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P r e s e n t a t i o n / S u b j e c t :

The T U T O R I A L on

The Numerical Solution of Partial Differencial Equations. The two strategies : Finite **Differences** or Finite **Elements**,

which one to choose?

Conference in Numerical Analysis -NUMAN 2007

by

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1. THE BACKGROUND

The various **laws of Nature**(in Physics, Chemistry, Biology, e.t.c.)as well as the various natural phenomena around us, **may be described in terms of mathematical functions** of several variables, which satisfy either certain differential equations(d.e.) with partial derivatives, or systems of such equations. In this way, we obtain the various **named Mathematical Models** which express the corresponding theories such as **the Maxwell Equations of Classic Electrodynamics** or the **Einstein Equations of the General Theory of Relativity**,e.t.c.The fact is that there is **a strong connection** of Science with partial differential equations and **the need to produce solutions of any kind** (analytic or numerical) satisfying certain conditions **is quite critical**, although the analytic solutions are **scarcely possible**.So, **numerical solutions** are **the only feasible** alternatives for the problems ,in hand.

2. The two strategies: Finite Differences (F.D.) versus Finite Elements (F.E.)

To this end , **two main strategies exist** for facing problems including differential equations; **the first one replaces the derivatives** involved by a suitable finite difference approximation of certain accuracy -**this is the Finite Difference approach**, while **the second one transforms**, through a proper principle, **the initial differential problem into an integral one**, taking thus advantage of **the better stability** properties of **integration** against **differentiation**. Along these lines, a series of methods were developed, like the **Weighted Residual Method**, the **Subdomain Method**, the **Least Squares Method**, the **Ritz Method**, the **Galerkin Method**, the **Variational Methods** and finally the **Finite Elements Method**, which proved a **very successful evolution** of the previous ones.

3. The Second Order D.E. as a Study Case

Let us now consider the following equation, in two variables x and y, as a study case, in order **to demonstrate the peculiarities and the subtleties of p.d.e.'s.**:

$$a \cdot \frac{\partial^2 \varphi}{\partial x^2} + \beta \cdot \frac{\partial^2 \varphi}{\partial x \cdot \partial y} + \gamma \cdot \frac{\partial^2 \varphi}{\partial y^2} + \delta \cdot \frac{\partial \varphi}{\partial x} + \varepsilon \cdot \frac{\partial \varphi}{\partial y} + \zeta \cdot \varphi + \eta = 0,$$

where : a, β, γ, δ, ε, ζ and η could be functions of x and y, **with** $a^2 + \beta^2 + \gamma^2 \not\equiv 0$**, and** $\varphi = \varphi(x, y)$ **the unknown function**.

4. The Classical Cases of P.D.E.'s

If we write the above equation in a domain Ω, in the simpler form :

$$(1) \qquad Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = H,$$

where the coefficients: A, B, C, D, E, F and H **could be piecewise continuous functions** of x and y, for $\forall (x, y) \in \Omega$, then :

Definition: In the domain Ω, the P.D.E. (1) with $A^2 + B^2 + C^2 \neq 0$ is called:

(a) **Elliptic** if $B^2 - AC < 0$ for $\forall (x,y) \in \Omega$,
(b) **Parabolic** if $B^2 - AC = 0$ for $\forall (x,y) \in \Omega$,
(g) **Hyperbolic** if $B^2 - AC > O$ for $\forall (x,y) \in \Omega$.

The classical representatives of the above equations are :

(2)	$u_{xx} + u_{yy} = 0$	(**Laplace's** equation-	**Elliptic** case
(3)	$u_{xx} + u_{yy} = f(x,y)$	(**Poisson's** equation)	
(4)	$u_{xx} - u_y = 0$	(Heat transfer - **Heat conduction**)	**Parabolic** case
(5)	$u_{xx} - u_{yy} = 0$	(Wave equation- **Wave propagation**)	**Hyperbolic** case

Finally, **the Tricomi equation**- with variable coefficients:

$$yu_{xx} + u_{yy} = 0,$$

(describes the transition from a conventional to a supersonic state-in aerodynamics) is :

Elliptic in the upper half plane ($y > 0$),
Parabolic on the axis ($y = 0$), and
Hyperbolic in the lower half plane ($y < 0$).

Note: The above classification **is not just technical but substantial**,because it describes the general properties of the solutions of the models pertaining to natural phenomena. So, the **Elliptic equations** describe **steady-state phenomena**, while the **parabolic and the hyperbolic equations** refer to **diffusion states and processes**.

5. General Solutions of P.D.E.'s

Before getting on, it will be helpful to clarify that the **"General" Solution** of a P.D.E. does not have any practical use for finding a particular solution satisfying certain conditions for a problem, as it includes **arbitrary functions**. So, **a particular solution** of a P.D.E. problem **has to be created** on the basis of **the conditions(Boundary or/and Initial) which has to satisfy**, that is something quite different from what happens in the O.D.E.'s cases.

6. Examples

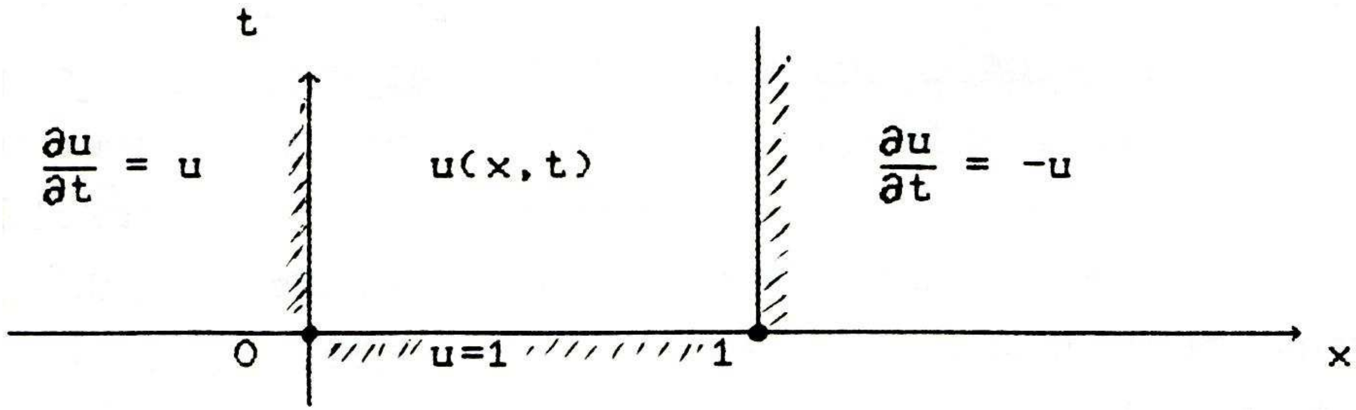
1. For the following **heat conduction** problem (see fig.1):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad |x \in (0,1), \qquad u = u(x,t),$$

under the conditions : (**initial**) $u = u(x,0) = 1 \quad | \quad x \in [0,1],$

and (**boundary**) : $\dfrac{\partial u(0,t)}{\partial x} = u(0,t), \ \dfrac{\partial u(1,t)}{\partial x} = -u(1,t) \ \forall t,$

Figure 1.



the following **analytical solution** can be produced, in some way :

(6) $$u(x,t) = 4\sum_{n=1}^{\infty}\left\{\frac{\sec a_n}{\left(3+4a_n^2\right)}\left(e^{-4a_n^2.t}\right)\cos 2a_n\,(x-0.5)\right\} \quad \Big|\ 0 < x < 1,$$

where a_n are the positive roots of the following equation:

(7) $$a \cdot \tan(a) = 0.5.$$

2. **For Laplace's equation** on a circular disc :

$$u_{xx} + u_{yy} = 0 \mid \left\{(x,y) : \; x^2 + y^2 \leq R^2\right\},$$

under the Dirichlet boundary conditions:

$$u(x,y) = g(x,y) \quad \mid \left\{(x,y) : \; x^2 + y^2 = R^2\right\}, with \quad g \quad a \quad given \quad function$$

we can prove the existence of the following **unique solution** (in polar coordinates):

$$u(x,y) = \frac{R^2 - \rho^2}{2\pi} \cdot \int_0^{2\pi} \frac{g(Re^{i\varphi})d\varphi}{R^2 + \rho^2 - 2R\rho\cos(\vartheta - \varphi)}, \tag{8}$$

where $x = \rho\cos\vartheta$ and $y = \rho\sin\vartheta$.

