_3 \\
\hline
\Gamma_{s1} I_1 = 1 & \Gamma_{s1} I_2 = 0 & \Gamma_{s1} I_3 = 0 \\
\Gamma_{s2} I_1 = 0 & \Gamma_{s2} I_2 = 1 & \Gamma_{s2} I_3 = 0 \\
\Gamma_{s3} I_1 = 0 & \Gamma_{s3} I_2 = 0 & \Gamma_{s3} I_3 = 1 \\
\hline
\Gamma_{sl1} I_1 = -I_2 & \Gamma_{sl1} I_2 = -I_3 & \Gamma_{sl1} I_3 = 0 \\
\Gamma_{sl2} I_1 = I_1 & \Gamma_{sl2} I_2 = 0 & \Gamma_{sl2} I_3 = -I_3 \\
\Gamma_{sl3} I_1 = 0 & \Gamma_{sl3} I_2 = 2I_2 & \Gamma_{sl3} I_3 = 2I_2 \\
\hline
\Gamma_{ic1} I_1 = -\frac{1}{2} I_2^2 & \Gamma_{ic1} I_2 = -I_2 I_3 & \Gamma_{ic1} I_3 = -I_3^2 \\
\Gamma_{ic2} I_1 = I_1 I_2 & \Gamma_{ic2} I_2 = \frac{1}{2} I_2^2 + I_1 I_3 & \Gamma_{ic2} I_3 = I_2 I_3 \\
\Gamma_{ic3} I_1 = -I_1^2 & \Gamma_{ic3} I_2 = -I_1 I_2 & \Gamma_{ic3} I_3 = -\frac{1}{2} I_2^2 .
\end{array} \tag{8}$$

We see that the pattern with the exceptional symmetries, Γ_{icj}, $j = 1, 3$, does not have the simplicity of that shown by the Γ_{ncj}, $j = 1, 2$, of (1) in (4).

6. A Standardised Form for the Exceptional Symmetries

The exceptional symmetries, the noncartan, Γ_{ncj}, of (1) and the intrinsically contact, Γ_{icj}, of (5) have a generally quadratic effect on the fundamental first integrals.

However, there is insufficient evidence to support a postulate of a general formula.

We propose a set of symmetries to replace the exceptional symmetries for (1) and (5) to produce simpler actions than those listed in (4) and (8). We propose that the symmetries have the property

$$\Gamma_{ei} I_j = I_i I_j, \tag{9}$$

where Γ_{ei} is the ith exceptional symmetry and I_j is the jth fundamental first integral.

We take the exceptional symmetries of (1) and (5) to have the form

$$\Gamma_{ei} = \xi_i \partial_x + \eta_i \partial_y, \; i = 1, n, \tag{10}$$

in which the variable dependence in the coefficient functions, ξ_i and η_i has yet to be specified and $n = 2, 3$ respectively.

The First Simple Equation

We commence with (1). The first exceptional symmetry, Γ_{e1}, is required to have the properties

$$\begin{aligned}
&\eta_1'' - y'\xi_1'' = 0 \\
&\eta_1 - y'\xi_1 - (\eta_1' - y'\xi_1')\, x = (y - xy')^2 \\
&\eta_1' - y'\xi_1' = y'\,(y - xy')\,.
\end{aligned} \tag{11}$$

When we substitute (11c) into (11b), we obtain

$$\eta_1 - y'\xi_1 = y(y - xy') = yI_1,$$

where we have made use of the definition of I_1. Note that (11a) and (11c) are differential consequences.

The noncartan symmetry, Γ_8, is recovered by the choice $\xi_1 = xy$ and $\eta_1 = y^2$.

We recover Γ_7 in a similar fashion with the choice $\xi_2 = -y$ and $\eta_2 = 0$.

In the case of the second-order equation the status quo is preserved, apart from the sign, as one would expect from (4).

Alternatively we could forget about sticking to point symmetries and simply write

$$\begin{aligned}
\eta_1 &= y(y - xy') = yI_1 \\
\eta_2 &= y' = I_1.
\end{aligned}$$

These are generalised symmetries.

