Bounding the Zeros of an Interval Equation

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Abstract

In this paper, we consider the problem of finding reliably and with certainty all zeros of an interval equation within a given search interval for continuously differentiable functions over real numbers. We propose a new formality of interval arithmetic which is treated in a theoretical manner to develop and prove a new method, lying on the context of interval Newton methods. Some important theoretical aspects of the new method are stated and proved. Finally, an algorithmic realization of our method is proposed to be applied on a set of test functions, where the promising theoretical results are verified.

Key words: Interval equation, Interval zeros, Hull interval arithmetic, Interval Newton method, Interior part of interval zero
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1 Introductory notions on interval equations

In this work we deal with the often occurring problem of finding reliably and with certainty all zeros of the equation

$$f(x) = 0, \tag{1.1}$$

for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers.

In many fields of science (e.g. chemical engineering: [2], computer graphics: [3], robotics: [12], control theory: [16], etc) the determination of all zeros of (1.1) remains a crucial and well-timed issue. Interval methods have shown a

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great performance in solving it, resulting with guaranteed sharp bounds for each zero. However, since experimental measurements are not accurate, a great necessity arises to express them by quantities that contain the actual values of problem's parameters. For this reason, we enable the use of interval numbers to enclose and bound those inaccurate measurements. Thus, we can restate the problem definition (1.1), employing interval parameters: *Find reliably and with certainty all zeros of equation*

$$f(x;p) = 0, \tag{1.2}$$

for all $p \in [p]$, with $[p] \in \mathbb{IR}$, where $\mathbb{IR}$ denotes the set of real intervals.

Now, if we consider the natural interval extension of $f(x;p)$ with respect to the parameter vector p, we arrive to $f(x;[p])$. So, we can define the notion of interval equation.

Definition 1.1 *Let $f : \mathbb{R}\times\mathbb{IR} \to \mathbb{IR}$ be a continuously differentiable function, $[x]_0$ a search interval and $[p]$ an interval vector containing all the parameters involved in f. The equation of the form*

$$f(x;[p]) = 0 \tag{1.3}$$

will be called interval equation by the meaning of (1.2).

The set of all zeros of (1.3) will be the union of the zeros of all equations (1.2). The solution formulation of the above equation will be either the empty interval (no zero case), or a degenerate interval (a point-zero case - e.g. multiple zero), or a non-degenerate interval. Apparently, for the last case, the number of point-zeros that have to be enclosed are of infinite number. But, since the parameters of (1.3) are varied continuously (closed and compact intervals) the point-zeros of $f(x;[p]) = 0$ will vary continuously too and the number of those continuum of zeros will be of finite number. Thus, instead of searching for all the infinite point-zeros of (1.3), we search for intervals where point-zeros are varied continuously in them.

Remark 1.2 *From now on, for convenience and without misunderstanding we will denote an interval equation as $f(x) = 0$ instead of $f(x;[p]) = 0$.*

Definition 1.3 *Every continuous set of point-zeros of $f(x,p) = 0$, for every $p \in [p] \in \mathbb{IR}$ constitutes a solution of the corresponding interval equation $f(x) = 0$ and will be called interval zero. If the interval zero contains a single point-zero will be called degenerate interval zero.*

The first attempt in solving (1.3) was at the origin of interval analysis, in the book of Ramon Moore "Interval Analysis" ([8, pp. 60–61]), where a simple example of interval equation is given and it is treated as in most equations, applying an early scheme of interval Newton method to enclose its zeros.

In 1992, Hansen ([6, pp. 73–74]) proposed a heuristic stopping criterion to solve interval equations using interval Newton method, by accepting parts of search interval as enclosures of interval zeros.

Recently, Nikas et al. ([11]) proposed a method which is conceptualized on a boundary approach in solving interval equations, with promising results.

In this work we propose an interval Newton-based method for solving interval equations. In order to provide and derive the proposal idea we state and prove an equivalent formulation of interval arithmetic utilizing the interval hull operator. Some theoretical results are provided, while the algorithm of the new method is proposed. The main concept of our proposal consists in the isolation of the endpoints of interval zeros, excluding guaranteed inner parts of it. The immediate result is the shrinking of the search interval, and the method, finally, provides sharp bounds on every interval zero. The presented numerical results verify our expectations.

Briefly, in §2 we give some necessary prerequisites and definitions, while in §3 we describe the "classic" interval Newton method, stating some serious drawbacks in solving (1.3). In §4 we derive our method and in §5 we establish our proposal in a theoretical manner. In §6 we propose an algorithm for our method and in §7 we present some numerical results. Finally, in §8 we conclude, discussing our results.

2 On intervals

In the following, we give some basic definitions on intervals and related terms. A more thoroughly study can be found in [1], [6] and [10].

2.1 Elementary definitions and notation

A closed, compact interval is a set of the form

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x}\},$$

where $\underline{x}, \overline{x} \in \mathbb{R}$ denote the lower and upper bound of interval $[x]$, respectively. The set of all closed and compact intervals is denoted by $\mathbb{IR}$, while $[/]$ denotes the empty interval: $[/] \subseteq [x]$ for all $[x] \in \mathbb{IR}$.

An interval $[x]$ is said to be contained in the interior of interval $[y]$ if $\underline{y} < \underline{x}$ and $\overline{y} > \overline{x}$. It is written $[x] \overset{\circ}{\subset} [y]$ and it is called inner inclusion relation.

The width of an interval is given by a real-valued function $w : \mathbb{IR} \to \mathbb{R}$, which is defined by

$$w([x]) = \overline{x} - \underline{x}.$$

Similarly, the midpoint of an interval $[x]$ is given by a real-valued function $m : \mathbb{IR} \to \mathbb{R}$, defined by

$$m([x]) = \frac{\overline{x} + \underline{x}}{2}.$$

An interval vector $[a] \in \mathbb{IR}^n$ is a vector whose components are, in general, intervals.

The hull of a non-empty subset $S \subset \mathbb{R}$, denoted by the unary operator $\square$, is defined as the tightest interval enclosing S. Thus, if $S = \{a_1, a_2, \ldots, a_n\}$, $n \in \mathbb{N}$, then

$$\square \{a_1, a_2, \ldots, a_n\} = [\min \{a_1, a_2, \ldots, a_n\}, \max \{a_1, a_2, \ldots, a_n\}]$$

Now, for the case of intervals, a binary operator is defined, the hull of two intervals, as it is given in [4]:

$$[x] \sqcup [y] = \left[\min \left\{\underline{x}, \underline{y}\right\}, \max \{\overline{x}, \overline{y}\}\right],$$

where $[x],\ [y] \in \mathbb{IR}$.