Now, it is clear from (6)-(8) **that** both analytical solutions **have no practical use**, since if one wishes to obtain a number of values of the solution $u(x,t)$ of (6) for certain points (x,t) of the domain of definition of the problem, one, first of all, has to determine the positive roots of (7), and then, by using the included **quite slow converging series** in (6), to calculate the values with the required accuracy, something which is not an easy cup of ...tea!. In addition, as we 'll see later on, **there are numerical methods** which can compute the required values **faster and cheaper**.

7. Particular Solutions of a P.D.E.

Regarding now the meaning of a particular solution of a P.D.E., satisfying certain conditions which secure a unique solution to the whole problem, **two points should be stressed**: the first refers to **the general solution** (g.s.) of the equation involved, which for the cases we'll deal with, **one can produce forms of g.s. easily obtainable**, and the real difficulty will be **how to manipulate the arbitrary functions** such as to satisfy the required conditions. The second point regards the need in the P.D.E.'s case **to build the particular solution** on the basis of the conditions imposed and **not via the g.s.** of the equation, something which is pronounced in the method of **Separation of Variables**. The examples that follow, from the **Euler equation**, will clarify the points just made.

8. The Euler's equation

As is known ,the Euler's equation has the following form :

$$a \cdot \frac{\partial^2 \varphi}{\partial x^2} + 2\beta \cdot \frac{\partial^2 \varphi}{\partial x \cdot \partial y} + \gamma \cdot \frac{\partial^2 \varphi}{\partial y^2} = 0, \tag{9}$$

where a, β, γ are constants and $\varphi = \varphi(x, y)$. The equation **can be normalized** by the transformation:

$$\begin{cases} \xi = \kappa x + \lambda y \\ \eta = \mu x + \nu y, \end{cases} \tag{10}$$

with κ, λ, μ and ν parameters to be determined.
Three cases can show up (after proper manipulation):

(i) if $\beta^2 - a\gamma > 0$ (Hyperbolic case)
then,the equation (9) takes the form:

$$\frac{\partial^2 \varphi}{\partial \xi \cdot \partial \eta} = 0,$$

with the following **General Solution**:

$$\varphi(\xi, \eta) = A(\xi) + B(\eta),$$

or going back to the initial variables:

$$\varphi(x, y) = A(x + x_1 y) + B(x + x_2 y), \tag{11}$$

where $A(z)$ and $B(z)$ **are arbitrary functions and twice differentiable** and x_1 and x_2 are the roots of the equation :

$$\gamma x^2 + 2\beta x + a = 0, \tag{12}$$

which obviously, in this case, are real and unequal.

Application :

In the following **wave equation**:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial^2 \varphi}{\partial y^2} = 0, \quad (\kappa - \text{parameter}) \tag{13}$$

the equation (12) has the form:

$$1 - \frac{1}{\kappa^2}x^2 = 0,$$

and the two roots are: $x_1 = \kappa$ and $x_2 = -\kappa$; therefore **the General solution** of (13) now becomes:

$$\varphi(x, y) = A(x + \kappa y) + B(x - \kappa y),$$

which, of course **is quite useless** for finding any particular solution of the equation.

(ii) If $\beta^2 - a\gamma < 0$ (Elliptic case)

then, working in the same way as before, **we come up**, this time, with the following **General Solution**:

$$\varphi(x,y) = A(x + \sigma y + i\tau y) + B(x + \sigma y - i\tau y),$$

where $x_1 = \sigma + i\tau$ and $x_2 = \sigma - i\tau$, are the complex roots of (12),which is the **characteristic property of an elliptic p.d.e.** and the functions A and B are, again, arbitrary and twice differentiable.

Application: In the case of the well known **Laplace's** equation:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

the equation (12) now becomes:

$$x^2 + 1 = 0,$$

with the following **two complex roots**:

$$x_1 = i \quad \text{and} \quad x_2 = -i.$$

Therefore, **the General Solution**,in this case , will be:

$$\varphi(x, y) = A(x + i.y) + B(x - i.y),$$

which, again, **has no practical use**, at all.

(iii)If $\beta^2 - a\gamma = 0$ (Parabolic case)
then, the normalized equation now becomes:

$$\frac{\partial^2 \varphi}{\partial \eta^2} = 0,$$

and the **General Solution** takes the form:

$$\varphi(\xi, \eta) = A(\xi) + \eta B(\xi),$$

or, going back to the initial variables :

(14) $$\varphi(x, y) = A(x + x^* y) + y.B(x + x^* y),$$

where x^* is the double root of (12),and $A(z)$ - $B(z)$ are **arbitrary functions** having second order derivatives.

Application :
In the following p.d.e.:

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0,$$

we can easily check up that it is **of parabolic type**, and it has the following **General Solution**:

$$u(x, y) = A\left(x - \frac{y}{2}\right) + y \cdot B\left(x - \frac{y}{2}\right),$$

as, in this case, equation (12) takes the form :

$$4x^2 + 4x + 1 = 0,$$

or

$$(2x + 1)^2 = 0,$$

and has **the double root**:

$$x^* = -\frac{1}{2},$$

with the functions **A and B to be arbitrary and twice differentiable.**

9. Uniqueness of a P.D.E. Solution and its Stability

In this presentation the study of P.D.E.'s **is application oriented**. So, we are forced to deal with P.D.E. problems by the mathematical models we set up in order to explain physical phenomena. For example, **Laplace's** equation can explain **steady state phenomena**, as we said before. The above mathematical description in order to be valid has to satisfy **the uniqueness** criterion , as well as **the need for the solution to be stable**, which means that small changes in the accompanying conditions should only cause small changes in the resulting solutions (**the stability criterion)**, and then the problem is said to be **well-posed, or well formulated**. Otherwise, is said to be **ill-posed**.

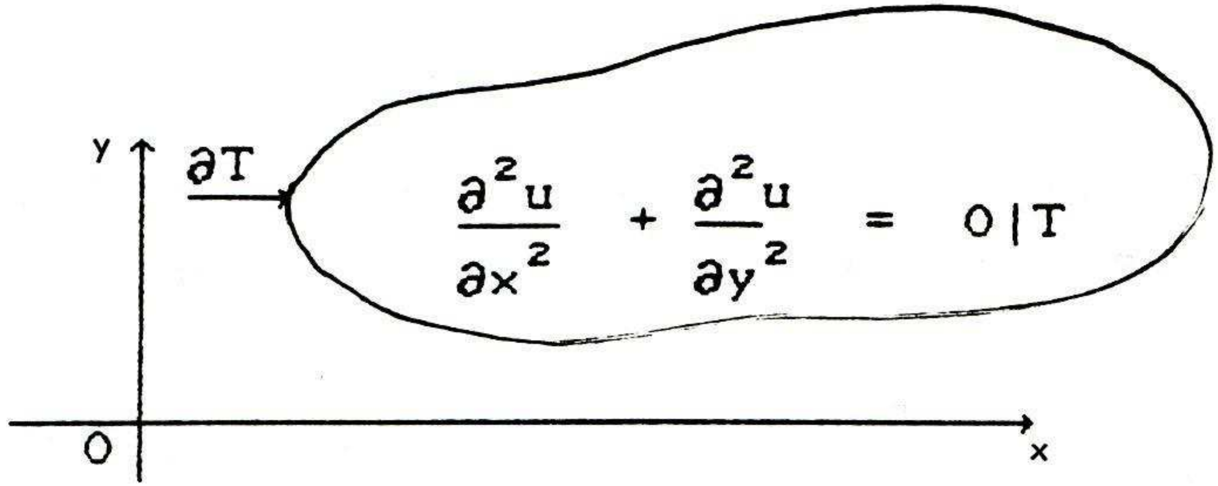
10. Initial and Boundary conditions for P.D.E.'s

There are various **named** Initial and Boundary conditions or a mixture of them, which **can secure the uniqueness of solution** to the pertaining P.D.E. So, we have:

1. Boundary Conditions **Dirichlet**, 2. Boundary Conditions **Neumann**, 3. Boundary Conditions **Robbin** and 4. **Initial Conditions Cauchy**.
More analytically for **the Dirichlet** conditions, they demand the solution function $u(x, y)$,at every point of the boundary ∂T of the domain of definition T of the p.d.e., to obtain given values. So, in this case, we can secure the **uniqueness of the solution** of the general Elliptic equation (1), provided that $F \leq 0 \ \forall \ (x, y) \in \Omega$ - **this is the coefficient of the function term in the equation**(see relation (1)).

