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An identity relating Fibonacci and Lucas numbers of order k

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Abstract

The following relation between Fibonacci and Lucas numbers of order k,

$$\sum_{i=0}^{n} m^i \left[L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2)F_{i-j+1}^{(k)} \right] = m^{n+1}F_{n+1}^{(k)} + k - 2,$$

is derived by means of colored tiling. This relation generalizes the well-known Fibonacci - Lucas identities, $\sum_{i=0}^{n} 2^i L_i = 2^{n+1}F_{n+1}$, $\sum_{i=0}^{n} 3^i (L_i + F_{i+1}) = 3^{n+1}F_{n+1}$ and $\sum_{i=0}^{n} m^i (L_i + (m-2)F_{i+1}) = m^{n+1}F_{n+1}$ of A.T. Benjamin and J.J. Quinn, D. Marques, and T. Edgar, respectively.

Keywords: Fibonacci numbers, Lucas numbers, order k, color tiling, generalization.

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1 Introduction

Let k be a fixed integer greater than or equal to 2, and let n be a nonnegative integer, unless otherwise specified. We define the Fibonacci and Lucas numbers of order k as follows (see also [3, 5, 9–12]).

Definition 1.1 The sequence $\{F_n^{(k)}\}_{n\geq -k+1}$ is said to be the sequence of Fibonacci numbers of order k, if $F_n^{(k)} = 0$ for $-k+1 \leq n \leq 0$, $F_1^{(k)} = 1$, and $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ for $n \geq 2$.

Definition 1.2 The sequence $\{L_n^{(k)}\}_{n\geq 0}$ is said to be the sequence of Lucas numbers of order k, if $L_0^{(k)} = k$, $L_1^{(k)} = 1$, $L_n^{(k)} = n + \sum_{j=1}^{n-1} L_{n-j}^{(k)}$ for $2 \leq n \leq k$, and $L_n^{(k)} = \sum_{j=1}^{k} L_{n-j}^{(k)}$ for $n \geq k+1$.

Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then,

$$F_n^{(2)} = F_n \quad \text{and} \quad L_n^{(2)} = L_n.$$

Benjamin and Quinn [1] used tiling and, among other identities, derived the following one (see also [2, 6, 13]),

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1}. \tag{1}$$

Recently, Marques [7] (see also Martinjak [8]) derived the new identity

$$\sum_{i=0}^{n} 3^i (L_i + F_{i+1}) = 3^{n+1} F_{n+1}, \tag{2}$$

and Edgar [4] stated and proved the following one encompassing both (1) and (2), that is

$$\sum_{i=0}^{n} m^i (L_i + (m-2)F_{i+1}) = m^{n+1} F_{n+1}, \tag{3}$$

where m is an integer greater than or equal to 2.

In this paper, we employ a known lemma (see Lemma 2.1) and a new lemma (see Lemma 2.2) to derive the following generalization of the above identities

$$\sum_{i=0}^{n} m^i \left[L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2) F_{i-j+1}^{(k)} \right] = m^{n+1} F_{n+1}^{(k)} + k - 2,$$

utilizing tiling with m colors and tiles of length at most k. Our proof depends on the following preliminary results.

2 Preliminary Results

We consider the numbers $f_n^{(k)}$ and $c_n^{(k)}$ of an n-board and an n-bracelet tiling, respectively, using j-squares, $j = 1,2,\ldots,k$. A j-square is a tile of length j. Notice that, an 1-square is simply called a *square* and a 2-square a *domino*. Then, the following lemma is a special case of Combinatorial Theorem 4 of Benjamin and Quinn [2] (see pp. 36).

Lemma 2.1 *Let $f_n^{(k)}$ be the number of an n-board tilings using j-squares ($j = 1,2,\ldots,k$) and $F_n^{(k)}$ be the n-th Fibonacci number of order k. Then, $f_n^{(k)} = F_{n+1}^{(k)}$.*

Next, we state and prove a lemma of our own.