The range of a real continuously differentiable function f over a closed and compact interval $[x] \subset \mathbb{R}$ is defined as:

$$f_{rg}([x]) = \square \{f(x) \mid x \in [x]\} = \{f(x) \mid x \in [x]\},$$

but, through the continuity of $[x]$ we result to

$$f_{rg}([x]) = \left[\underline{f}, \overline{f}\right],$$

where $\underline{f}$ and $\overline{f}$ denote the minimum and maximum value in the range of f. Sometimes, a sufficient tight range may be adequate by using an interval extension F of f, in conjunction with interval arithmetic. Interval extensions provide enclosures of the range of a real function

$$f_{rg}([x]) \subseteq F([x]).$$

Finally, an interval function $F : \mathbb{IR} \to \mathbb{IR}$ is called inclusion isotone if for all $[x],\ [y] \in \mathbb{IR}$

$$[x] \subseteq [y] \Rightarrow F([x]) \subseteq F([y]).$$

The elementary operations $\circ \in \{+, -, \cdot, /\}$ can be extended to the interval operands under the following definition:

$$[a] \circ [b] = \square \{a \circ b \mid \forall a \in [a],\ \forall b \in [b]\} = \{a \circ b \mid \forall a \in [a],\ \forall b \in [b]\}\,, \quad (2.1)$$

where $[a],\ [b] \in \mathbb{IR}$. However, the above definition is equivalent to the following four rules ([13]),

$$\begin{aligned} [a] + [b] =& [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \\ [a] - [b] =& [\underline{a} - \overline{b}, \overline{a} - \underline{b}] \\ [a] \cdot [b] =& \left[\min\left\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\right\}, \max\left\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\right\}\right] \\ \frac{[a]}{[b]} =& \left[\min\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}, \max\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}\right], \quad 0 \notin [b] \end{aligned} \quad (2.2)$$

Theorem 2.1 *The elementary operations $\circ \in [+, -, \cdot, /]$ are inclusion isotonic.*

PROOF. The proof is in [14]. $\square$

Let us now introduce a new formulation of interval arithmetic as well as a proof of its inclusion isotonicity. We suppose that $\circ \in [+, -, \cdot, /]$ denotes the elementary operations. Similarly, as above, we can extend them to operands where the first one is real number and the second one an interval:

$$a \circ [b] = \square \{a \circ b \mid \forall b \in [b]\} = \{a \circ b \mid \forall b \in [b]\}\,,$$

where $[b],\ a \circ [b] \in \mathbb{IR}$. Using this definition we can rewrite (2.1) obtaining,

$$[a] \circ [b] = \square \{a \circ [b] \mid \forall a \in [a]\}$$

where $[a] \in \mathbb{IR}$. The endpoints of the interval $[a] \circ [b]$ will be contained either in $\underline{a} \circ [b]$ or in $\overline{a} \circ [b]$. Consequently, we have

$$[a] \circ [b] = \square \{a \circ [b] \mid \forall a \in [a]\} = \{\underline{a} \circ [b]\} \sqcup \{\overline{a} \circ [b]\}\,, \quad (2.3)$$

which constitutes an alternative definition of interval arithmetic utilizing the interval hull operator (for the needs of this work we will refer to this expression of interval arithmetic as *Hull Interval Arithmetic*). Now, we can easily convert definition (2.3) to its implementable form as it follows:

$$
\begin{aligned}
[a] + [b] &= \{\underline{a} + [b]\} \sqcup \{\overline{a} + [b]\}, \\
[a] - [b] &= \{\underline{a} - [b]\} \sqcup \{\overline{a} - [b]\}, \\
[a] \ \cdot \ [b] &= \{\underline{a} \cdot [b]\} \sqcup \{\overline{a} \cdot [b]\}, \\
\frac{[a]}{[b]} &= \left\{ \frac{\underline{a}}{[b]} \right\} \sqcup \left\{ \frac{\overline{a}}{[b]} \right\}, \quad 0 \notin [b].
\end{aligned}
\tag{2.4}
$$

Lemma 2.2 *Interval arithmetic defined in* (2.2) *and interval arithmetic defined in* (2.4) *are equivalent.*

PROOF. The proof of the above lemma is trivial and it is omitted.

Proposition 2.3 *The elementary operations* $\circ \in \{+, -, \cdot, /\}$ *as defined in* (2.4) *are inclusion isotonic.*

PROOF. The proof comes easy from Lemma 2.2.

Remark 2.4 *In this paper, we will use the hull interval arithmetic only in a theoretical point of view, since its numerical study consists a part of a future work.*

3 Classic interval Newton method and drawback issues

The derivation of classic interval Newton arise from the application of Mean Value Theorem in combination with the Fundamental Theorem of Interval Arithmetic ([8]), resulting in the following relation:

$$
x^* \in m - \frac{F(m)}{F'([x])} =: N([x]), \tag{3.1}
$$

with $x^* \in [x]$, being a zero of $f(x) = 0$, $m \in [x]$, F and F' interval extensions of f and f', respectively, and $N([x])$ denoting the interval Newton operator over interval $[x]$. The corresponding iterative scheme of (3.1) is given by

$$
\begin{aligned}
m_k &= m([x]_k), \\
N([x]_k) &= m_k - \frac{F(m_k)}{F'([x]_k)}, \\
[x]_{k+1} &= [x]_k \cap N([x]_k).
\end{aligned}
\tag{3.2}
$$

Many theoretical aspects of interval Newton methods, concerning, mostly, issues of zero existence and uniqueness of $f(x) = 0$ and the accuracy sharpness of the obtained enclosures of those zeros, have been discussed widely in literature (see mainly [1], [5], [6], [8], [10]).

Hansen in [5] used the extended interval arithmetic to define an extension of interval Newton method, to handle, in general, the case where $0 \in F'([x])$.

The main defection of the method arises when, the midpoint of the search interval $[x]$ is a zero of the equation $f(x) = 0$ and at the same time $0 \in F'([x])$. The only, so far, efficient treatment consists in the adoption of heuristics, e.g. a bisection scheme ([8]).

It is known that, if we apply the interval Newton method to an interval equation $f(x) = 0$ we will get some wider results, as it would be expected. For example, suppose we have a simple quadratic interval equation,

$$f(x) \equiv x^2 - [-1, 1] = 0, \tag{3.3}$$

and we want to isolate all interval zeros of (3.3) over the search interval $[-3, 3]$, accepting intervals as solution with tolerance $\varepsilon = 10^{-3}$ $(w([x] \leq \varepsilon)$. Taking a

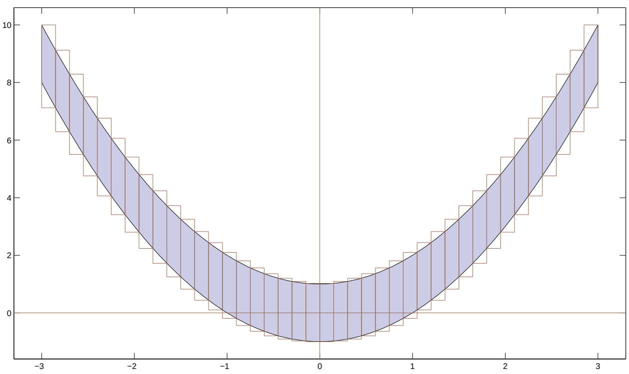


Fig. 1. Interval equation $x^2 - [-1, 1]$ over [-3,3].

short look to Figure 1 we can deduce that interval equation (3.3) has a unique interval zero $([x]^* = [-1, 1])$ over $[-3, 3]$. Applying once the iterative scheme (3.2), we get:

$$N([x]) = m([x]) - \frac{F(m([x]))}{F'([x])} = 0 - \frac{0}{[-26, 26]} = [-\infty, +\infty].$$

As we can see, the algorithm needs only one iteration, revealing its weakness to reduce the initial search interval to a smaller one.