For example, the next problem of fig. 2 **has a unique solution**.

Figure 2. $u(x,y) = a(x,y)|\partial T$ **with** $a(x,y)|\partial T$ **a given function**



(ii) **The Neumann conditions** require the value of the derivative in the direction of the outward normal v, of the domain of definition T, at every point of the boundary ∂T,to be given; that is $\frac{\partial u}{\partial v} = B \mid \partial T$, where B is a given function defined on the boundary ∂T of the domain T. **We can prove**, that for Elliptic P.D.E.'s defined in a domain T with a boundary ∂T a closed curve, **the Neumann conditions produce solutions unique up to an additive constant**.

(iii) **The Robbin (or Cauchy) conditions** require that **a linear expression** of the unknown function $u(x, y)$ and its derivative in the direction of the outward normal, at every point on the boundary of the domain of definition, is given :

$$\frac{\partial u}{\partial v} + \lambda u = \gamma \mid \partial T,$$

where λ is a given constant and γ is a given function defined both on the boundary ∂T. We can prove that for Elliptic equations defined in a domain T with a closed boundary, **the Robbins conditions produce unique solutions**.

(iv) **The Cauchy initial conditions** require that the function and all its derivatives with respect the t variable (**which is usually the time variable**) up to an order by **one less** the order of the differential equation, are given, at a certain point $t = t_0$.

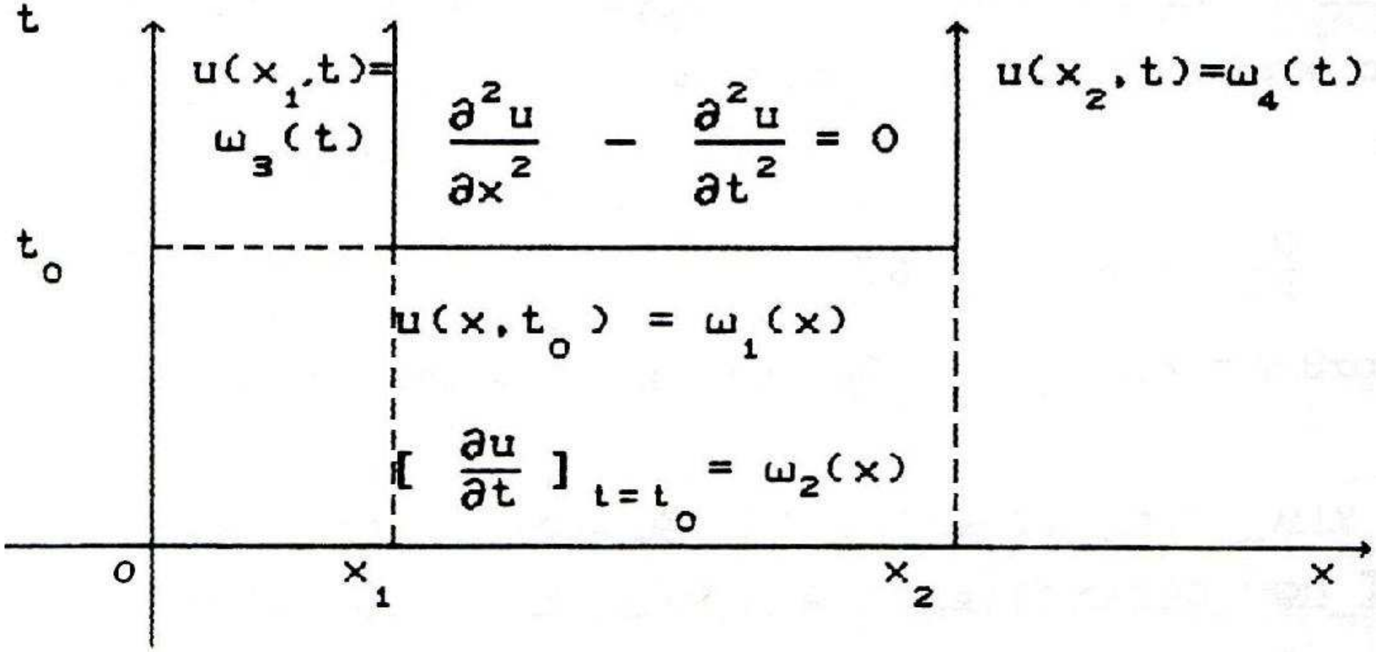
For example, **The Cauchy conditions**, for a second order P.D.E. for the function $u(x,t)$ can be of the following short:

$$u(x,t_0) = \delta(x), \quad \left[\frac{\partial u(x,t)}{\partial t}\right]_{t=t_0} = \varepsilon(x),$$

where t_0 is given and $\delta(x)$ with $\varepsilon(x)$ are given functions when $t = t_0$. **We can prove that the Cauchy conditions**, produce **unique** solutions for **Hyperbolic** P.D.E.'s.

Finally, there are problems with conditions of **mixed type**. For example, in the following **wave equation** problem, a solution $u(x,t)$, should be sought for the p.d.e., satisfying the following conditions: **Cauchy (initials)** at $t = t_0$, and **Dirichlet (boundary)**for $x = x_1$ and $x = x_2$, as is shown in Fig. 3, that follows :

Figure 3.


t
$u(x_1,t)=$
$\omega_3(t)$
$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$
$u(x_2,t)=\omega_4(t)$
t_0
$u(x,t_0) = \omega_1(x)$
$\left[\frac{\partial u}{\partial t}\right]_{t=t_0} = \omega_2(x)$
o
x_1
x_2
x

11. A case of a no unique solution

Let us, now, take the following problem:

$$y'' + y = 0 \quad | \; [0, 2\pi], \tag{15}$$

$$y(0) = y(2\pi) = 0. \tag{16}$$

As is known, the general solution of (15) is given by :

$$y(x) = A \cdot \sin(x) + B \cdot \cos(x), \qquad \text{where A,B are parameters.}$$

Now, for conditions (16) to be satisfied, we have to have:

$$\begin{cases} A \cdot \sin(0) + B \cdot \cos(0) = 0 \\ \\ A \cdot \sin(2\pi) + B \cdot \cos(2\pi) = 0. \end{cases} \tag{17}$$

The solution of the resulting linear system (17) is easily computed to be:

$$A = \text{arbitrary} \quad , B = 0;$$

and therefore the problem (15)-(16) has the following **infinity** of solutions:

$$y(x) = A.sin(x), \tag{18}$$

and consequently, **we can not solve it numerically**.

Note: H.B.Keller in his book entitled **"Numerical Method of Two-Point Boundary Value problems"** (Blaisdell, Waltham, Mass, 1968), presents the following **theorem**:

The **two point boundary value problem**:

$$y'' + P(x) \cdot y' + Q(x) \cdot y = R(x) \quad | \; x \in [a, b]$$
$$y(a) = A, \; y(\beta) = B,$$

where A, B are given constants, with $P(x)$, $Q(x)$ and $R(x) \mid (a, b)$ are continuous functions, **has a unique solution** in (a, β), if $Q(x) \leq 0 \mid \forall x \in (a, \beta)$.

Note: The above problem (15) - (16), obviously, **does not satisfy** the requirements of the previous theorem.

In the rest of this presentation, all the problems we shall be dealing with, will have a unique solution.

