Dual and annihilator topological algebras in $H$-structures

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The annihilator operators play an important role in Wedderburn decompositions of pseudo-$H$-algebras. Through them the notions of annihilator and dual topological algebra are defined. A dual algebra is annihilator. The converse is not in general true. Our aim is to present conditions that through $H$-(topological) structures, the notions dual and annihilator coincide.
Throughout all vector spaces and algebras are taken over the field of complexes.

**Definition**

- A **pseudo $H$-space** is a vector space $E$ equipped with a family $(<, >_\alpha)_{\alpha \in A}$ of positive semi-definite (pseudo-) inner products such that the induced topology makes $E$ into a locally convex space.
- A **pseudo-$H$-algebra** is a pseudo $H$-space and an algebra (which is locally convex) with separately continuous multiplication (or even $m$-convex).
The topology of a pseudo-$H$-algebra $E$ is defined by a family $(p_\alpha)_{\alpha \in A}$ of seminorms so that $p_\alpha(x) = \langle x, x \rangle^{1/2}_\alpha$ for every $x \in E$. The “$m$-convex” case will be referred each time it is used, otherwise, we shall always employ the locally convex case for the term **pseudo-$H$-algebra**. Such a topological algebra will be denoted by

$$(E, \langle, \rangle_\alpha)_{\alpha \in A}.$$ 

The **orthogonal** $S^\perp$ of a non-empty subset $S$ in $E$ is defined by

$$S^\perp = \{ x \in E : \langle x, y \rangle_\alpha = 0 \text{ for every } y \in S, \alpha \in A\} \quad (1)$$

In the sequel, we denote by $\mathcal{M}_1(E)$ the set of all closed maximal regular left ideals of a topological algebra $E$ (separately continuous multiplication).
Definition

A pseudo-H-algebra $E$ is called

(i) **left modular complemented H-algebra** if it satisfies the conditions:

Any left or right ideal $I$ in $E$ with $I^\perp = (0)$ is dense in $E$ (2)  

(The density property).

\[ \bigcap_{M \in \mathcal{M}_l(E)} M = (0), \text{ and } M^\perp \text{ is a left ideal for each } M \in \mathcal{M}_l(E) \]  

(The intersection property).

(ii) **properly left precomplemented H-algebra** if

\[ E = M \oplus M^\perp \text{ for every maximal regular left ideal } M \text{ of } E. \]  

(3)

(iii) **anti-properly left precomplemented H-algebra** if

\[ E = I \oplus I^\perp \text{ for every minimal left ideal } I \text{ of } E. \]  

(5)
Peirce property

We shall say that a pseudo-$H$-algebra $E$ has the **Peirce property** (on the left (right)) if it satisfies the condition:

\[
\text{If } x \text{ is a right (left) unit for } E \text{ modulo a maximal regular left (right) ideal } M \text{ of } E,
\]
\[
\text{then } x \in M^\perp, \text{ and } M^\perp \text{ is a left (right) ideal.}
\]

The last terminology is justified by the fact that the Peirce property leads to the **Peirce decomposition** for Hausdorff pseudo-$H$-algebras $E$. Namely, $E = E(1 - x) \oplus Ex$ (resp. $E = (1 - x)E \oplus xE$), where $E(1 - x) = \{y - yx : y \in E\}$ (resp. $(1 - x)E = \{y - xy : y \in E\}$).
Annihilator and dual topological algebras

Let $E$ be an algebra. If $(\emptyset \neq) S \subseteq E$, $A_L(S)$ (resp. $A_R(S)$) denotes the left (right) annihilator of $S$. Further, $\mathcal{L}_L$ (resp. $\mathcal{L}_R$) denotes the set of all closed left (right) ideals in a topological algebra $E$.

**Definition**

A topological algebra $E$ is said to be an **annihilator algebra**, if it is preannihilator (viz. $A_L(E) = A_R(E) = (0)$) with $A_R(I) \neq (0)$ for every $I \in \mathcal{L}_L$, $I \neq E$ and $A_L(J) \neq (0)$ for every $J \in \mathcal{L}_R$, $J \neq E$.

A topological algebra $E$ satisfying $A_L(A_R(I)) = I$ for all $I \in \mathcal{L}_L$ and $A_R(A_L(J)) = J$ for all $J \in \mathcal{L}_R$ is called a **dual algebra**.
Example
Consider the normed algebra $\mathcal{AP}(G)$ of all almost periodic functions on a topological group $G$ the underlying linear space being a pre-Hilbert space. Then every (not necessarily closed) 2-sided ideal in $\mathcal{AP}(G)$ is a dual algebra.