If we adopt Hansen's termination criterion (given in [6]) the algorithm will return the initial interval $[-3, 3]$ as an interval zero, which is too overestimated with respect to the actual interval zero $([x]^* = [-1, 1])$.

The choice of a bisection scheme ([9, pp. 77–81]), to overcome the aforementioned impasse result, will make the algorithm to return 3072 result intervals! Furthermore each result interval will contain at least a point zero of (3.3). However, the union of the resulted intervals gives an interval that encloses sharply the interval zero $[-1, 1]$. From a theoretical point of view, the union of these intervals is not with certainty an interval zero, but an interval containing at least 3072 point zeros of (3.3). The computational effort of the method comes up to 9254 function evaluations, 3091 derivative evaluations and 3071 bisections.

Interesting and important conclusions may be arisen, regarding the behavior of classic interval Newton method in solving an interval equation:

- It is not able to find reliably and with certainty an interval zero.
- It is not able to guarantee the continuity of a found interval zero.
- There is no sufficient and efficient termination criterion dedicated to interval zeros.
- The existent heuristic techniques are either ineffective or prohibitively expensive.

Our numerical experiments show that things are getting worse in solving interval equations with several interval zeros and/or with wide interval zeros.

The basic idea of interval Newton method in solving a non-interval equation is to reduce the initial search interval to almost degenerate intervals. This results to the reduction of range overestimation and finally to sharp bounds for every point-zero. Generally, in interval equations we have interval zeros of non-zero width. Thus, we are able to reduce the initial search interval only to the bounds of interval zero, which means that we have a serious amount of overestimation in our calculations and, therefore, loosely bounds on the found interval zeros.

Proposition 3.1 *Let f be a continuously differentiable interval function as defined in Definition 1.1, $[x]$ the search interval and $[x]^* \subset [x]$ the interval zero of interval equation $f(x) = 0$. If $N([x]) \subset [x]$ and $w([x]^*) > 0$ then*

$$|w(N([x])) - w([x]^*)| = \mathcal{O}(w([x])).$$

PROOF. *The proof comes easy:*

$$|w(N([x])) - w([x]^*)| \leq w(N([x])) + w([x]^*) < 2w([x]).$$

The above result strengthens our assertion that we should not hope for sharp bounds of $[x]^*$, since we always have $w([x]) > 0$.

4 The proposed method

We consider the interval equation $f(x) = 0$ and we want to find all interval zeros over a given search interval $[x]$. If we write $F(m) = [\underline{f_m}, \overline{f_m}]$ and $F'([x]) = [\underline{d}, \overline{d}]$, where $m \in [x]$, the classic interval Newton method becomes

$$N([x]) := m - \frac{\left[\underline{f_m}, \overline{f_m}\right]}{\left[\underline{d}, \overline{d}\right]}. \tag{4.1}$$

The constant m is usually set to be the midpoint of search interval $[x]$. If we make use of the proposed hull interval arithmetic (2.4), the relation (4.1) takes the following form:

$$N([x]) := m - \left(\frac{\underline{f_m}}{\left[\underline{d}, \overline{d}\right]} \sqcup \frac{\overline{f_m}}{\left[\underline{d}, \overline{d}\right]}\right) = \left(m - \frac{\underline{f_m}}{\left[\underline{d}, \overline{d}\right]}\right) \sqcup \left(m - \frac{\overline{f_m}}{\left[\underline{d}, \overline{d}\right]}\right) \tag{4.2}$$

Thus, we can restate interval Newton iterative scheme (3.2) as:

$$\begin{aligned} m_k &= m([x]_k), \\ H([x]_k) &= N_L([x]_k) \sqcup N_U([x]_k), \\ [x]_{k+1} &= [x]_k \cap H([x]_k), \end{aligned}$$

where N_L and N_U denote the left and right operand in relation (4.2) and H denotes the proposed operator (the “Hull” of intervals produced by intervals N_L and N_U).

4.1 Interior part of interval zero

The main concept of this new approach is the isolation of the endpoints of an interval zero. The hull of these enclosures will derive an interval containing the whole interval zero. According to Proposition 3.1 we are not able to bound sharply an interval zero. In order to overcome the arisen overestimation issues we propose a technique of reducing the current search interval, providing at the same time better enclosures for the interval zeros. The following statements prove the existence of an interior part of an interval zero which is used to produce the above mentioned better enclosures.

Proposition 4.1 *Let f be a continuously differentiable interval function as defined in Definition 1.1, $[x]$ the search interval, and $[x]^* \subseteq [x]$ an interval zero of $f(x) = 0$. If $0 \in f(m([x]))$ the interval $[r]$, defined by*

$$[r] = \left[\min\left\{\overline{N}_L, \overline{N}_U\right\}, \max\left\{\underline{N}_L, \underline{N}_U\right\}\right] \tag{4.3}$$

always exists and is contained in the interval zero $[x]^*$.

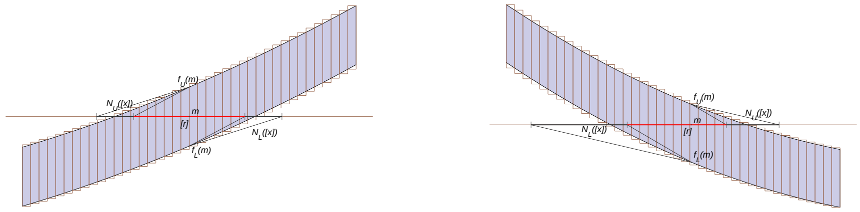


Fig. 2. Example graphs in Proposition 4.1 - Monotone functions.