12. The Numerical Algorithms

We come now to **the main theme** of this presentation which is **the ways we produce approximating algorithms** to the solution of P.D.E. problems that possess a unique solution. Two strategies have **a commanding role** in the area; the **Finite Differences Technique - F.D.**, and the **Finite Elements Method - F.E.**. The interesting point with them is that even though **they both start up from a different philosophy**, they both **end up with a similar outcome**. To be more specific, I have to underline the fact that F.D. works with **the Differentiation Operator**, while the F.E. deals with **the Integration Operator**, something **totally different** to each other. The other noticeable point is that they both end up with **a similar system of equations**, which in the case of linear P.D.E.'s become similarly linear, with **the unknowns to be the solution values at predetermined points**, of the domain of definition of the problem. Finally, there is **a second interesting similarity** of the two approaches- in the case again of a linear P.D.E.: that refers to **the structure of the resulting linear system**, which is **symmetric, banded** (most of the elements of the coefficient matrix are zeroes) **with positive eigenvalues**. These properties of the coefficient matrix are quite favorable to the existing **Fast Solvers**, for obtaining the desired numerical values at a low cost.

13. The Method of Finite Differences

In this approach, as we said before, we replace the **existing** derivatives of the unknown function of the problem, by a finite difference approximation, **at each point** of the domain of definition of the problem, where we wish to find a solution value. These points are distributed **in a canonical way** in the domain of definition, by **imposing a mesh of proper size**, the nodes of which consist of all the points where solution values are required. To be more specific let us assume the following problem for Laplace's equation on the unit square under Dirichlet boundary conditions (this is the so called **the Model Problem**).

14. The Model Problem :

Find the solution of the following differential system:

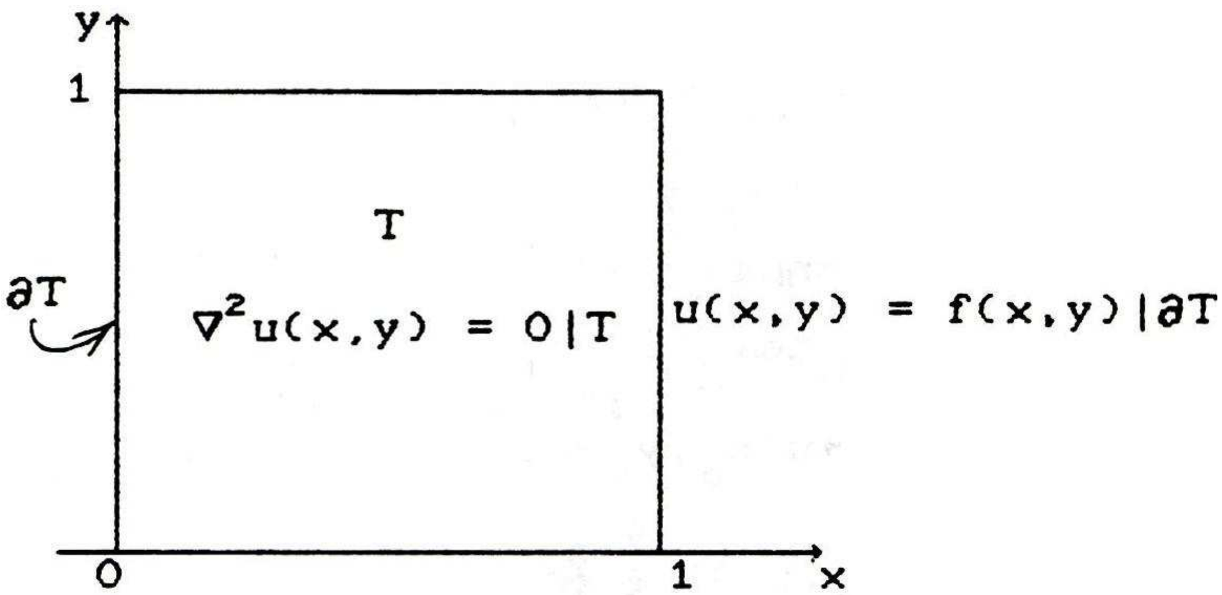
$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad | \; T = \{(x,y) \mid 0 < x,\ y < 1\} \tag{19}$$

under the Dirichlet conditions:

$$u(x,y) = f(x,y) \quad | \; \forall (x,y) \in \partial T, \tag{20}$$

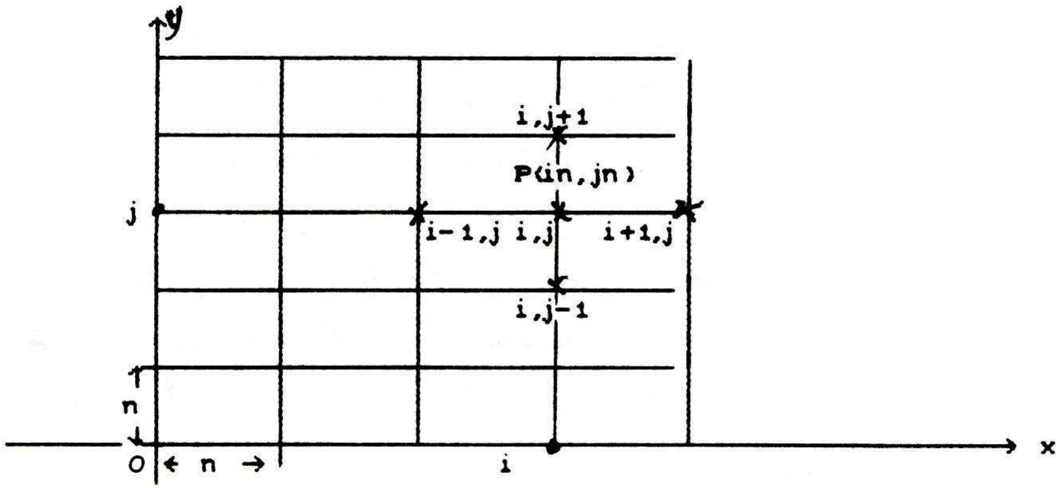
where $f(x,y)$ is a given function defined on the boundary of the square (see figure 4).

Figure 4.



To this end, we impose a proper mesh of size n, as it appears in figure 5,

Figure 5.



and at each internal mesh point we find the central difference analogue of the differential equation, by using the Taylor series expansion of $u(x,y)$. To be more specific, let us suppose that the coordinates of a point $P(in,jn)$ of the square are given by: $(x_0,\ y_0)$ and let us try to produce the required difference analogue of the differential equation at the point $P(in,jn)$, by using the Taylor series expansion of the function u along the two coordinate axis :

$$(21)\quad u(x_0 \pm n, y_0) = u(x_0,y_0) \pm n u_x(x_0,y_0) + \frac{n^2}{2} u_{xx}(x_0,y_0) \pm \\ \pm \frac{n^3}{6} u_{xxx}(x_0,y_0) + \frac{n^4}{24} u_{xxxx}(x_0,y_0) \pm \cdots ,$$

$$(22)\quad u(x_0, y_0 \pm n) = u(x_0, y_0) \pm n u_y(x_0, y_0) + \frac{n^2}{2} u_{yy}(x_0, y_0) \pm \pm \frac{n^3}{6} u_{yyy}(x_0, y_0) + \frac{n^4}{24} u_{yyyy}(x_0, y_0) \pm \cdots ,$$

where ,of course, the point (x_0, y_0) and its 4 neighboring points $(x \pm n, y)$ and $(x, y \pm n)$ must be points of the set $T \cup \partial T$.

