An Interval Branch-and-Bound Algorithm for Global Optimization using Pruning Steps

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Abstract— **We present a new method for computing verified enclosures for the global minimum value and all global minimum points of univariate functions subject to bound constrains. The method works within the branch and bound framework and incorporates inner and outer pruning steps by using first order information of the objective function by means of an interval evaluation of the derivative over the current interval. First order information aids fourfold. Firstly, to check monotonicity. Secondly, to determine optimal centers which, along with the mean value form, are used to improve the enclosure of the function range. Thirdly, to prune the search interval using the current upper bound of the global minimum, and finally, to apply a more sophisticated splitting strategy. Results of numerical experiments are also presented.**

Keywords— **Global optimization, interval arithmetic, branch-and-bound, optimal center, mean value form, pruning technique.**

I. Introduction

Let $f : \mathcal{D} \to \mathbb{R}$ be a C^1 function, where $\mathcal{D}$ is the closure of a non-empty bounded open subset of $\mathbb{R}$. Our underlying problem can be formulated as follows: Given a compact interval $X \subseteq \mathcal{D}$, find the global minimum

$$f^* = \min_{x \in X} f(x) \qquad (1)$$

and the set $X^* = \{x^* \in X : f(x^*) = f^*\}$ of all global minimizer points of f.

For what it follows we need some notations and notions. Let $\mathbb{I} = \{[a,b] \mid a \leq b,\ a,b \in \mathbb{R}\}$ be the set of compact intervals. If $X = [\underline{x}, \overline{x}]$ is a given interval, where $\underline{x}$ and $\overline{x}$ are the lower and upper endpoints, respectively, we define as $mid(X) = (\underline{x}+\overline{x})/2$ the midpoint and $rad(X) = (\overline{x}-\underline{x})/2$ the radius of X. We call a function $F\colon \mathbb{I} \to \mathbb{I}$ an *inclusion function* of $f\colon \mathbb{R} \to \mathbb{R}$ if $f_{\mathrm{rg}}(X) \subseteq F(X)$ for any $X \in \mathbb{I}$, where $f_{\mathrm{rg}}(X) = \{f(x) : x \in X\}$ be the range of f over X. The inclusion function of the derivative of f is denoted by F'. Inclusion functions can be produced in a number of ways such as natural extension, mean value forms, and Taylor expansion. Each of these forms have slightly different properties and convergence order. For a more thorough discussion on these issues, see [1].

Interval methods for global optimization are based on the branch-and-bound principle [2, Section 5.1]. These methods subdivide the search interval in subintervals (branches) and use bounds for the objective function to eliminate subintervals which cannot contain a global minimizer of f. For a more thorough discussion, see [3], [4], [2]. Interval arithmetic ([5]) is able to provide rigorous bounds on the range of the function over the subintervals. However it does not necessarily provide the exact range, but only an interval which is guaranteed to contain the range. Finding sharper bounds on $f_{\mathrm{rg}}(X)$ with a reasonable amount of computation is a central problem in interval analysis. When the objective function is continuously differentiable, the interval derivative evaluation together with a mean value form is often used to improve the enclosure of the function range. First order information is also used for checking monotonicity (see [6], [2]) in order to discard subregions of the search space. Monotonicity test is the only available accelerating device that uses only first order information (see [3]).

In this paper we present a new interval branch-and-bound algorithm which incorporates a new accelerating device, called *pruning steps*. This accelerating device uses the already available first order information to eliminate regions from the search space once it has been proved that they do not contain the optimal solution. The *inner* pruning step is similar to slope pruning step given by Ratz for nonsmooth functions (see, [7], [8]). The *outer* pruning step is inspired by the technique of Casado et al. [9] that prunes outwardly the search interval with no function evaluations at the endpoints.

Moreover, first order information is utilized to determine the optimal center of a mean value form with respect to the lower bound. This center is also used to improve the current upper bound of the global minimum and to apply a more sophisticated splitting strategy (when it is necessary since in most cases the partitioning process is carried out implicitly from the inner pruning step).