PROOF. We will prove the existence of interval $[r]$ examining the following three cases:
$\underline{d} > 0$ (Left part of Figure 2): Since $0 \in F(m)$, we have

$$N_L = \left[m - \frac{\underline{f_m}}{\overline{d}}, m - \frac{\underline{f_m}}{\underline{d}}\right] \quad \text{and} \quad N_U = \left[m - \frac{\overline{f_m}}{\underline{d}}, m - \frac{\overline{f_m}}{\overline{d}}\right].$$

We calculate the bounds of interval (4.3):

$$\min\left\{\overline{N}_L, \overline{N}_U\right\} = \overline{N}_U \quad \text{and} \quad \max\left\{\underline{N}_L, \underline{N}_U\right\} = \underline{N}_L.$$

But,

$$\overline{N}_U = m - \frac{\overline{f_m}}{\overline{d}} < m - \frac{\underline{f_m}}{\overline{d}} = \underline{N}_L,$$

that is,

$$\underline{N}_U \leq \overline{N}_U < \underline{N}_L \leq \overline{N}_L \Leftrightarrow N_U \cap N_L = [/],$$

which proves that interval $[r]$ as defined in (4.3) always exists for $\underline{d} > 0$.
$\overline{d} < 0$ (Right part of Figure 2): Similarly, we can conclude to the existence of $[r]$ for $\overline{d} < 0$:

$$\underline{N}_L \leq \overline{N}_L < \underline{N}_U \leq \overline{N}_U \Leftrightarrow N_L \cap N_U = [/],$$

$0 \in [\underline{d}, \overline{d}]$ (Figure 3): For the case where $0 \in [\underline{d}, \overline{d}]$ either the f_L boundary

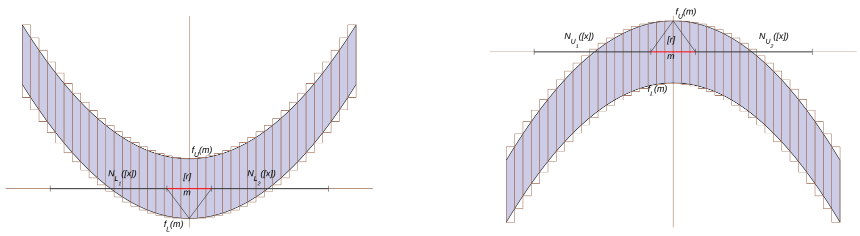


Fig. 3. Example graphs in Proposition 4.1 - Non monotone functions.

function or the f_U boundary function will enclose both the endpoints of $[x]^*$.

That is, interval Newton method will result a union of intervals and each operand of the union will bound an endpoint of interval zero. Thus,

$$N_L = N_{L_1} \cup N_{L_2} \quad \text{or} \quad N_U = N_{U_1} \cup N_{U_2},$$

and the existence of interval $[r]$ is emerged by the next relations,

$$N_{L_1} \cap N_{L_2} = [/] \quad \text{or} \quad N_{U_1} \cap N_{U_2} = [/]. \tag{4.4}$$

For the second part of the proposition we work as follows:
$\underline{d} > 0$:

$$\left.\begin{array}{l} \underline{x}^* \in N_U \\ \overline{x}^* \in N_L \end{array}\right\} \Rightarrow \underline{x}^* \leq \overline{N}_U = \underline{r} < \overline{r} = \underline{N}_L \leq \overline{x}^* \Rightarrow [r] \subseteq [x]^*$$

$\overline{d} < 0$:

$$\left.\begin{array}{l} \underline{x}^* \in N_L \\ \overline{x}^* \in N_U \end{array}\right\} \Rightarrow \underline{x}^* \leq \overline{N}_L = \underline{r} < \overline{r} = \underline{N}_U \leq \overline{x}^* \Rightarrow [r] \subseteq [x]^*$$

$\overline{0} \in [\underline{d}, \overline{d}]$: The proof for this case comes easy from (4.4).

$$\left.\begin{array}{l} \underline{x}^* \in N_{L_1} \\ \overline{x}^* \in N_{L_2} \end{array}\right\} \Rightarrow \underline{x}^* \leq \overline{N}_{L_1} = \underline{r} < \overline{r} = \underline{N}_{L_2} \leq \overline{x}^* \Rightarrow [r] \subseteq [x]^*,$$

or

$$\left.\begin{array}{l} \underline{x}^* \in N_{U_1} \\ \overline{x}^* \in N_{U_2} \end{array}\right\} \Rightarrow \underline{x}^* \leq \overline{N}_{U_1} = \underline{r} < \overline{r} = \underline{N}_{U_2} \leq \overline{x}^* \Rightarrow [r] \subseteq [x]^*.$$

The question that arises from the above proposition is what is happening with this interval $[r]$ when $0 \notin F(m)$. The answer to this question follows.

Proposition 4.2 *Let the assumptions of Proposition 4.1 hold. If* $0 \notin f(m([x]))$ *and* $0 \notin F'([x])$*, and if interval* $[r]$*, as defined in* (4.3)*, exists then it is contained in* $[x]^*$*.*

PROOF. Firstly we will deal with the matter of existence of interval $[r]$. Thus we distinguish the following four cases:
$\underline{d} > 0$ and $\underline{f_m} > 0$: Since $0 \in F(m)$, we have

$$N_L = \left[m - \frac{\underline{f_m}}{\underline{d}}, m - \frac{\underline{f_m}}{\overline{d}}\right] \quad \text{and} \quad N_U = \left[m - \frac{\overline{f_m}}{\underline{d}}, m - \frac{\overline{f_m}}{\overline{d}}\right].$$

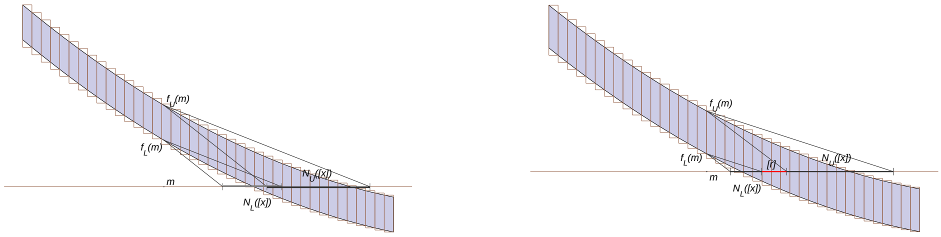


Fig. 4. Example in Proposition 4.2: The left figure shows a case where the [r] interval does not exist, while the right one the opposite case.