Lemma 2.2 *Let $c_n^{(k)}$ be the number of an n-bracelet tilings using j-squares ($j = 1,2,\ldots,k$) and $L_n^{(k)}$ be the n-th Lucas number of order k. Then,*

$$c_n^{(k)} = L_n^{(k)}.$$

Proof. We note that $c_n^{(k)}$ is equal to the sum of the number of in phase n-bracelets and the number of out of phase n-bracelets. The number of in phase n-bracelets is $f_n^{(k)}$. An out of phase n-bracelet may end in a 2-square, or a 3-square,..., or a k-square. The number of out of phase n-bracelets ending in a j-square ($j = 2,3,\ldots,k$) is $(j-1)f_{n-j}^{(k)}$. Therefore,

$$\begin{aligned} c_n^{(k)} &= f_n^{(k)} + \sum_{j=2}^{k}(j-1)f_{n-j}^{(k)} = F_{n+1}^{(k)} + \sum_{j=2}^{k}(j-1)F_{n-j+1}^{(k)}, \text{ by Lemma 2.1} \\ &= F_{n+1}^{(k)} + \sum_{j=2}^{k} jF_{n-j+1}^{(k)} - \sum_{j=2}^{k} F_{n-j+1}^{(k)} = L_n^{(k)}, \end{aligned}$$

by Definition 1 and the relation $L_n^{(k)} = \sum_{j=1}^{k} jF_{n-j+1}^{(k)}$ for $n \geq 1$ (see [3, 12]). □

We proceed now to state and prove our main result, using colored tiling.

3 Main result

Theorem 3.1 *Let $\{F_n^{(k)}\}_{n\geq -k+1}$ and $\{L_n^{(k)}\}$ be the Fibonacci sequence of order k and the Lucas sequence of order k, respectively. Then, for $m \geq 2$, and $n \geq 0$,*

$$\sum_{i=0}^{n} m^i \left[L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k}(j-2)F_{i-j+1}^{(k)} \right] = m^{n+1}F_{n+1}^{(k)} + k - 2.$$

Proof. We consider colored tiling with m colors. Let $A_{n,m}^{(k)}$ and $B_{n,m}^{(k)}$ be the sets of an n-board colored tiling and an n-bracelet colored tiling, respectively, using j-squares ($j=2,3,\ldots,k$) in m^j different colors. Clearly,

$$|A_{n,m}^{(k)}| = m^n f_n^{(k)}, \qquad \text{and} \qquad |B_{n,m}^{(k)}| = m^n c_n^{(k)}. \tag{4}$$

Without loss of generality, we suppose that one of the first m colors is white, and we consider the following subsets of $A_{n,m}^{(k)}$ and $B_{n,m}^{(k)}$.

(a) $A_{n,m}^{(k,j_i)}$ denotes the set of n-board colored tiling with a non-white j-square ($j = 1,2,\ldots,k$) ending on the i-th cell and white 1-squares on the cells $i+1, i+2,\ldots,n$, ($i=1,2,\ldots,n$).

(b) $B_{n,m}^{(k_l)}$ denotes the set of in-phase and out of phase n-bracelet colored tilings ending in an l-square, $l=1,2,\ldots,k$ (the distinction between in-phase and out of phase n-bracelet tilings originates from Benjamin and Quinn [1]).

Clearly,

$$\sum_{i=1}^{n}\sum_{j=1}^{k} |A_{n,m}^{(k,j_i)}| = |A_{n,m}^{(k)}| - 1, \tag{5}$$

since the left hand-side of the equality does not include the unique all-white squares tiling of a board of length n.

Consider the subsets $A_{n,m}^{(k,1_i)}$ ($i=1,2,\ldots,n$) of $A_{n,m}^{(k)}$. Removing white tiles from $i+1, i+2,\ldots,n$ and gluing cells i and 1 together, we get i-bracelet colored tilings ending in squares of $m-1$ different colors ($m \geq 2$). Since the bracelets ending in squares can end in m different colors, we obtain

$$|B_{i,m}^{(k_1)}| = \frac{m}{m-1}|A_{n,m}^{(k,1_i)}|. \tag{6}$$

Next, let us consider the following subsets $A_{n,m}^{(k,j_i)}$ ($j \leq i \leq n$ and $j=2,3,\ldots,k$) of $A_{n,m}^{(k)}$. Removing white tiles from $i+1, i+2,\ldots,n$ and gluing cells i and 1 together, we get i-bracelet colored tilings ending in j-squares. Taking into consideration that for each in-phase i-bracelet ending in a j-square (the j-square covers cells $i-j+1$, $i-j+2$, …, i) there are $j-1$ more out of phase j-squares, we get