The rest of the paper is organized as follows: In Section II we give some issues about the optimal center and the optimal mean value forms. We next present in algorithmic form the pruning steps, while in Section IV we present our model algorithm. We evaluate the proposed algorithm and compare it with the traditional method that uses bisection and monotonicity test in Section V.

II. Optimal Mean Value Forms

Reducing overestimation in the evaluation of a function $f \in C^1$ can be done by the *mean value form* derived by Moore [5]: Let c be the midpoint of X, $c \equiv mid(X)$, and

$x \in X$. Then

$$f(x) = f(c) + f'(\xi)(x - c), \qquad \xi \in X. \tag{2}$$

This yields

$$f(x) \in F_m(X, c) \equiv f(c) + F'(X) \cdot (X - c), \tag{3}$$

where $F'(X)$ is an interval extension of f'. Since (3) applies for all $x \in X$ the range $f_{\text{rg}}(X)$ must be contained in $F_m(X, c)$. The interval function $F_m\colon \mathbb{I} \to \mathbb{I}$ defined in (3) is called *the mean value form* of f on X with center c. It has been shown in [10] that the mean value form is inclusion monotone, i.e. $X \subseteq Y$ implies $F_m(X) \subseteq F_m(Y)$. As relation (3) shows, to bound f it suffices to bound another function f'. Thus, different expressions of f for the same real function do not affect the value of the mean value form, something that does not hold for the derivative f'.

The following theorem states the main result for the quadratic convergence of mean value form to $f_{\text{rg}}(X)$, i.e., $w\left(F_m(X, c)\right) - w\left(f_{\text{rg}}(X)\right) = \mathcal{O}(w(X)^2)$. A proof can be found in [1, pp. 71].

Theorem 1: Let $f\colon \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ and let F' be an inclusion function for f'. Then, the mean value form (3) is quadratically convergent if the estimation F' satisfies a Lipschitz condition.

In general, the center c in the mean-value form (3) is chosen to be the midpoint of the interval X. However, Baumann in [11] introduced the notion of the "optimal" center of a mean value form and proved that, given the optimal center, the lower bound of the mean value form has the greatest value among all other possible centers within a given interval. In this paper, we study the opposite direction: We determine the center for which the lower bound of the mean value form attains its maximum and prove that this point is identical to the optimal center defined by Baumann. We next give some formal definitions and state some theoretical results.

Definition 1: The mean value form (3) is called *optimal* with respect to the lower bound if

$$\underline{F_m}(X, c) \leq \underline{F_m}(X, c_\star), \tag{4}$$

for any $c \in X$, while the point $c_\star$ which yields the greatest lower bound among all centers is called *optimal center*.

It has been shown in [10] that the mean value form is inclusion monotone. A weaker property that holds for all optimal mean value forms has been introduced in [11], called *one-sided monotonicity*. One-sided monotonicity requires only that at least one bound is improved over smaller intervals while inclusion monotonicity implies that both lower and upper bound are improved simultaneously.

Theorem 2: Let $f : \mathcal{D} \to \mathbb{R}$ be a C^1 function, $X = [\underline{x}, \overline{x}] \in \mathbb{I}$ and $F'(X) = [\underline{d}, \overline{d}]$ be an enclosure of the derivative of f over X. Then,
(i) the lower bound of the mean value form attains its maximum at the center

$$c_\star = \begin{cases} \overline{x}\,, & \text{if } \overline{d} \leq 0, \\ \underline{x}\,, & \text{if } \underline{d} \geq 0, \\ mid(X) - rad(X) \cdot \dfrac{\overline{d} + \underline{d}}{\overline{d} - \underline{d}}\,, & \text{if otherwise.} \end{cases}$$

(ii) the upper bound of the mean value form attains its minimum at the center

$$c^\star = \begin{cases} \underline{x}\,, & \text{if } \overline{d} \leq 0, \\ \overline{x}\,, & \text{if } \underline{d} \geq 0, \\ mid(X) + rad(X) \cdot \dfrac{\overline{d} + \underline{d}}{\overline{d} - \underline{d}}\,, & \text{if } \underline{d} < 0 < \overline{d}. \end{cases}$$