By adding equations (21) and (22) we have :

$$u(x_0 + n, y_0) + u(x_0 - n, y_0) + u(x_0, y_0 + n) + u(x_0, y_0 - n) = 4u(x_0, y_0) + + n^2 \{u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0)\} + \frac{n^2}{12} \{u_{xxxx}(x_0, y_0) + u_{yyyy}(x_0, y_0)\} + \cdots$$

or, finally, **the required relationship** :

$$(23)\quad u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = \frac{1}{n^2} \{u(x_0 + n, y_0) + u(x_0 - n, y_0) + + u(x_0, y_0 + n) + u(x_0, y_0 - n) - 4u(x_0, y_0)\} - \frac{n^2}{12} \{u_{xxxx}(x_0, y_0) + + u_{yyyy}(x_0, y_0)\} - \cdots$$

15. The Difference Analogue Production

Equation (23) **is fundamental and in essence provides the desired difference analogue of the Laplacian operator** $\nabla^2 u(x_0, y_0)$ **in connection with the function values at the original point** $u(x_0, y_0)$ **and its 4 neighboring points, plus the derivatives of higher order of the unknown function, at** (x_0, y_0) **.**
Consequently, a first approximation of the Laplacian at the point (x_0, y_0) **of order of accuracy** n^2 **is given by**:

$$(24)\ \ \nabla^2 u(x_0, y_0) = \frac{1}{n^2}\{u(x_0 + n, y_0) + u(x_0 - n, y_0) + \\ + u(x_0, y_0 + n) + u(x_0, y_0 - n) - 4u(x_0, y_0)\},$$

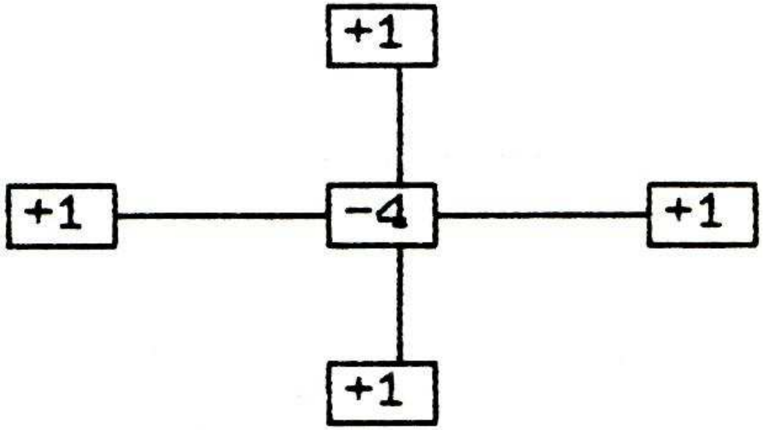
which is valid for every neighboring quintic points of $T \cup \partial T$, like the ones shown in fig. 5, before.

Moreover, because of (19), relation (23) gives:

$$(25)\quad u(x_0+n,y_0)+u(x_0-n,y_0)+u(x_0,y_0+n)+u(x_0,y_0-n)- \\ -4u(x_0,y_0)=0.$$

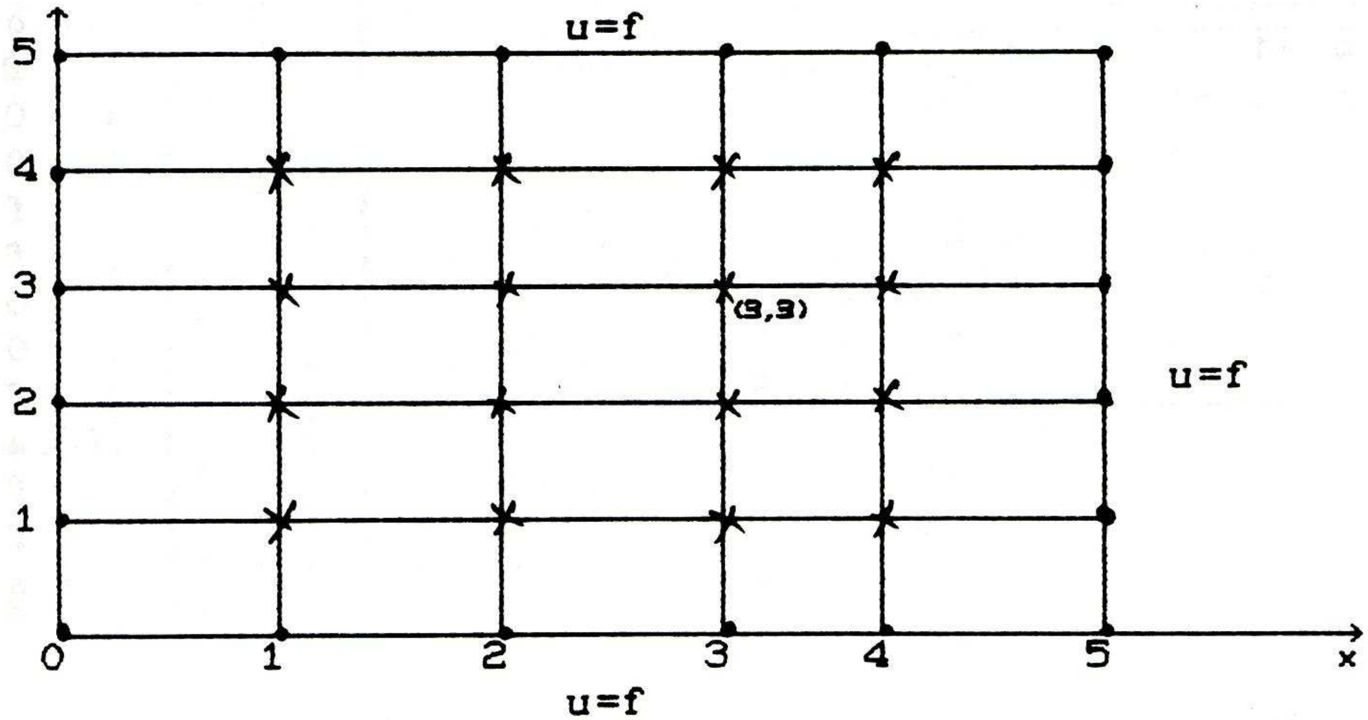
Equation (25) is a relation connecting 5 points, that is the reason why it is called **the five-point formula** for approximating the Laplacian $\nabla^2 u$ with the order of accuracy n^2, and for reasons of simplicity it is described by the following **computational cell**:

Figure 6. The 5-point formula computational cell



In addition, if we apply equation (25) **to every internal point**(a total of 16), of the mesh in fig.7, we obtain for every point a linear algebraic equation, the totality of which results in the following system :

Figure 7.



16. The Linear System of the Finite Difference Method :

$$
\begin{array}{lccccccccccl}
(1,1) & 4u_{1,1} & - & u_{1,2} & - & u_{2,1} & & & & & = & f_{1,0}+f_{0,1} \\
(2,1) & 4u_{2,1} & - & u_{1,1} & - & u_{3,1} & - & u_{2,2} & & & = & f_{2,0} \\
(3,1) & 4u_{3,1} & - & u_{2,1} & - & u_{4,1} & - & u_{3,2} & & & = & f_{3,0} \\
(4,1) & 4u_{4,1} & - & u_{3,1} & - & u_{4,2} & & & & & = & f_{4,0}+f_{5,1} \\
\hline
(1,2) & 4u_{1,2} & - & u_{1,1} & - & u_{3,1} & - & u_{2,2} & & & = & f_{0,2} \\
(2,2) & 4u_{2,2} & - & u_{2,1} & - & u_{2,3} & - & u_{1,2} & - & u_{3,2} & = & 0 \\
(3,2) & 4u_{3,2} & - & u_{3,1} & - & u_{3,4} & - & u_{3,1} & - & u_{3,3} & = & 0 \\
(4,2) & 4u_{4,2} & - & u_{4,1} & - & u_{4,3} & - & u_{3,2} & & & = & f_{5,2} \\
\hline
(1,3) & 4u_{1,3} & - & u_{1,2} & - & u_{1,4} & - & u_{2,3} & & & = & f_{0,3} \\
(2,3) & 4u_{2,3} & - & u_{2,2} & - & u_{2,4} & - & u_{1,3} & - & u_{3,3} & = & 0 \\
(3,3) & 4u_{3,3} & - & u_{3,2} & - & u_{3,4} & - & u_{2,3} & - & u_{4,3} & = & 0 \\
(4,3) & 4u_{4,3} & - & u_{4,2} & - & u_{4,4} & - & u_{3,3} & & & = & f_{5,3} \\
\hline
(1,4) & 4u_{1,4} & - & u_{2,4} & - & u_{1,3} & & & & & = & f_{0,4}+f_{1,5} \\
(2,4) & 4u_{2,4} & - & u_{1,4} & - & u_{3,4} & - & u_{2,3} & & & = & f_{2,5} \\
(3,4) & 4u_{3,4} & - & u_{2,4} & - & u_{4,4} & - & u_{3,3} & & & = & f_{3,5} \\
(4,4) & 4u_{4,4} & - & u_{4,3} & - & u_{3,4} & & & & & = & f_{4,5}+f_{5,4} \\
\hline
\end{array}
$$

and in matrix form:

$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
& -1 & 4 & -1 & & & -1 & & & & & & & & & \\
& & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
& -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
& & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
& & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
& & & & -1 & & & & 4 & -1 & & & -1 & & & \\
& & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
& & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
& & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
& & & & & & & & -1 & & & & 4 & -1 & & \\
& & & & & & & & & -1 & & & -1 & 4 & -1 & \\
& & & & & & & & & & -1 & & & -1 & 4 & -1 \\
& & & & & & & & & & & -1 & & & -1 & 4
\end{array}\right]
\cdot
\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3} \\ u_{1,4} \\ u_{2,4} \\ u_{3,4} \\ u_{4,4} \end{bmatrix}
=
\begin{bmatrix} f_{1,0} + f_{0,1} \\ f_{2,0} \\ f_{3,0} \\ f_{4,0} + f_{5,1} \\ f_{0,2} \\ 0 \\ 0 \\ f_{5,2} \\ f_{0,3} \\ 0 \\ 0 \\ f_{5,3} \\ f_{0,4} + f_{1,5} \\ f_{2,5} \\ f_{3,5} \\ f_{4,5} + f_{5,4} \end{bmatrix}.
\tag{26}
$$

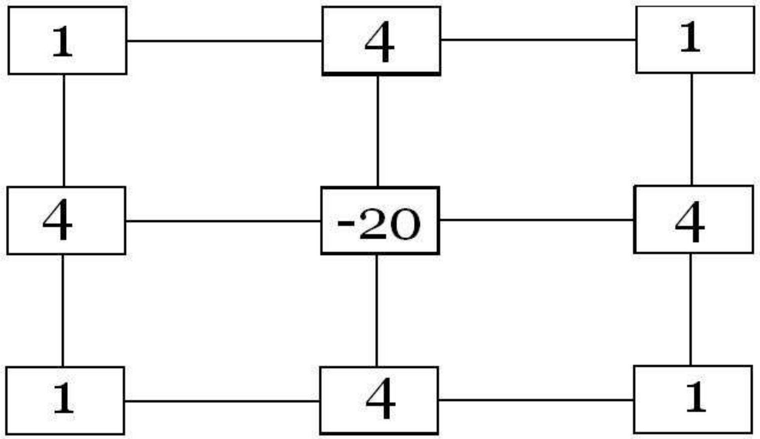
In system (26), the coefficient matrix is clearly **sparse,symmetric** and easily proved **with positive eigenvalues**, a fact very helpful **for a fast computation** of the numerical solution of the original problem.

Note: An improved accuracy scheme of a 9-point formula is easily obtained working in a similar way and **involving 9** - instead of 5 - **neighboring points**. Thus, we can produce **a better approximation** to the Laplacian equation - which will be of order $O(n^6)$ - and for the central point (i,j) **will be expressed** by the following equation:

$$-20u_{i,j} + 4(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}) + u_{i+1,j+1} + u_{i+1,j-1} + \\ + u_{i-1,j+1} + u_{i-1,j-1} = 0,$$

which is **the analogous equation to (25)**, of the 5 - point scheme and it is described by the following **computational cell** :

Figure 8. The 9-point formula computational cell



17. The Technique of Finite Elements

In this approach, **we end up again in a similar linear system** like the one in (26), for the same problem (19)-(20), the only difference being that the nodes are not forced to be canonically distributed within the domain of definition, but they can be chosen according to the needs of the problem. This is quite helpful especially in the cases of complicated boundaries ∂T. The starting point of the method is **totally different**, as it asks for a suitable principle **in order to reform** the original differential problem into an equivalent **integral one**. To this end, the following theorem is decisive :

THEOREM: *Let $k_1(x,y)$, $k_2(x,y)$, $\rho(x,y)$ and $f(x,y)$ be given functions in the region T and $a(s)$ and $y(s)$ be given functions on the boundary segments ∂T of T. The function $u(x,y)$ which makes* **the integral expression**:

$$(27)\quad I = \int\int_T \left\{ \frac{1}{2}\left[k_1(x,y)u_x^2 + k_2(x,y)u_y^2\right] - \frac{1}{2}\rho(x,y)u^2 + f(x,y)u \right\} dxdy + \\ + \oint_{\partial T} \left\{ \frac{1}{2}a(s)u^2 - \gamma(s)u \right\} ds,$$

stationary, *under the associated condition (***essential***):*

$$(28)\qquad u(x,y) = \varphi(x,y) \quad |\partial T_1,$$

necessarily solves the boundary value problem :

$$(29)\qquad \frac{\partial}{\partial x}\left(k_1(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_2(x,y)\frac{\partial u}{\partial y}\right) + \rho(x,y)u = f(x,y) \quad | \; T,$$

under the Dirichlet boundary condition (28) *and the general (* **natural***) Robbin boundary conditions:*

$$(30)\qquad k_1\frac{\partial u}{\partial x}n_x + k_2\frac{\partial u}{\partial y}n_y + a(s)u = \gamma(s) \quad |\partial T_2,$$

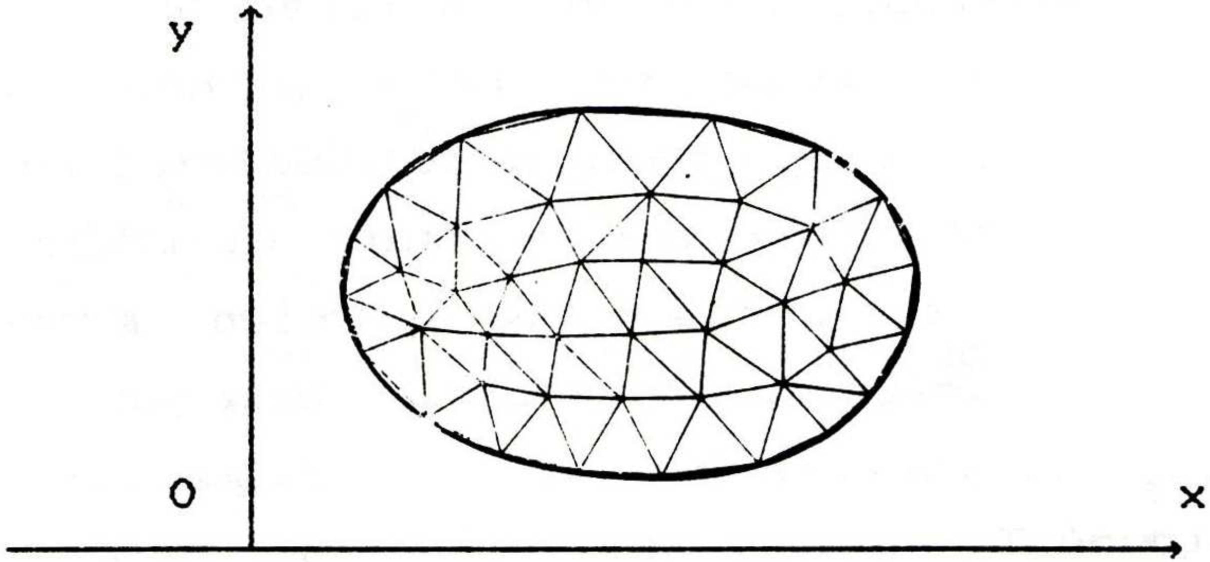
where n_x and n_y are the direction cosines of the outward normal n to the boundary ∂T, while ∂T_2 is the rest of the boundary that satisfies the property:

$$\partial T_1 \cup \partial T_2 = \partial T \quad \text{and} \quad \partial T_1 \cap \partial T_2 = \emptyset.$$

Note: It is clear that the F.E. method instead of seeking the solution of the differential system (29)-(28)-(30), **is looking for a function** $u(x, y)$ **which will make the functional** (27) **stationary, under only the essential condition** (28), **and that with the following steps:**

Step 1^{st}:
The domain T **is discretized** by dividing the total space domain into simple subdomains, **the elements**. In 2-dim problems, for example, the domain is divided into triangles, parallelograms, curved - sided triangles or quadrilaterals, as is shown in figure 9 .

Figure 9. Discretization of the 2- dim region T



Note: This type of discretization is very flexible and can be adjusted to suit the region of any problem. It has to be taken into account, however, that adjacent small angles or highly obtuse angles should be avoided, as they create numerical difficulties.

The region is replaced, in the calculation, by the union of the approximating elements.