$$|B_{i,m}^{(k_j)}| = j|A_{n,m}^{(k,j_i)}|, \tag{7}$$

where $j \leq i \leq n$ and $j=2,3,\ldots,k$. Using (4) and Lemma 2.2, we get

$$\sum_{i=1}^{n} |B_{i,m}^{(k)}| = \sum_{i=1}^{n} m^i c_i^{(k)} = \sum_{i=1}^{n} m^i L_i^{(k)}. \tag{8}$$

Next, using (6) and (7) as well as (5) and the equality $|A_{n,m}^{(k,j_i)}| = m^j |A_{i-j,m}^{(k)}|$, which holds true since the j-square can be tiled in m^j different ways and the rest of the board in $|A_{i-j,m}^{(k)}|$ ways, we get

$$\begin{aligned}\sum_{i=1}^{n} |B_{i,m}^{(k)}| &= \sum_{i=1}^{n}\sum_{l=1}^{k} |B_{i,m}^{(k_l)}| = \sum_{i=1}^{n}\left(\frac{m}{m-1}|A_{n,m}^{(k,1_i)}| + \sum_{j=2}^{k} j|A_{n,m}^{(k,j_i)}|\right)\\
&= \frac{1}{m-1}\sum_{i=1}^{n}\left(m|A_{n,m}^{(k,1_i)}| + (m-1)\sum_{j=2}^{k} j|A_{n,m}^{(k,j_i)}|\right)\\
&= \frac{1}{m-1}\sum_{i=1}^{n}\left(m|A_{n,m}^{(k,1_i)}| + \sum_{j=2}^{k} m|A_{n,m}^{(k,j_i)}| + \sum_{j=2}^{k}\left((m-1)j-m\right)|A_{n,m}^{(k,j_i)}|\right)\\
&= \frac{1}{m-1}\left(m\sum_{i=1}^{n}\sum_{j=1}^{k}|A_{n,m}^{(k,j_i)}| + \sum_{i=1}^{n}\sum_{j=2}^{k}\left((j-1)m-j\right)|A_{n,m}^{(k,j_i)}|\right)\\
&= \frac{1}{m-1}\left(m\left(|A_{n,m}^{(k)}|-1\right) + \sum_{i=1}^{n}\sum_{j=2}^{k} m^j\left((j-1)m-j\right)|A_{i-j,m}^{(k)}|\right). \qquad (9)\end{aligned}$$

Noting that $|A_{n,m}^{(k)}| = m^n F_{n+1}^{(k)}$ and $|A_{i-j,m}^{(k)}| = m^{i-j} F_{i-j+1}^{(k)}$ because of (4) and Lemma 2.1, relations (8) and (9) imply

$$\sum_{i=1}^{n} m^i L_i^{(k)} = \frac{1}{m-1}\left(m^{n+1}F_{n+1}^{(k)} - m + \sum_{i=1}^{n}\sum_{j=2}^{k} m^i\left((j-1)m-j\right)F_{i-j+1}^{(k)}\right). \qquad (10)$$

Rearranging relation (10) and using some simple calculations and the relations

$$L_n^{(k)} = \sum_{j=1}^{k} jF_{n-j+1}^{(k)} \quad (n \geq 1) \qquad \text{and} \qquad F_{i+1}^{(k)} = \sum_{j=1}^{k} F_{i-j+1} \quad (i \geq 1),$$

we get

$$\sum_{i=1}^{n} m^i\left[L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=2}^{k}(j-2)F_{i-j+1}^{(k)}\right] = m^{n+1}F_{n+1}^{(k)} - m,$$

from which the theorem follows. □

We end this paper by noting that, for $k = 2$, Theorem 3.1 readily reduces to identity (3). If in addition $m = 2$ (respectively 3), Theorem 3.1 reduces to identity (1) (respectively identity (2)).

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