Proof: For a proof, see [12]. ■

Corollary 1: Let $f : \mathcal{D} \to \mathbb{R}$ be a C^1 function, $c_\star \in X = [\underline{x}, \overline{x}]$ be the optimal center with respect to the lower bound and $F'(X) = [\underline{d}, \overline{d}]$ be an enclosure of the derivative of f over X. Then,

$$\text{(i) } \underline{F_m}(X, c_\star) = \begin{cases} f(c_\star) + w(X) \cdot \underline{d}\,, & \text{if } \overline{d} \leq 0, \\ f(c_\star) - w(X) \cdot \overline{d}\,, & \text{if } \underline{d} \geq 0, \\ f(c_\star) + w(X) \cdot \dfrac{\underline{d} \cdot \overline{d}}{\overline{d} - \underline{d}}\,, & \text{otherwise.} \end{cases}$$

$$\text{(ii) } \overline{F_m}(X, c_\star) = \begin{cases} f(c_\star) - w(X) \cdot \underline{d}\,, & \text{if } \overline{d} \leq 0, \\ f(c_\star) + w(X) \cdot \overline{d}\,, & \text{if } \underline{d} \geq 0, \\ f(c_\star) + w(X) \dfrac{\max\{\underline{d}^{\,2}, \overline{d}^{\,2}\}}{\overline{d} - \underline{d}}\,, & \text{otherwise.} \end{cases}$$

The proof comes from the definition of the mean value form along with Theorem 2 and it is omitted.

It is worthy to notice that when $0 \in F'(X)$, the points $c_\star$ and $c^\star$ are symmetric with respect to the midpoint of the interval X. Moreover, their position within a given interval X depends only on the sign of the sum $\overline{d} + \underline{d}$, since the denominator $\overline{d} - \underline{d}$ is always positive. Specifically,
a) If $\overline{d} + \underline{d} > 0$, then $c_\star < mid(X) < c^\star$.
b) If $\overline{d} + \underline{d} < 0$, then $c^\star < mid(X) < c_\star$.
c) If $\overline{d} + \underline{d} = 0$, then $c_\star \equiv mid(X) \equiv c^\star$.

Neumaier has pointed out that a better enclosure for the range can be obtained by using *bicentered forms* [13, pp. 59]. This improvement is achieved by intersecting the mean value forms with centers $c_\star$ and $c^\star$. The price to pay for this improvement is an extra function evaluation at the point $c^\star$. In our problem, however, the upper bound of the mean value form is of no interest since a better upper bound $\widetilde{f}$ for the global minimum can be obtained by evaluating the function at any point c, i.e. $f(c) \leq \overline{F_m}(X, c)$, $\forall\ c \in X$.

III. Pruning the search interval

In this section we briefly present the theoretical background for the pruning (inner and outer) steps of the algorithm. These steps are based on first-order information and serve as an accelerating device in the main part of the algorithm. As we next show, these steps come from the solution of a linear interval inequality which obtained from Taylor expansion around the optimal center $c_\star$.

Let $\widetilde{f}$ denotes the smallest currently known upper bound of f^*. Taking into account that $f \in C^1$, the bound $\widetilde{f}$ can be used in an attempt to prune the current subinterval X. We would like to find an interval enclosure Y of the set of points y for which $f(y) \leq \widetilde{f}$.

Expanding f about the optimal center $c \equiv c_\star \in X$, we obtain that $f(y) = f(c) + (y - c) \cdot f'(\xi)$, where ξ is a point between c and y. Since $c \in X$ and $y \in X$, then $\xi \in X$. Thus,

$$f(y) \in f(c) + F'(X) \cdot (y - c) \leq \widetilde{f}. \tag{5}$$

Let $z = y - c$, $f_c = f(c)$ and $D = [\underline{d}, \overline{d}] = F'(X)$. Then, the interval inequality becomes:

$$\left(f_c - \widetilde{f}\right) + D \cdot z \leq 0. \tag{6}$$

The solution set Z of the above inequality is determined as follows (see,[14],[15]):