Example
The (Arens) algebra $L^\omega$ is a real unital commutative semisimple annihilator (topological) algebra.
Let \((E, <, >_{\alpha})_{\alpha \in A}\) be a pseudo-\(H\)-algebra. An element \(x^l\) is a \textit{left adjoint} of \(x \in E\) if \(<xy, z>_\alpha = <y, x^lz>_\alpha\) for all \(y, z \in E, \alpha \in A\). \(x^l\) is unique, if there exists. If the algebra \(E\) is preannihilator and \(x \neq 0\), then \(x^l \neq 0\). A \textit{right adjoint} is defined, analogously.

**Theorem**

Let \(E\) be a (Jacobson) semisimple Hausdorff anti-properly, and properly precomplemented \(H\)-algebra with continuous quasi-inversion, satisfying the density property. Then the following are equivalent:

1. \(E\) is an annihilator algebra.
2. \(E\) is a left annihilator algebra.
3. Every element in the socle \(\mathcal{S}_E\) of \(E\) has a left adjoint.
4. Every nonzero right ideal of \(E\) contains an element with left adjoint.
Steps of the proof

(1) $\Rightarrow$ (2): It is obvious by the very definitions.

(2) $\Rightarrow$ (3):

- If $I$ is a minimal right ideal of $E$, $I^\perp$ is a closed maximal right ideal.
- $\mathcal{A}_I(I^\perp) \neq (0)$ then
- $I^\perp = (1 - x)E \equiv \{ y - xy : y \in E \}$ with $x$ a minimal idempotent element in $E$.
- $I = xE$.
- The elements in $xE$ have left adjoints, a fortiori this will be true for the elements in $\mathcal{S}_E$. 
Steps of the proof, cont.

(3) $\Rightarrow$ (4): Let $J$ be a right ideal of $E$.
- It is proved $E = \overline{S_E}$ hence
- $J$ contains an element with a left adjoint.

(4) $\Rightarrow$ (2): Let $I$ be a proper closed right ideal of $E$.
- By the density property, $I^\perp \neq (0)$.
- By hypothesis, $I^\perp$ has a nonzero element, say $x$, with a left adjoint $x^l$. This leads to
- $\mathcal{A}_I(I) \neq (0)$. Actually, $x^l I = (0)$. 

(2) \(\Rightarrow\) (1): The annihilation property is proved on the right.

- Consider a minimal left ideal \(J\) of \(E\); by a result mentioned before any element of \(J\) has a right adjoint.
- The same is true for any element in \(\mathcal{S}_E\), which by the first Wedderburn structure theorem for properly precomplemented \(H\)-algebras, is dense in \(E\).
- The continuity of the pseudo-inner product in both variables, the separate continuity of multiplication in \(E\) and the density of the socle imply \(\langle Ex, z \rangle_\alpha = \{0\}\) for all \(z\) in a proper closed left ideal \(I\) of \(E\) and \(0 \neq x \in I^\perp\). Finally,
- \(0 \neq x^r \in A_r(I)\), where \(x^r\) is the right adjoint of \(x\). Hence the assertion.
Every dual algebra is an annihilator one

The converse is not in general true even in the normed case; However, we do know special cases for which the converse is also true. For no-normed contexts, we have proved that

A Hausdorff locally $C^*$-algebra is annihilator if and only if, it is dual, if and only if, it is complemented with left (right) complementor $\perp_l = * \circ A_r$. (resp. $\perp_r = * \circ A_l$).

In the sequel, we shall give one more no-normed framework in which the two classes of “annihilator” and “dual” algebras coincide.
In the sequel, we employ a class of algebras we introduced in 1994, in connection with annihilator (non-normed) topological algebras. We called them deep algebras.

A **deep algebra** is an algebra in which every non-zero left (right) ideal contains a minimal left (right) ideal.

**Example**

The locally $C^*$-algebra $C_c(X)$ of all $\mathbb{C}$-valued continuous functions on a discrete space $X$, with the standard algebraic-topological structure, is a deep algebra.
A (left, right) precomplemented $H$-algebra is a pseudo-$H$-algebra $E$, such that

$$E = I \oplus I^\perp \text{ for every closed left (resp. right) ideal } I \text{ of } E.$$ 

**Theorem**

Let $E$ be a semisimple Hausdorff precomplemented, and anti-properly precomplemented $H$-algebra with continuous quasi-inversion. Moreover, suppose that $E$ has the Peirce and the density properties. Then the following are equivalent:

1. $E$ is a dual algebra.
2. $E$ is a annihilator algebra.
Steps of the proof

- Any semisimple Hausdorff pseudo $H$-algebra, under the Peirce property, is a properly precomplemented $H$-algebra, and modular annihilator one.
- We only have to prove that (2) $\Rightarrow$ (1):
  - By the first structure theorem, $E$ has a dense socle and it is a $Q'$-algebra (viz. any maximal regular left (right) ideal is closed).
  - $E$ as a semisimple annihilator $Q'$-algebra is a deep one.
  - Therefore, for a proper closed left ideal $I$, $I^\perp$ contains a minimal left ideal, say $L$.
  - We have $I \subseteq L^\perp$ and $L^\perp$ is a closed maximal regular left ideal.
Steps of the proof, cont.

- Put $S = \bigcap M$, the intersection is taken over all closed maximal regular left ideals of $E$ containing $I$, which by the previous argument is non-empty. Then $I \subseteq S$.
- Using among others the precomplementation, the deepness of $E$ and the Peirce property, we get $I = S = \bigcap M \subseteq M$. Each $M$ has the form $M = E(1 - x), x \in \mathcal{I}d(E)$, from which $M = A_l(A_r(M))$.
- $I \subseteq A_l(A_r(I)) \subseteq A_l(A_r(M)) = M$ for all $M$'s.
- Thus, $I \subseteq A_l(A_r(I)) \subseteq \bigcap M = S = I$,
- and $I = A_l(A_r(I))$.
- Similarly, $E$ is a right dual algebra.
A class of pseudo-Hilbert algebras, which are dual algebras

We introduce a generalization of Hilbert algebras in the following sense.

**Definition**

An algebra $E$ is called a **pseudo-Hilbert algebra** if it is a pseudo-$H$-space equipped with an involution $x \rightarrow x^*$ having the properties:

- $\langle xy, z \rangle_\alpha = \langle y, x^*z \rangle_\alpha$ for all $x, y, z \in E$.
- $\langle x, y \rangle_\alpha = \langle y^*, x^* \rangle_\alpha$ for all $x, y \in E$.
- The left multiplication is continuous.
- The set $\{xy, x, y \in E\}$ is dense in $E$. 
Theorem

Every Hausdorff orthocomplemented pseudo-Hilbert algebra $E$ is a dual algebra.

Its proof is based on the following facts:

- Any pseudo-Hilbert algebra is proper.
- **Proposition.**- Let $E$ be a proper algebra and $I$ a left ideal of $E$ such that there exists a left ideal $I'$ of $E$ with $E = I \oplus I'$. If $Ex \subseteq I$ (resp. $Ex \subseteq I'$) for some $x \in E$, then $x \in I$ (resp. $x \in I'$).
- Take a (closed) left ideal $I$ which is orthocomplemented, then we prove that $Ex \subseteq I$ (resp. $Ey \subseteq I'$) for some $x \in E$ (resp. $y \in E$), and thus $x \in I$ (resp. $y \in I'$).
- For any Hausdorff pseudo-Hilbert algebra $E$, and any (closed) left (resp. right) orthocomplemented ideal $I$ (resp. $J$) of $E$, we get $A_r(I) = (I^\perp)^*$ (resp. $A_l(J) = (J^\perp)^*$).
Example

Let $I$ be an arbitrary set of elements. Consider the set $\mathbb{C}^{I \times I}$ of all complex-valued functions $a$ on $I \times I$, such that $\sum_{i,j} |a(i,j)|^2 \in \mathbb{R}_+$. The latter, endowed with “point-wise” defined operations becomes a vector space and an algebra with “matrix” multiplication

$$(ab)(i,j) = \sum_k a(i,k)b(k,j),$$

for all $a, b \in \mathbb{C}^{I \times I}$. Take a family of real numbers $(t_\alpha)_{\alpha \in \Lambda}$, such that $t_\alpha \geq 1$. For each $\alpha \in \Lambda$, the mapping $\langle \cdot, \cdot \rangle_\alpha : \mathbb{C}^{I \times I} \times \mathbb{C}^{I \times I} \to \mathbb{C}$ given by

$$\langle a, b \rangle_\alpha = t_\alpha \sum_{i,j} a(i,j)\overline{b(i,j)}$$

defines a pseudo-inner product on $\mathbb{C}^{I \times I}$, where “$-$” denotes complex conjugation. Thus $A \equiv (\mathbb{C}^{I \times I}, (\langle \cdot, \cdot \rangle_\alpha)_{\alpha \in \Lambda})$ becomes a pseudo-Hilbert algebra with an involution given by $a^*(i,j) = \overline{a(j,i)}$. 