We calculate the bounds of interval $[r]$:

$$\min\left\{\overline{N}_L, \overline{N}_U\right\} = \overline{N}_U \quad \text{and} \quad \max\left\{\underline{N}_L, \underline{N}_U\right\} = \underline{N}_L.$$

But, we must have $\overline{N}_U - \underline{N}_L < 0$, thus

$$\overline{N}_U - \underline{N}_L = m - \frac{\overline{f_m}}{\underline{d}} - \left(m - \frac{\underline{f_m}}{\overline{d}}\right) = \frac{\underline{f_m}}{\overline{d}} - \frac{\overline{f_m}}{\underline{d}},$$

for which it is not able to decide whether $\overline{N}_U - \underline{N}_L$ is greater or less than zero. In similar way, the same results are proved for the rest of the cases:

$$\underline{d} > 0 \text{ and } \overline{f_m} < 0: \quad \min\left\{\overline{N}_L, \overline{N}_U\right\} - \max\left\{\underline{N}_L, \underline{N}_U\right\} = \frac{\underline{f_m}}{\overline{d}} - \frac{\overline{f_m}}{\underline{d}}$$

$$\overline{d} < 0 \text{ and } \underline{f_m} > 0: \quad \min\left\{\overline{N}_L, \overline{N}_U\right\} - \max\left\{\underline{N}_L, \underline{N}_U\right\} = \frac{\overline{f_m}}{\underline{d}} - \frac{\underline{f_m}}{\overline{d}}$$

$$\overline{d} < 0 \text{ and } \overline{f_m} < 0: \quad \min\left\{\overline{N}_L, \overline{N}_U\right\} - \max\left\{\underline{N}_L, \underline{N}_U\right\} = \frac{\overline{f_m}}{\overline{d}} - \frac{\underline{f_m}}{\underline{d}}$$

For all three cases it is not able to decide whether the formulated interval $[r]$ exists or not.

If interval $[r]$ exists ($\min\{\overline{N}_L, \overline{N}_U\} - \max\{\underline{N}_L, \underline{N}_U\} < 0$) then $N_L \cap N_U = [/]$ always holds, and intervals N_L and N_U will enclose the endpoints of interval zero. Thus, the derived interval $[r]$ will be contained in $[x]^*$.

For the case of $0 \notin f(m)$ and $0 \in F'([x])$, method's behavior is the same with the non-interval case: the gap created by the application of the method on the upper boundary function (the left part of Figure 5) or the lower boundary function (the right part of Figure 5) of the interval function is excluded for further consideration.

The existence of an interior part of interval zero allow us to carry out an abstraction of this part, resulting in the reduction of search interval. So, we can

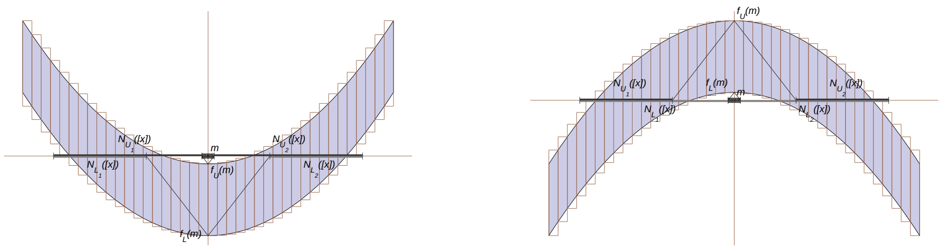


Fig. 5. The case where $0 \notin F(m)$ and $0 \in F'([x])$.

summarize the above observations and propositions to the following iterative scheme of interval Newton method for interval equations:

$$\begin{aligned} m_k &= m([x]_k), \\ H([x]_k) &= N_L([x]_k) \sqcup N_U([x]_k) = N_L([x]_k) \cup [r] \cup N_U([x]_k), \\ [x]_{k+1} &= [x]_k \cap (N_L([x]_k) \cup N_U([x]_k)), \end{aligned} \tag{4.5}$$

where N_L, N_U denote the left and right operand in (4.2), $[r]$ is defined in (4.3) and $k = 0, 1, 2, \ldots$.

In general, the application of the above proposed iterative scheme will result in three intervals: N_L and N_U intervals that aim to bound the endpoints of an interval zero and interval $[r]$, an inner part of interval zero, that will be excluded from the search procedure. In particular, intervals that guarantee to be contained in an interval zero or to contain no zero of the equation are excluded.

The exclusion procedure used in this work is running in the literature as "pruning step", and is defined as an exclusion mechanism of parts of search interval that, in general, contain no solution. Our purpose is to extend this pruning property to intervals with existence guarantee. Specifically, we prune intervals that are contained with certainty in an interval zero. In addition, we do not discard the pruned intervals that are assigned as zero sectors, but we keep them to build the interval zeros.

For the case of non-interval functions, that is, equations with simple or multiple point-zeros, we can easily prove that the proposed method is similar with interval Newton method.

Proposition 4.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and $[x]$ the initial search interval. For the function f the proposed iterative scheme* (4.5) *and the classic interval Newton scheme* (3.2) *are equivalent.*

PROOF. The function f can be considered as degenerate interval function. Thus, $f(m) = [\underline{f_m}, \overline{f_m}]$, with $\underline{f_m} = \overline{f_m}$. So, we have:

$$H([x]) = N_L([x]) \sqcup N_U([x]) = \left(m - \frac{\underline{f_m}}{[\underline{d}, \overline{d}]} \right) \sqcup \left(m - \frac{\overline{f_m}}{[\underline{d}, \overline{d}]} \right) =$$

$$= \left(m - \frac{f(m)}{[\underline{d}, \overline{d}]} \right) \sqcup \left(m - \frac{f(m)}{[\underline{d}, \overline{d}]} \right) = m - \frac{f(m)}{[\underline{d}, \overline{d}]} = N([x]).$$

4.2 Pathological case

A zero in a denominator consists an important stage in interval calculations and needs special treatment. In our proposal, a pathological situation with zero denominator occurs and it does need a special treatment too. Specifically, when we have a non-monotone interval function and the midpoint of search interval is either a zero of the lower boundary function f_L of f (the left part of Figure 6) or a zero of the upper boundary function f_U of f (the right part of Figure 6), the proposed method results the real line. In mathematical terms,

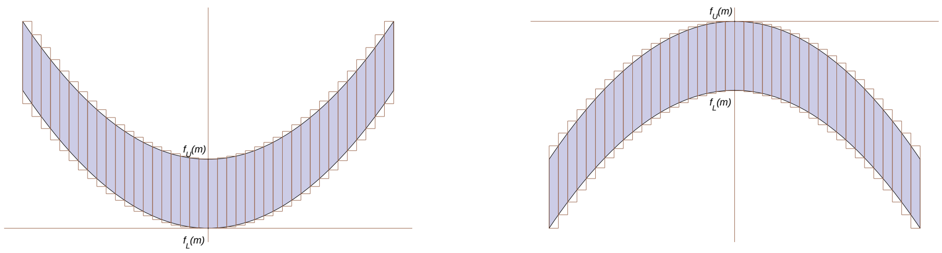


Fig. 6. Pathological cases when $\underline{f}_m = 0$ and $\overline{f}_m = 0$.

when $0 \in F'([x])$ and $\underline{f}_m = 0$ or $\overline{f}_m = 0$ then we have no improvement to the search interval. For these cases, a heuristic technique is adopted. Particularly, we adopt a bisection scheme in order to make an artificial improvement of the search interval, aiming to avoid the conditions causing this pathological case.

5 Theoretical Results

In this section, we state and prove some important properties of the proposed method. Firstly we prove the existence of an interval zero in the interval derived by the proposed operator.