$$Z = \begin{cases} [-\infty,\ (\widetilde{f} - f_c)/\overline{d}] \cup & \\ \quad [(\widetilde{f} - f_c)/\underline{d},\ +\infty], & \text{if } \widetilde{f} < f_c,\ \underline{d} < 0 < \overline{d}, \\ \quad [-\infty,\ (\widetilde{f} - f_c)/\overline{d}], & \text{if } \widetilde{f} < f_c,\ \underline{d} \geq 0 \text{ and } \overline{d} > 0, \\ \quad [(\widetilde{f} - f_c)/\underline{d},\ +\infty], & \text{if } \widetilde{f} < f_c,\ \underline{d} < 0 \text{ and } \overline{d} \leq 0, \\ \qquad \emptyset\ , & \text{if } \widetilde{f} < f_c,\ \underline{d} = \overline{d} = 0, \\ \quad [-\infty, +\infty], & \text{if } \widetilde{f} \geq f_c,\ \underline{d} \leq 0 \leq \overline{d}, \\ \quad [-\infty,\ (\widetilde{f} - f_c)/\underline{d}], & \text{if } \widetilde{f} \geq f_c, \underline{d} > 0, \\ \quad [(\widetilde{f} - f_c)/\overline{d},\ +\infty], & \text{if } \widetilde{f} \geq f_c,\ \overline{d} < 0. \end{cases} \tag{7}$$

Recall that $z = y - c$. Therefore the set Y of points $y \in X$ that satisfy (6) is $Y = c + Z$. Since we are only interested in points $y \in X$, we compute the interval enclosure Y as

$$Y = (c + Z) \cap X. \tag{8}$$

The last two cases of (7) is no of interest since the function is strictly monotone in the entire subinterval and so it cannot contain any stationary point in its interior. Thus, the cases of interest are the first five of them. If Z is composed of two intervals Z_1 and Z_2 (first case), then the solution set may be composed by the intervals

$$Y_i = (c + Z_i) \cap X,\ i = 1, 2. \tag{9}$$

If Z is a single interval, then the desired solution of the inequality (6) is an interval Y and as a consequence, the current subinterval X is pruned either from the right or from the left.

The fourth case is a special one since it occurs only when the function f is constant ($\underline{d} = \overline{d} = 0$) within an subinterval X. In this case, X can not contain any global minimizer and is discarded since the current upper bound $\widetilde{f} < f_c$, $\forall\ c \in X$.

In each case the global minimum does not lie in the gap produced by the pruning. Thus, deleting the complement of the computed set never deletes a global minimizer point. If the produced interval Y from (9) is empty, then f exceeds $\widetilde{f}$ and X can be deleted.

As it will be seen from the first four conditions, a pruning step is possible only when $\widetilde{f} < f_c$ holds. In different case (this is the last case of interest), it seems that a pruning is not possible and it is necessary to subdivide the interval. We next give an algorithmic formulation of the pruning steps.

A. Algorithmic formulation of pruning steps

Algorithm 1 implements the outer pruning step and takes as input the subinterval $X = [\underline{x}, \overline{x}]$, $D = [\underline{d}, \overline{d}] = F'(X)$, the current upper bound $\widetilde{f}$ and the lower bounds $\underline{f_L}$ (left) and $\underline{f_R}$ (right) of $f(\underline{x})$ and $f(\overline{x})$, respectively. The algorithm returns the outwardly pruned interval X.

Algorithm 1: OuterPrune$(X, \underline{f_L}, \underline{f_R}, D, \widetilde{f})$

if $\widetilde{f} < \underline{f_L}$ **then** {pruning from left is possible}
    if $\underline{d} < 0$ **then**
        $q = \underline{x} + (\widetilde{f} - \underline{f_L})/\underline{d}$
        if $q \leq \overline{x}$ **then**
            $X = [q, \overline{x}]$;
        else $X = \emptyset$;
if $\widetilde{f} < \underline{f_R}$ **then** {pruning from right is possible}
    if $\overline{d} > 0$ **then**
        $p := \overline{x} + (\widetilde{f} - \underline{f_R})/\overline{d}$
        if $p \geq \underline{x}$ **then**
            $X = [\underline{x}, p]$;
        else $X = \emptyset$;
return X;