The second simple equation

In the case of (5) we obtain three sets of equations which are similar in structure to those in (11). By the same process, this time a twofold substitution, we find that

$$\begin{aligned} \eta_1 - y'\xi_1 &= y\left(y - xy' + \tfrac{1}{2}x^2y''\right) = yI_1 \\ \eta_2 - y'\xi_2 &= y\left(y' - xy''\right) = yI_2 \\ \eta_3 - y'\xi_3 &= yy'' = yI_3. \end{aligned} \tag{12}$$

The expressions in (12) contain the second derivative and can be reexpressed in terms of the intrinsically contact symmetries. However, we may repeat what we did for the first simple equation, $y'' = 0$, set the ξ_i, $i = 1, 3$, to zero and simply write

$$\begin{aligned} \eta_1 &= y\left(y - xy' + \tfrac{1}{2}x^2y''\right) = yI_1 \\ \eta_2 &= y\left(y' - xy''\right) = yI_2 \\ \eta_3 &= yy'' = yI_3. \end{aligned} \tag{13}$$

The Lie Brackets of the three symmetries are zero and, as was the case of the intrinsically contact symmetries, these generalised symmetries constitute an abelian subalgebra.

Note that a symmetry $\Gamma = \xi\partial_x + \eta\partial_y$ can always be written as $\Gamma = (\eta - y'\xi)\,\partial_y$ provided that one does not wish to remain within the class of point or contact symmetries.

7. Higher-order equations

Now that we have freed ourselves from the necessity to think only in terms of point or contact symmetries we can deal with higher-order equations. We now enunciate some propositions for the exceptional symmetries, which have now become nonexceptional, for higher-order equations of maximal symmetry.

Proposition 1. *The differential operator*

$$\Gamma_{ej} = \xi_j\partial_x + \eta_j\partial_y \quad \Leftrightarrow \quad (\eta_j - y'\xi_j)\,\partial_y = yI_j\partial_y, \quad j = 1, n, \tag{14}$$

where I_j, $j = 1, n$, is a fundamental first integral of the nth-order ordinary differential equation of maximal order, ie belonging to the equivalence class of $y^{(n)} = 0$ under transformation, is a Lie symmetry of that equation.

Proof The nth extension of Γ_{ej} is

$$\Gamma_{ej}^{[n]} = I_j \sum_{i=0}^{n} y^{(i)}\partial_{y^{(i)}}, \tag{15}$$

whence the result follows by the application of (15) to the differential equation. ■

Remark 1. *We observe that*

$$\Gamma_{ej}^{[k]} = I_i\Gamma_h^{[k]}, \tag{16}$$

where Γ_h is the homogeneity symmetry.

Proposition 2. *The set of n symmetries of the form (14) constitutes an abelian algebra.*

Proof Since the symmetries are generalized, the $(n-1)$th extension is required. The Lie Bracket of Γ_{ei} and Γ_{ej} is

$$\begin{aligned} \left[\Gamma_{ei}, \Gamma_{ej}\right]_{LB} &= \left[\Gamma_{ei}^{[n-1]}, \Gamma_{ej}^{[n-1]}\right]_{LB} = \left[I_i\Gamma_h^{[n-1]}, I_j\Gamma_h^{[n-1]}\right]_{LB} = I_i\left(\Gamma_h^{[n-1]}I_j\right)\Gamma_h^{[n-1]} - I_j\left(\Gamma_h^{[n-1]}I_i\right)\Gamma_h^{[n-1]} \\ &= I_iI_j\Gamma_h^{[n-1]} - I_jI_i\Gamma_h^{[n-1]} = 0. \end{aligned} \tag{17}$$

Proposition 3. *The action of Γ_{ei} on I_j is*

$$\Gamma_{ei}^{[n-1]}I_j = I_iI_j. \tag{18}$$

Proof We have

$$\Gamma_{ei}^{[n-1]}I_j = I_i\Gamma_h^{[n-1]}I_j = I_iI_j. \tag{19}$$

We note that there are general results for the standard homogeneity, solution and special linear symmetries. In the case of the first the results are somewhat obvious, but we state them for the purpose of completion. ■

Proposition 4. *The representative nth-order ordinary differential equation, $y^{(n)} = 0$, possesses the homogeneity symmetry, $\Gamma_h = y\partial_y$, with the property that*

$$\Gamma_h^{[n-1]}I_j = I_j, \tag{20}$$

where I_j is a fundamental first integral.

Proof The proof is evident.

Proposition 5. *The representative nth-order ordinary differential equation possesses n solution symmetries of the form $\Gamma_{si} = s_i\partial_y$, $i = 1, n$, where*

$s_i^{(n)} = 0$, *with the property that*

$$\Gamma_{si}^{[n-1]} I_j = \delta_{ij},\ i, j = 1, n. \qquad (21)$$

Proof Again the proof is trivial.