Lemma 5.1 *Let a, b be real numbers such that $a, b \in [x] \in \mathbb{IR}$. If $a \leq b$ then $[a, b] \subseteq [\underline{x}, \overline{x}]$.*

PROOF. The proof of the above lemma is trivial and it is omitted.

Theorem 5.2 *Let f be a continuously differentiable interval function as defined in Definition 1.1 and $[x] \in \mathbb{IR}$ the search interval. Moreover, let f_L and f_U be boundary functions of f, and N_L, N_U the corresponding interval Newton operators over f_L, f_U, respectively, as they deifined in (4.2). If an interval zero $[x]^* \subset [x]$ of interval equation $f(x) = 0$ exists, then it is also a subset of interval $H([x])$.*

PROOF. Suppose that an interval zero $[y]^*$, with $[y]^* \not\subset H([x])$ exists. Then, $\underline{y}^* \in N_L$ or N_U and $\overline{y}^* \in N_L$ or N_U too. That is, $\underline{y}^*, \overline{y}^* \in N_L \sqcup N_U$ and from Lemma 5.1 $[\underline{y^*}, \overline{y^*}] \subseteq N_L \sqcup N_U = H([x])$, which does not hold and the theorem is proved.

The next theorem proves that the method is able to prove automatically the nonexistence of an interval zero.

Theorem 5.3 *Let the assumptions of Theorem 5.2 hold. If $[x] \cap H([x]) = [/]$, there exists no interval zero of $f(x) = 0$ in $[x]$.*

PROOF. Suppose the existence of an interval zero $[x]^* \subset [x]$. From Theorem 5.2 we have $[x]^* \subseteq H([x])$ and consequently $[x] \cap H([x]) \neq [/]$, which is not true and the theorem is proved.

Proposition 5.4 *Let the assumptions of Theorem 5.2 hold. The endpoints of interval zero $[x]^* \subset [x]$ can be bounded to arbitrary accuracy.*

PROOF. The proof of the above statement comes easy if we consider that the N_L and N_U operators are applied on the non-interval functions f_L and f_U, respectively, to isolate the bounds of interval zero. Hansen in [6] proved that a point zero of a non-interval function can be bounded to arbitrary accuracy. Therefore, both the endpoints can be bounded to arbitrary accuracy.

Furthermore, as it is proved in [8] the rate of convergence in the case of monotone boundary functions is quadratic.

Corollary 5.5 *Let the assumptions of Theorem 5.2 hold. An interval zero* $[x]^* \subset [x]$ *can be bounded to arbitrary accuracy.*

PROOF. Since we are able to bound sharply the endpoints of an interval zero, we are able to bound sharply an interval zero too.

Theorem 5.6 *Let the assumptions of Theorem 5.2 hold. Furthermore we suppose that* $w([x]^*) > 0$. *If* $H([x]) \overset{\circ}{\subset} [x]$ *and* $0 \in F([x])$, *then there exists a unique interval zero of* $f(x) = 0$ *in interval* $[x]$.

PROOF. Hansen in [6] proved the existence of an interval zero for interval equations. We suppose, here, the existence of a second one over the same search interval. This means that f is a non monotone function, that is $0 \in F'([x])$. The application of the proposed iteration scheme will answer a union of intervals, $[-\infty, c] \cup [d, +\infty]$, where $c, d \in \mathbb{R}$. This is not true since we have $H([x]) \overset{\circ}{\subset} [x]$ and therefore the original assumption $0 \in F'([x])$ does not hold. Consequently, we have a monotone function and a unique interval zero of $f(x) = 0$ in $[x]$.

6 Algorithmic formulation

In this section, we will present an algorithmic formulation of the method, described in the previous sections. The following algorithm, called HIN (Hull Interval Newton) takes as input the interval natural extension of interval function f, the searching interval $[x]_0$, the tolerance ε and returns a result list $\mathcal{L}$ containing the found interval zeros. At the following lines a brief description of the proposed algorithm is given. Firstly, in Step 2 we initialize the result list $\mathcal{L}$ to be the empty list, and the working list $\mathcal{S}$ with the initial search interval $[x]_0$. The main iterative scheme of the method is processed in Steps 3 to 32, while the working list $\mathcal{S}$ stays non-empty. In particular, in Step 4 the first item of the working list is popped and removed from it, and is assigned with $[x]$. As concerns the working list, a LIFO scheme (stack) is adopted and implemented.

Since an interval estimation of the range of f over the interval $[x]$ is evaluated, it is feasible to examine whether $0 \in F([x])$ or not (Step 5). In case where $0 \notin F([x])$, the non-existence of a zero in $[x]$ is proved and the current search interval is discarded, proceeding to the next candidate interval. In the opposite case, a termination criterion on search interval is applied (Steps 6, 7), and if it is fulfilled we push the search interval to the result list. If this does not hold we proceed with evaluation of f on the midpoint of $[x]$ and the evaluation of

Algorithm 1 *Hull Interval Newton*

1: ***function*** HIN$(F, [x]_0, \varepsilon, \mathcal{L})$
2: $\quad \mathcal{L} = \{\}$; $\mathcal{S} = \{[x]_0\}$;
3: $\quad$ ***while*** $(\mathcal{S} \neq \{\})$ ***do***
4: $\quad\quad [x] := Pop(\mathcal{S})$; $\mathcal{S} := \mathcal{S} - \{[x]\}$
5: $\quad\quad$ ***if*** $(0 \in F([x]))$ ***then***
6: $\quad\quad\quad$ ***if*** $(w([x]) \leq \varepsilon)$ ***then***
7: $\quad\quad\quad\quad \mathcal{L} := \mathcal{L} \uplus \{[x]\}$
8: $\quad\quad\quad$ ***else***
9: $\quad\quad\quad\quad F_m := F(m([x]))$
10: $\quad\quad\quad\quad H := (N_L([x]) \sqcup N_U([x])) \cap [x]$
11: $\quad\quad\quad\quad [r] = \left[\min\left\{\overline{N}_L, \overline{N}_U\right\}, \max\left\{\underline{N}_L, \underline{N}_U\right\}\right]$
12: $\quad\quad\quad\quad$ ***if*** $(0 \in F'([x]))$ ***then***
13: $\quad\quad\quad\quad\quad$ ***if*** $\left(\underline{F}_m = 0 \ \vee \ \overline{F}_m = 0 \ \vee \ H \supseteq [x]\right)$ ***then***
14: $\quad\quad\quad\quad\quad\quad$ BISECT$([x], [x]_1, [x]_2)$
15: $\quad\quad\quad\quad\quad\quad \mathcal{S} := \mathcal{S} \uplus \{[x]_1, [x]_2\}$
16: $\quad\quad\quad\quad\quad\quad$ *Pick the next item from* $\mathcal{S}$.
17: $\quad\quad\quad\quad\quad$ ***else if*** $(0 \notin F_m)$ ***then***
18: $\quad\quad\quad\quad\quad\quad [r] = [/]$
19: $\quad\quad\quad\quad\quad$ ***end if***
20: $\quad\quad\quad\quad$ ***end if***
21: $\quad\quad\quad\quad [x]_1 := N_L \cap [x]$
22: $\quad\quad\quad\quad [x]_2 := N_U \cap [x]$
23: $\quad\quad\quad\quad$ ***if*** $[r]$ *is defined* ***then***
24: $\quad\quad\quad\quad\quad [r] := [r] \cap [x]$
25: $\quad\quad\quad\quad$ ***else***
26: $\quad\quad\quad\quad\quad [r] := [/]$
27: $\quad\quad\quad\quad$ ***end if***
28: $\quad\quad\quad\quad \mathcal{S} := \mathcal{S} \uplus \{[x]_1, [x]_2\}$
29: $\quad\quad\quad\quad \mathcal{L} := \mathcal{L} \uplus \{[r]\}$
30: $\quad\quad\quad$ ***end if***
31: $\quad\quad$ ***end if***
32: $\quad$ ***end while***
33: ***return*** $\mathcal{L}$
34: ***end function***