The Algorithm 2 takes as input the subinterval $X = [\underline{x}, \overline{x}]$, the center $c \in X$, $f_c = f(c)$, $D = [\underline{d}, \overline{d}] = F'(X)$, the current upper bound $\widetilde{f}$ of the global minimum, and the upper bound $\widehat{f}$ at the moment of creation of the subinterval X, and returns the pruned (or, subdivided) subset $Y_1 \cup Y_2 \subseteq X$ as well as the value of $\widehat{f}$ for the generated subintervals.

Algorithm 2: PruningSteps$(X, \widehat{f}, c, f_c, D, \widetilde{f}, Y_1, Y_2)$

$Y_1 = \emptyset$; $Y_2 = \emptyset$;
if $\widetilde{f} < f_c$ **then** {interior pruning is possible}
    if $\overline{d} > 0$ **then** {pruning from center to the left}
        $p = c + (\widetilde{f} - f_c)/\overline{d}$
        if $p \geq \underline{x}$ **then**
            $Y_1 = [\underline{x}, p]$;
            OuterPrune$(Y_1, \widehat{f}, \widetilde{f}, D, \widetilde{f})$;
    if $\underline{d} < 0$ **then** {pruning from center to the right}
        $q = c + (\widetilde{f} - f_c)/\underline{d}$
        if $q \leq \overline{x}$ **then**
            $Y_2 = [q, \overline{x}]$
            OuterPrune$(Y_2, \widetilde{f}, \widehat{f}, D, \widetilde{f})$;
    $\widehat{f} = \widetilde{f}$;
else { subdivide the interval at point c}
    $Y_1 = [\underline{x}, c]$; $Y_2 := [c, \overline{x}]$;
    OuterPrune$(Y_1, \widehat{f}, f_c, D, \widetilde{f})$;
    OuterPrune$(Y_2, f_c, \widehat{f}, D, \widetilde{f})$;
    $\widehat{f} = f_c$;
return Y_1, Y_2, $\widehat{f}$;

The following theorem summarizes the properties of Algorithm 2. The proof is straightforward from the above analysis.

Theorem 3: Let $f : \mathcal{D} \to \mathrm{I\!R}$, $c \in Y \subseteq X \subseteq \mathrm{I\!R}$. Let also $f_c = f(c)$, $D = F'(Y)$, and $\widehat{f} \geq \widetilde{f} \geq \min_{x \in X} f(x)$, then Algorithm 2 applied as PruningSteps$(Y, \widehat{f}, c, f_c, D, \widetilde{f}, Y_1, Y_2)$ has the following properties:

i) $Y_1 \cup Y_2 \subseteq Y$.

ii) Every global optimizer x^* of f in X with $x^* \in Y$ satisfies $x^* \in Y_1 \cup Y_2$.

iii) If $Y_1 \cup Y_2 = \emptyset$, then there exists no global optimizer of f in Y.

IV. The model algorithm using pruning steps

We have already presented a number of tools that use first-order information and can be incorporated into a simple branch-and-bound model algorithm. The most common technique for the subdivision of the interval into two subintervals uses the midpoint of it. In this work we define a new subdivision strategy which is based on the position of the optimal center $c_\star$. By this way, the interval is splitted into two unequal parts. The optimal center $c_\star$ tends to be a rather good approximation for global minimum points.

We exploit first-order information twofold: Firstly, to check monotonicity. Monotonicity test determines whether the function f is strictly monotone in an subinterval Y. If this is the case, i.e., $0 \notin F'(Y)$, then no stationary point occurs in Y and, hence, Y can be deleted. In different case, the derivative enclosure is used to determine the optimal center $c_\star$. The point $c_\star$ is used to obtain an optimal mean value form with respect to the lower bound and improve the enclosure of the function range. The value $f(c_\star)$ serves to update the current upper bound $\widetilde{f}$. Both $c_\star$ and $f(c_\star)$ are supplied to Algorithm 2 which is responsible for the pruning as well as for the splitting process.