Proposition 6. *The representative nth-order ordinary differential equation possesses the three-element algebra $sl(2, R)$ in the representation*

$$\begin{aligned} \Gamma_{sl1} &= \partial_x \\ \Gamma_{sl2} &= x\partial_x + \tfrac{1}{2}(n-1)y\partial_y \\ \Gamma_{sl3} &= x^2\partial_x + (n-1)xy\partial_y \end{aligned} \qquad (22)$$

with actions on the fundamental first integrals of

$$\begin{aligned} \Gamma_{sl1}^{[n-1]} I_j &= -I_{j+1}, \quad I_{n+1} = 0 \\ \Gamma_{sl2}^{[n-1]} I_j &= \tfrac{1}{2}(n+1-2j)I_j \\ \Gamma_{sl3}^{[n-1]} I_j &= (j-1)(n+1-j)I_{j-1}, \quad I_0 = 0. \end{aligned} \qquad (23)$$

Proof The proof follows from direct computation. ■

Finally we present the general structures based upon a symmetry of the form $\Gamma = \xi\partial_x + \eta\partial_y$ which follows from the requirements (17), (19), (21) and (23).

Proposition 7. *The general forms of a symmetry $\Gamma = \xi\partial_x + \eta\partial_y$ possessing the characteristic properties of the homogeneity symmetry, the solution symmetries, the special linear symmetries and the exceptional symmetries are*

$$\begin{aligned} \Gamma_h &= \xi_h\partial_x + \eta_h\partial_y &&: \eta_h - y'\xi_h = y, \\ \Gamma_{si} &= \xi_{si}\partial_x + \eta_{si}\partial_y &&: \eta_{si} - y'\xi_{si} = s_i,\ i = 1, n, \\ \Gamma_{sli} &= \xi_{sli}\partial_x + \eta_{sli}\partial_y &&: \eta_{sli} - y'\xi_{sli} = \begin{cases} -y', & i = 1, \\ \tfrac{1}{2}(n-1)y - xy', & i = 2, \\ (n-1)xy - x^2y', & i = 3, \end{cases} \\ \Gamma_{ei} &= \xi_{ei}\partial_x + \eta_{ei}\partial_y &&: \eta_{ei} - y'\xi_{ei} = yI_i,\ i = 1, n, \end{aligned}$$

where s_i, $i = 1, n$, and I_i, $i = 1, n$, are the solutions and fundamental first integrals of the differential equation $y^{(n)} = 0$ respectively, determined from the requirements that

$$\begin{aligned} \Gamma_h^{[n-1]} I_i &= I_i \\ \Gamma_{si}^{[n-1]} I_j &= \delta_{ij} \\ \Gamma_{sli}^{[n-1]} I_j &= \begin{cases} -I_{j+1}, & i = 1 \\ \tfrac{1}{2}(n+1-2j)I_j, & i = 2 \\ (j-1)(n+1-j)I_{j-1}, & i = 3 \end{cases} \end{aligned} \qquad (24)$$

$$\Gamma_{ei}^{[n-1]} I_j = I_iI_j, \qquad (25)$$

for which in the cases of Γ_{si} and Γ_{ei} the index i runs from 1 to n, $j = 1, n$ and we set I_0 and I_{n+1} equal to zero.

Proof Trivial.

8. Concluding Comments

A central issue of this paper is the following lesson. A mathematical theory is developed to deal with a specific class of situations. It is then applied to a wider class of situations and, as has happened here, apparent discrepancies arise. It is important to realise that the discrepancies may be a consequence of limitations in the original theory which do not impact the specific class in which the theory was founded. One must ever be ready to examine carefully the assumptions made in the initial formulation. This is what we have done here and the result has indeed been the removal of the exceptional.

The idea of a generalisation of the symmetries, Γ_{ei}, defined in (14) is easy to develop. For example we may define

$$\Gamma_{eij} = yI_iI_j\partial_y \tag{26}$$

which clearly has the properties

$$\Gamma_{eij}^{[n-1]}I_k = I_iI_jI_k, \quad [\Gamma_{eij}, \Gamma_{elk}]_{LB} = 0. \tag{27}$$

Further generalisations are evident.

Finally we observe that the subalgebras of the different classes of symmetry are abelian with the exception of the three-dimensional subalgebra, $sl(2, R)$. This subalgebra, as represented by point symmetries, is essential for all equations of maximal symmetry and yet is not sufficient for the linearity which is characteristic of all equations of maximal symmetry.

Future research: The properties of the generalised symmetries considered in this work suggest that other generalised symmetries may also have interesting algebraic properties.

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