Step 2^{nd}:

For each of the elements, **a suitable approximation** to the function sought, has to be chosen. For 1-dim cases appropriate choices can be **1st, 2nd, or 3rd degree polynomials**, or occasionally even higher degree polys depending on the needs of the problems. In two dimensional problems **linear, quadratic or bilinear forms** of the following short may be appropriate:

(31) $u(x, y) = a_1 + a_2 x + a_3 y,$ $\quad a_1,\ a_2,$ and $a_3,$ are constants

(32) $u(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$

$a_1,\ a_2,\ a_3,\ a_4,$ and $a_5,$ are constants

(33) $u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy,$ $\ a_1,\ a_2,\ a_3,$ and $a_4,$ are constants.

The exact form of the approximation **depends** on the **shape** of the element, together with the **type** of the problem being treated. There are **definite continuity conditions** which have to be satisfied by the approximating functions across inter - element boundaries; for example , if-u represents the displacement of a continuous medium in the z - direction, it must be continuous across the common boundary of 2 elements, in order to guarantee continuity of the material.

Elements with approximating functions **which satisfy the continuity conditions** are said to be **conforming**.

Note: When linear transformations from one Cartesian coordinate system to another are involved and after that the approximating function must still be suitable for the problem,then, **the polys to be used should be complete** of a specific degree, as the ones shown before.

In addition, in each element σ, if the approximating function is $u^{(\sigma)}$, then **its value is controlled** by function values u_i at certain points, **the nodes**. These nodal function values, together with the values of the partial derivatives at the nodes, are called **the nodal variables**, of the element. The **important fact** is that **the approximating function can be expressed with the help of these nodal variables** as a linear expression of **the basis functions**, with **the nodal values as parameters**. If we now suppose that **the function values** are the only nodal variables and that they are of multitude κ, then the **approximating function** in the domain T, **in each element**, will be :

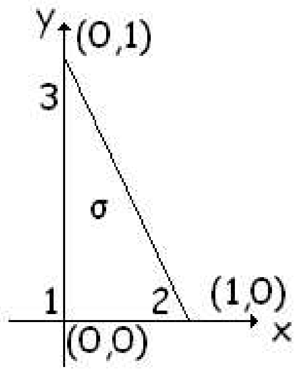
(34) $$u^{(\sigma)}(x,y) = \Sigma_{i=1}^{\kappa} u_i^{(\sigma)} \cdot N_i^{(\sigma)}(x,y),$$

where $N_i^{(\sigma)}(x,y)$ **are the basis functions**. But the above expression (34) **should be valid** for any nodal variable, therefore the basis functions have to possess the **interpolation characteristic**, and consequently to satisfy the property:

$$N_i^{(\sigma)}(x_j^{(\sigma)},y_j^{(\sigma)}) = \begin{cases} 1, & \text{when } i = j, \\ \\ 0, & \text{when } i \neq j, \end{cases}$$

where$(x_j^{(\sigma)},y_j^{(\sigma)})$ is one of the κ nodes of the element σ.

Example:



Let us suppose the element on the left- the orthogonal triangle σ - and let us further assume the following approximating function (31):

$$u(x,y) = a_1 + a_2 x + a_3 y,$$

then, three conditions will be enough for determining the values of the coefficients: a_1, a_2 and a_3, as functions of the **nodal variables** at the three points**(vertices)**:

1(0,0), 2(1,0), and 3(0,1).

So, we obtain the relationships(one on each vertix):

(35)
$$\left.\begin{array}{l} u_1 = a_1 \\ u_2 = a_1 + a_2 \\ u_3 = a_1 + a_3 \end{array}\right\} \rightarrow$$

The solution of the system (35) determines the values of a_1, a_2 and a_3 as functions of the nodal variables; that is :

$$\begin{aligned} a_1 &= u_1, \\ a_2 &= u_2 - u_1, \\ a_3 &= u_3 - u_1, \end{aligned}$$

and if we replace the found values in the approximating function, we get:

$$u(x,y) = u_1 + (u_2 - u_1)x + (u_3 - u_1)x$$

which, in turn, can be written as:

$$u(x,y) = (1 - x - y)u_1 + xu_2 + yu_3 \equiv N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3,$$

and therefore **the basis functions, in this case, become**:

$$N_1(x,y) = 1 - x - y,$$

$$N_2(x,y) = x,$$

$$N_3(x,y) = y.$$

Note: Now, it is easy to see **the interpolating property** of the above basis functions: $N_i(x,y)$, $i = 1,2,3$, at the three vertices of the element, as it obviously holds:

$$\begin{aligned}
N_1(0,0) &= 1,\ N_1(1,0) = 0 \ \text{ and } N_1(0,1) = 0,\\
N_2(0,0) &= 0,\ N_2(1,0) = 1 \ \text{ and } N_2(0,1) = 0,\\
N_3(0,0) &= 0,\ N_3(1,0) = 0 \ \text{ and } N_3(0,1) = 1.
\end{aligned}$$

Finally, expression (34) holds for every element σ, which is **the local expression of the function** u, whose **the global expression** will be:

(36) $$u(x, y) = \Sigma_{\lambda=1}^{n} u_\lambda N_\lambda(x, y), \quad (n: \quad \textbf{all kind of nodes}),$$

which is the piecewise combination of the approximations of $u^{(\sigma)}(x, y)$ over all the elements, and is therefore the union of approximations (34) over all the elements, on **the whole region**.

In expression (36), the function $N_\lambda(x, y)$ is the union of all the individual element basis functions $N_i^{(\sigma)}(x, y)$ that take the value 1 at the node (x_λ, y_λ), which is connected with the nodal variable u_λ. **Consequently, the global basis function** $N_\lambda(x, y)$ **takes a non zero value only in elements which contain the nodal point** (x_λ, y_λ).
Note: In expression (36), it is an easy matter to incorporate any essential conditions on the global approximating function, **by just replacing** the function values with the given values, by the condition.

Step 3**rd**:

After that, expressions (36) are replaced in (27), where the integration can be materialized in parts **on each one element**, and then by summing them all up, we get a **quadratic expression** which, for stationary problems, has the general form:

$$I = \frac{1}{2}\bar{u}^{\top} A \cdot \bar{u} + \bar{\beta}^{\top} \cdot \bar{u} + \gamma, \tag{37}$$

where:

$\bar{u}$**:** ,is the vector of the nodal variables, that is **the values of the unknown function**, that we are looking for, at the various nodes.

A**:** is the matrix of the quadratic terms of u (a **symmetric** and **positive - definite** matrix),

$\bar{\beta}$**:** is the vector of the linear terms of u, and

γ**:** is a constant.

Finally, if we impose the condition that **the functional be stationary**, we easily get the following system of linear equations:

$$A \cdot \bar{u} + \bar{\beta} = 0. \tag{38}$$

The linear system (38) **obtained by the method of Finite Elements is the corresponding to the system** (26) **derived by the method of the Finite Differences** and it has, in general, similar structure with it.

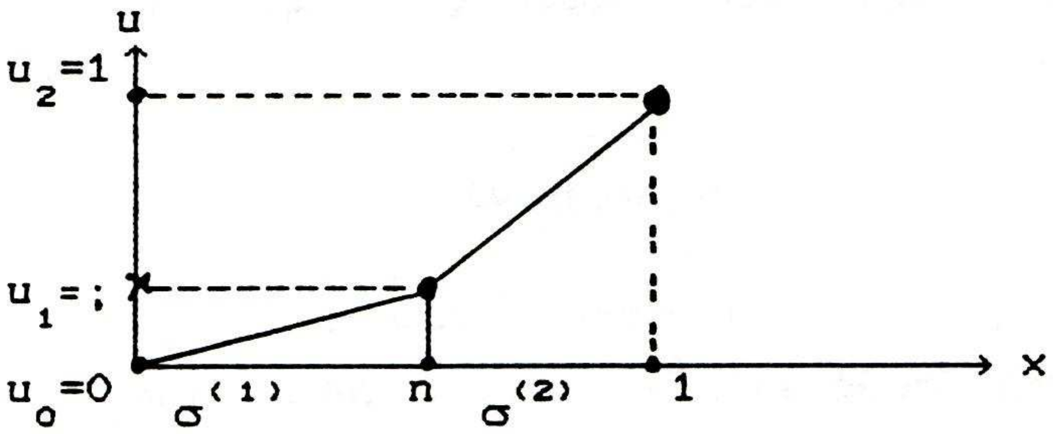
Example:

Let us take the 1- dim problem:

$$u'' - 2 = 0 \mid [0, 1]$$
$$u(0) = 0, \; u(1) = 1.$$

and let us suppose, for the ease of the application, only **two elements**, the first $[0, n]$ and the second $[n, 1]$, produced by the point n on the x- axis. It is clear that the only unknown value is that of $u(n)$ (see figure 10).