interval H (Steps 9, 10), while in Step 11 we construct interval $[r]$ (Definition (4.3)).

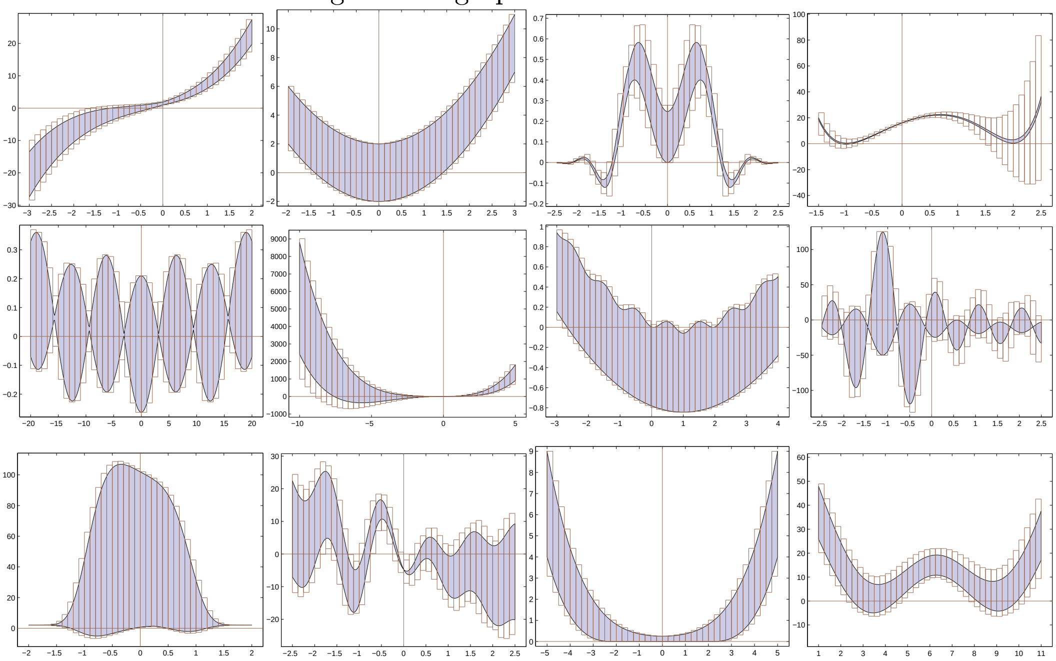
For the monotone case, an interior part $[r]$ of interval zero is always resulted. In the opposite, if a non-monotone case is occurred then either a bisection scheme is adopted (pathological cases described in paragraph 4.2 as well as the case where no improvement is achieved) or an empty interval $[r]$ is considered (when $0 \notin F(m([x]))$). These exceptions are formulated in algorithmic form through 12-20. In the next Steps 21, 22 and 23-27, the new intervals $[x]_1$, $[x]_2$ and $[r]$ are formulated, while in Steps 28 and 29 $[x]_1$, $[x]_2$ are pushed to working list $\mathcal{S}$ and interval $[r]$ is pushed to the result list $\mathcal{L}$, respectively. Finally, the algorithm answers the result list $\mathcal{L}$ containing all (interval and point) zeros of $f(x) = 0$.

7 Numerical Results

Table 1
Test Functions

No	$f(x)$
1.	$x^3 + [p]_2 x^2 + [p]_1 x + [p]_0$, $\quad [p]_0 = [1, 1.8907]$, $[p]_1 = [2.8749, 4.2501]$, $[p]_2 = [1.2499.2.2501]$
2.	$x^2 - [p]$, $\quad [p] = [-2, 2]$
3.	$sin([p]^2 + 2x^2)e^{-[p]^2 - x^2}$, $\quad [p] = [-0.5, 0.5]$
4.	$x^6 - p_5 x^5 + p_4 x^4 + [p]_3 x^3 - p_2 x^2 + [p]_1 x + [p]_0$ $[p]_0 = 16.1024$, $[p]_1 = [15.8448, 16.52]$, $p_2 = 7.872$, $[p]_3 = [-4.0388, -3.875]$, $p_4 = 1.0256$, $p_5 = 2$
5.	$\sum_{\substack{i=1 \\ i\neq 4}}^{7} \frac{[p]_i^2}{4000} + \frac{x^2}{4000} - \left(\prod_{\substack{i=1 \\ i\neq 4}}^{7} cos\left(\frac{[p]_i}{\sqrt{i}}\right) + \frac{x}{2} \right)$, $\quad [p]_{i=1,\dots,7} = [1, 2]$
6.	$x^4 + [p]_1^3 [p]_2 x^3 + [p]_1^2 [p]_2^2 [p]_3 x^2 + [p]_1 [p]_2^3 [p]_3^2 x + [p]_3$, $[p]_1 = [1.15, 1.65]$, $[p]_2 = [1.3, 1.7]$, $[p]_3 = [0.6, 1.0]$
7.	$\sum_{i=1}^{8} \{Y_i (1 + 10 Z_{i+1})\} + Y_9 (1 + 10 \cdot sin^2(\pi x)) + Z_1 + \left(\frac{x-1}{4}\right)^2 - c$, $Y_i = ([p]_i - 1)^2$, $Z_i = \sin^2(\pi [p]_i)$, $\quad [p]_{i=1,\dots,9} = [0.9, 1.1]$, $c = 0.5341615278415$
8.	$\sum_{j=1}^{5} \{j\, sin((j+1)[p] + j)\} \cdot \sum_{i=1}^{5} \{i \cdot sin((i+1)x + i)\} - 10$ $\quad [p] = [-0.1, 0.2]$
9.	$[(x + 10 \cdot [p]_0)^2 - 5 \cdot (x - [p]_1)^2 - (x - 2 \cdot [p]_2)^4)] \cdot e^{-x^4} + 4$, $[p]_0 = [-1, 0]$, $[p]_1 = [0, 0.5]$, $[p]_2 = [-0.25, 0.25]$
10.	$\sum_{j=1}^{5} j\, [sin\,((j+1)x + j) + xj\, sin\,((j+1)[p] + j)]$, $\quad [p] = [-0.1, 0.2]$
11.	$100 \cdot ([p] - x^2)^2 + (x - 1)^2$, $\quad [p] = [-5, 5]$
12.	$\left(\frac{5}{\pi} x - \frac{5.1}{4\pi^2} x^2 + [p] - 6\right)^2 + 10 \left(1 - \frac{1}{8\pi}\right) cos x$, $\quad [p] = [-2, 0]$