The above ideas are summarized in Algorithm 3. Cutoff test is additionally used to remove from the working list $\mathcal{L}$ all elements $(X, \underline{f_X})$ such that $\underline{f_X} > \widetilde{f}$. Search is performed by preferring the subintervals X according to the best-first strategy. This can be quickly determined by using and maintaining a priority queue or list sorted in nondecreasing order.

Algorithm 3: GlobalOptimize$(f, X, \epsilon, F^\star, \mathcal{L}^\star)$

1. $\underline{f_L} = \inf f(\underline{x})$; $\underline{f_R} = \inf f(\overline{x})$;
2. $c = \mathrm{OptimalCenter}(X, F'(X))$;
3. $\widetilde{f} = \min\{\underline{f_L}, \underline{f_R}, \sup f(c)\}$;
4. $\mathrm{OuterPrune}(X, \underline{f_L}, \underline{f_R}, F'(X), \widetilde{f})$;
5. $F_X = (f(c) + F'(X) \cdot (X - c)) \cap F(X)$;
6. $\mathcal{L} = \{(X, \underline{f_X}, \widetilde{f})\}$; $\mathcal{L}^\star = \{\}$;
7. **while** $\mathcal{L} \neq \{\}$ **do**
8. $\quad (Y, \underline{f_Y}, \widehat{f}) = \mathrm{Head}(\mathcal{L})$;
9. $\quad c = \mathrm{OptimalCenter}(Y, F'(Y))$; $f_c = \inf f(c)$;
10. $\quad \mathrm{PruningSteps}(Y, \widehat{f}, c, f_c, F'(Y), \widetilde{f}, Y_1, Y_2)$
11. $\quad$ **for** $i = 1$ **to** 2 **do**
12. $\quad\quad$ **if** $Y_i = \emptyset$ **then next** i;
13. $\quad\quad$ **if** $0 \notin F'(Y_i)$ **then next** i;
14. $\quad\quad c = \mathrm{OptimalCenter}(Y_i, F'(Y_i))$;
15. $\quad\quad \mathrm{AdjustCenter}(Y_i, \epsilon, c)$;
16. $\quad\quad$ **if** $\sup f(c) < \widetilde{f}$ **then** $\widetilde{f} = \sup f(c)$;
17. $\quad\quad F_Y = (f(c) + F'(Y_i) \cdot (Y_i - c)) \cap F(Y_i)$;
18. $\quad\quad$ **if** $\underline{f_Y} \leq \widetilde{f}$ **then**
19. $\quad\quad\quad$ **if** $d_{\mathrm{rel}}(F_Y) \leq \epsilon$ **or** $d_{\mathrm{rel}}(Y_i) \leq \epsilon$ **then**
20. $\quad\quad\quad\quad \mathcal{L}^\star = \mathcal{L}^\star \uplus (Y_i, \underline{f_Y})$
21. $\quad\quad\quad$ **else**
22. $\quad\quad\quad\quad \mathcal{L} = \mathcal{L} \uplus (Y_i, \underline{f_Y}, \widehat{f})$
23. $\quad$ **end for**
24. $\quad \mathrm{CutOffTest}(\mathcal{L}, \widetilde{f}\ )$;
25. **end while**
26. $(Y, \underline{f_Y}) = \mathrm{Head}(\mathcal{L}^\star)$; $F^\star = [\underline{f_Y}, \widetilde{f}]$;
27. $\mathrm{CutOffTest}(\mathcal{L}^\star, \widetilde{f}\ )$;
28. **return** $F^\star$, $\mathcal{L}^\star$.

The step 15 of Algorithm 3 is responsible for adjusting the optimal center when it tends to be at the endpoint of the interval. ϵ_m is the machine epsilon.