Figure 10. Two linear elements



Let us further suppose the **linear** approximating function:

$$u(x) = A \cdot x + \beta, \tag{39}$$

then, in **the first element**, the expression of the approximating function $u(x)$, as an expression of the nodal variables at 0 and n points - with corresponding values $u_0 = 0$(by the **essential conditions**) and u_1 (**the unknown value**) - is given by the Lagrange formula:

$$u^{(1)}(x) = \frac{1}{n} u_1 \cdot x. \tag{40}$$

Similarly, in **the second element** the approximating function $u(x)$ will be :

$$u^{(2)}(x) = \frac{u_2 - u_1}{1-n} \cdot x + \frac{u_1 - u_2 \cdot n}{1-n}, \qquad \text{where} \quad u_2 = 1(\textbf{by the essential conditio}$$

or, by substituting the given values in the equation, we get:

$$u^{(2)}(x) = \frac{1 - u_1}{1-n} \cdot x + \frac{u_1 - n}{1-n}. \tag{41}$$

The functional (27), in this case becomes:

$$I = \int_0^1 \left\{ \frac{1}{2} u_x^2 + 2u \right\} dx, \tag{42}$$

and by analyzing it in the two integrals we get:

$$I = \int_0^n \left\{ \frac{1}{2} u_x^2 + 2u \right\} dx + \int_n^1 \left\{ \frac{1}{2} u_x^2 + 2u \right\} dx, \tag{43}$$

where the function $u(x)$, in each element, is given by expressions (40) and (41). By replacing ,we get:

$$\int_0^n \left\{ \frac{1}{2} \left(\frac{1}{n} u_1 \right)^2 + \frac{2}{n} u_1 x \right\} dx = \frac{u_1^2}{2n} + u_1 \cdot n$$

$$\int_n^1 \left\{ \frac{1}{2} \left(\frac{1 - u_1}{1 - n} \right)^2 + 2 \left[\left(\frac{1 - u_1}{1 - n} \right) x + \frac{u_1 - n}{1 - n} \right] \right\} dx = \frac{(1 - u_1)^2}{2\,(1 - n)} + (1 + u_1)\,(1 - n)\,.$$

Therefore, the functional (43) finally becomes:

$$I = \frac{u_1^2}{2n} + u_1 \cdot n + \frac{(1-u_1)^2}{2\,(1-n)} + (1+u_1)\,(1-n)\,,$$

and in order to be stationary we should have :

$$\frac{dI}{du_1} = 0 \longrightarrow \frac{u_1}{n} + n + (1-n) - \frac{1-u_1}{1-n} = 0.$$

or, after simplification:

$$u_1 = n^2, \tag{44}$$

which is the sought value and it is quite interesting that **it coincides** with **the analytical solution** $u(x) = x^2$ of the problem.

Final remark

In the previous example, if we assume that n = 1/2 then we can use the 3- point **finite difference analogue substitute** to the second derivative, in the differential equation, and **obtain at once** the sought value, from the resulting algebraic equation :

$$(u_2 - 2.u_1 + u_0)/n^2 - 2 = 0,$$

or, after simplification and incorporation of the essential conditions of the problem:

$$u_0 = 0, u_2 = 1,$$

we easily get :

$$u_1 = n^2 = 1/4,$$

exactly, **the theoretical value**, again of the problem , in a very **simple and straightforward** way, which is the **characteristic** of the F.D. approach .

C O N C L U S I O N S

The final conclusion is easily reached: **the FINITE DIFFERENCE** approach is **quite simple** in the implementation, BUT it imposes **strict restrictions** on the **domain of definition** of the problem and the **choice** of the nodes. On the other hand, the **FINITE ELEMENT** approach is **very flexible** and can be adapted to a **very large variety** of domains and cover **sophisticated cases**, PROVIDED that **the required transformation** , in some way, can be effected.

So, for **the particular choice** one has to make, **it all depends on the problem in hand and its requirements.**

After all, to recall **Euclid's answer to King Ptolemeos • There is no Royal road in Mathematics•**.

All the results obtained during the Ages, are the outcome of **fierce fighting** against the unknown, and so, life gets on **improving all the way**.

My Best Wishes to all of you, for enjoying the rest of your stay in this beautiful part of GREEC, something that adverse circumstances prevented me of doing - FAREWELL...

1st APPLICATION - Finite Differences

Find the numerical solution of **Laplace's equation** for the model problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \mid (T \text{ unit square}),\ u = u(x,y)$$

under **the Dirchlet boundary condition**:

$$u(x,y) = f(x,y) \equiv x + y \mid \partial T,$$

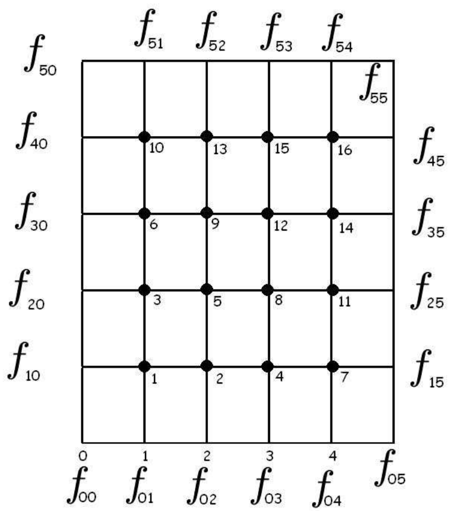
using Finite Differences and :

(a) **the 5 -point formula**

(b) **the 9-point formula.**

Finally, take the mesh size to be : $h = 0.20$.

(**SUGGESTION:** To solve the resulting linear systems, use a package like IMSL/MATHEMATICA, or any other, you can get by. For **naming the nodes** follow the next scheme:



The required values of $u = u(x,y)$ which have to be computed are : $u_1,\ u_2,\ u_3, \ldots,\ u_{13},\ u_{14},\ u_{15} and\ u_{16}$.

2nd Application - Finite Elements

Find the numerical solution of the differential system:

$$\frac{d^2u}{dx^2} - 2 = 0 \mid [0,1]$$

under **the essential boundary conditions** :$u(0) = 0.0$, $u(1) = 1.0$, using the Finite Elements technique. Take the nodal points: 0.2, 0.4, 0.6 and 0.8, and produce an **Element Information Matrix** of the following form , by adopting **a linear approximating function** :

Element	Nodal coordinates	Nodal values	Approximating function
$6^{(1)}$	$x_0 = 0$ and $x_1 = 0.2$	$u_0 = 0.0$, $u_1 = \ldots$	$u^{(1)} = \ldots$
$6^{(2)}$	$x_1 = 0.2$ and $x_2 = 0.4$	$u_1 =;$, $u_2 =;$	$\ldots$
$6^{(3)}$	$x_2 = 0.4$ and $x_3 = 0.6$	$u_2 =;$, $u_3 =;$	$\ldots$
$6^{(4)}$	$x_3 = 0.6$ and $x_4 = 0.8$	$u_3 =;$, $u_4 =;$	$\ldots$
$6^{(5)}$	$x_4 = 0.8$ and $x_5 = 1.0$	$u_4 =;$, $u_5 = 1.0$	$u^{(5)} = \ldots$

Hint: As the nodes are **equidistant** to each other, solve the same problem using THE F.D. method and the 3- point formula to approximate the second derivative and get the linear system which reads as follows:

$$u_0 - 2u_1 + u_2 = 2n^2, \quad u_1 - 2u_2 + u_3 = 2n^2, \quad u_2 - 2u_3 + u_4 = 2n^2, \quad u_3 - 2u_4 + u_5 = 2n^2.$$

Which one finally behaves better ?