Fig. 7. The graphs of test functions



Through the literature, according to our knowledge, there exist plenty of test problems concerning the problem of solving a non-interval equation, but a very small sample of interval equations either in the form of examples or in application problems. Thus, we put out a set of interval functions to cover the most important cases of the proposed method, as well as interval equations occurring in applications. In Table 1 we present a set of twelve interval functions with their corresponding graphs in Figure 7 (functions 3, 5, 7, 8, 9, 10, 11 and 12 are modifications of optimization problems found in [7], functions 1, 4 and 6 are modifications of interval polynomial problems found in [16], while function 2 is a simple quadratic function).

Table 2
Numerical Results, $\varepsilon = 10^{-14}$

No	SI	IT	F	F'	B	Z	Zeros
1.	[-3,2]	45	88	43	0	1	$[r] = [-1.173264124091346, -0.249991803609978]$
2.	[-2,3]	15	28	13	0	1	$[r] = [-1.414213562373095, 1.414213562373095]$
3.	[-2.5,2.5]	317	606	289	1	9	$[r]_1 = [-2.500000000000000, -2.481569121983017]$
							$[r]_2 = [-2.170803763674813, -2.141819082085289]$
							$[r]_3 = [-1.772453850905517, -1.736834089252561]$
							$[r]_4 = [-1.253314137315509, -1.202412710675869]$
							$[r]_5 = [-0.000000000000008, 0.000000000000008]$
							$[r]_6 = [1.202412710675869, 1.253314137315509]$
							$[r]_7 = [1.736834089252561, 1.772453850905517]$
							$[r]_8 = [2.141819082085289, 2.170803763674812]$
							$[r]_9 = [2.481569121983017, 2.500000000000000]$
4.	[-1.5,2.5]	55	103	48	0	1	$[r] = [-1.094076044826967, -0.908642763062733]$
5.	[-20,20]	151	268	117	0	7	$[r]_1 = [-20, -16.395305380213230]$
							$[r]_2 = [-15.253283748005453, -9.610838776749120]$
							$[r]_3 = [-9.201957555406935, -3.181124303417352]$
							$[r]_4 = [-3.111967834374116, 3.111967834374116]$
							$[r]_5 = [3.181124303417352, 9.201957555406935]$
							$[r]_6 = [9.610838776749120, 15.253283748005453]$
							$[r]_7 = [16.395305380213230, 20]$
6.	[-10,5]	69	134	65	0	1	$[r] = [-7.599154609171094, -0.074548001830275]$
7.	[-3,4]	216	416	200	0	4	$[r]_1 = [-2.672990110994471, 0.056902762968949]$
							$[r]_2 = [0.065000000092365, 0.735991988918098]$
							$[r]_3 = [1.264008011081902, 1.934999999907636]$
							$[r]_4 = [1.943097237031067, 4.000000000000000]$
8.	[-2.5,2.5]	231	423	192	0	7	$[r]_1 = [-2.467152980553360, -2.087460254410815]$
							$[r]_2 = [-1.958922899620498, -1.519342768690696]$
							$[r]_3 = [-1.436949439757614, -0.802257350337060]$
							$[r]_4 = [-0.729786967029618, -0.225747585941954]$
							$[r]_5 = [-0.120747964752167, 0.313580544722099]$
							$[r]_6 = [0.871714388244534, 1.280704105726763]$
							$[r]_7 = [1.858301514255557, 2.244666313984733]$
9.	[-2,2]	325	594	269	0	2	$[r]_1 = [-1.251045232674226, -0.111978757770101]$
							$[r]_2 = [0.484522198663111, 1.195334431624846]$
10.	[-2.5,2.5]	145	279	134	0	6	$[r]_1 = [-2.5, -1.920875742833900]$
							$[r]_2 = [-1.527234382559722, -1.258339862758130]$
							$[r]_3 = [-0.912042992031484, -0.755713118472338]$
							$[r]_4 = [-0.156163369857502, -0.120811879307023]$
							$[r]_5 = [0.321729033785959, 0.951474079195070]$
							$[r]_6 = [1.150136547412020, 2.5]$
11.	[-5,5]	101	151	50	0	1	$[r] = [0.999999999999996, 1.000000000000005]$
12.	[1,11]	62	118	56	0	2	$[r]_1 = [2.528141250340096, 4.488909517022945]$
							$[r]_2 = [8.122294734375085, 9.909824008961696]$

In the Table 2 we show the performance of the proposed method, where SI indicates the search interval, IT the number of iterations, F the number of interval function evaluations, F' the number of interval derivative evaluations, B the total number of required bisections, and Z the number of found zeros, using tolerance $\varepsilon = 10^{-14}$. In column $Zeros$ the found interval zeros are stated. The numerical performance of our method was measured on a Laptop PC with Intel Dual Core@2GHz processor, 2GB RAM, OS Ms Windows Vista, using Intlab package for verified computing [15]. The graphs were drawn using a plot-function, dedicated to interval equations, written in Matlab [1].

Remark 7.1 *An important task concerning the taken parameters of the above equations is arisen. The amount of overestimation in interval parameters affects seriously the width of resulted interval zeros. This does not suggest any defection of our proposal (the found interval zeros correspond to the given parameters), but implies a great necessity for writing them in suitable interval form. However, this is an issue beyond the scope of this work and consists a study case of a future work.*

8 Conclusions, Results and Future Work

In this work, we state the weaknesses of the existing methods and algorithms in solving, generally, the problem of finding reliably and with certainty all zeros of an interval equation. Even though the proposed method is, essentially, based on interval Newton method for non-interval equations, achieves to produce efficient solutions to the problem of solving interval equations.

The most significant result of our work is focused in generating an interval $[r]$ which consists a guaranteed interior part of the search interval zero. This formulation guide us to adopt a different approach in using interval methods. In particular, instead of discarding intervals that contains with certainty no zeros of $f(x) = 0$, we search for intervals that are interior parts of a solution (part of an interval zero). This seems to be a different aspect in using interval methods that requires further consideration and investigation.

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