Algorithm 4: AdjustCenter(X, ϵ, c)

if $w(X) \geq 2\epsilon$ **and** $w(X) > \sqrt{\epsilon_m}$ **then**
$\quad$ **if** $c - \underline{x} < \epsilon$ **then** $c = \underline{x} + \epsilon$; $\qquad$ {c tends to left endpoint}
$\quad$ **if** $\overline{x} - c < \epsilon$ **then** $c = \overline{x} - \epsilon$; $\qquad$ {c tends to right endpoint}
else
$\quad c = mid(X)$;
return c;

The properties of Algorithm 3 are summarized in the following theorem.

Theorem 4: Let $f : \mathcal{D} \to \mathbb{R}$, $X \subseteq \mathcal{D} \subseteq \mathbb{R}$, and $\epsilon > 0$. Then Algorithm 6 has the following properties:

i) $f^* \in F^*$.

ii) $X^* \subseteq \bigcup_{(Y, \underline{f_Y}) \in \mathcal{L}^\star} Y$.

V. Numerical results

Algorithm 3 presented in the previous section has been implemented and tested on the complete set of 20 smooth test functions given in [8]. The implementation has been carried out in C++ using the C-XSC library [16] and the basic toolbox modules for automatic differentiation [6].

We compare our algorithm which uses Pruning steps, with the corresponding method using only Monotonicity test and bisection. In our algorithm we have adopted the new subdivision strategy based on the position of optimal center within an interval.

Numerical results are summarized in Table 1 and are obtained with $\epsilon = 10^{-8}$. For each test function we report the number of function evaluations, the number of derivative evaluations, and the number of subdivisions. The last row of the table gives average values for the complete test set.

Numerical results indicate that the (P) method which uses pruning steps is better than the traditional method with monotonicity test. Moreover, average analysis shows that (P) method always outperforms (M). Compared with (M) method, (P) method exhibits an improvement of more than 49% in the function and derivative evaluations, and 85% in the number of subdivisions.

The new algorithm works on average two times faster in comparison with the traditional method on a set of 20 smooth test functions.

VI. Conclusions

In this paper we presented a new branch-and-bound algorithm for global optimization based on the existence of an optimal center. Optimal center is calculated by simple expressions with no extra computational effort and seems to be a good approximation for global minimum points. A good upper bound for the global minimum is obtained on early stages of the algorithm and, thus, no local search steps are necessary.

	Function eval.			Derivative eval.			Subdivisions		
No.	M	P	P/M	M	P	P/M	M	P	P/M
1	113	97	86%	71	55	77%	35	8	23%
2	296	172	58%	173	97	56%	86	3	3%
3	151	53	35%	97	31	32%	48	5	10%
4	112	35	31%	71	19	27%	35	4	11%
5	410	165	40%	263	100	38%	131	8	6%
6	1220	790	65%	617	429	70%	308	9	3%
7	951	650	68%	477	339	71%	238	9	4%
8	66	32	48%	41	18	44%	20	5	25%
9	86	22	26%	55	12	22%	27	1	4%
10	91	78	86%	53	42	79%	26	6	23%
11	261	172	66%	155	96	62%	77	9	12%
12	120	22	18%	75	11	15%	37	3	8%
13	50	10	20%	33	5	15%	16	2	13%
14	136	42	31%	83	21	25%	41	1	2%
15	68	43	63%	41	23	56%	20	9	45%
16	108	14	13%	71	7	10%	35	3	9%
17	254	148	58%	153	82	54%	76	11	14%
18	450	304	68%	291	182	63%	145	6	4%
19	75	52	69%	49	30	61%	24	8	33%
20	71	52	73%	41	25	61%	20	10	50%
$\sum$	5089	2953	58%	2910	1624	56%	1445	120	8%
Ø			51%			47%			15%

Taking advantage of the position of the optimal center, bisection can now be replaced with a more sophisticated splitting strategy. The results of this paper clearly establish that optimal centers are not only important in mean value forms but in the pruning steps of the algorithm, too. The pruning steps, along with the monotonicity test, offers the possibility to throw away large parts of the search interval. Thus, a new accelerating device is now available.

Acknowledgments

The authors would like to thank Dr. L.G. Casado for his valuable comments on an earlier version of this article.

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