Harmonic maps between rotationally symmetric manifolds

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Outline

1 Background

2 New results
   - Existence
   - Uniqueness

3 Final words
Harmonic Maps

Tension field: $\tau(u) = \text{Tr}(\nabla du)$, section of $u^{-1}TN$
In local coords:

$$\tau(u)^\alpha = \Delta_M u^\alpha + g^{kj} \Gamma^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^k} \frac{\partial u^\gamma}{\partial x^j}$$

$u: (M, g) \rightarrow (N, \gamma)$ harmonic $\iff \tau(u) = 0$

Examples:
- Harmonic functions
- Geodesics

Energy density: $e(u)(x) = |Du|^2(x) = g^{ij} \gamma_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}$
Dirichlet Problem

$M, N$ manifolds with (ideal) boundaries

$\phi: \partial M \to \partial N$ a given boundary map

Find $u: M \to N$ harmonic, such that $u|_{\partial M} = \phi$

Eells, Sampson, Hamilton: Compact manifolds
The results of Li and Tam

\[ \mathbb{H}^2 \text{ Hyperbolic space} = (\mathbb{R}^2, g = d\rho^2 + (\sinh \rho)^2 d\eta^2) \]

**Theorem (Existence)**
\[ \phi : \partial \mathbb{H}^2 \rightarrow \partial \mathbb{H}^2 \text{ } C^3 \text{ s.t. } \phi' \neq 0 \Rightarrow \exists u : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \text{ harmonic s.t. } u|_{\partial \mathbb{H}^2} = \phi \]

**Theorem (Uniqueness)**
\[ u, v : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \text{ proper harmonic s.t. } u|_{\partial \mathbb{H}^2} = v|_{\partial \mathbb{H}^2} = \phi, \phi' \neq 0 \Rightarrow u = v \]

We would like to solve the Dirichlet problem in other spaces.
Rotationally symmetric manifolds

\[ M = \mathbb{R}^2, \ g = d\rho^2 + f(\rho)^2 d\eta^2, \partial M \cong S^1, \]

where \( f(0) = 0, \ f'(0) = 1 \) and \( f(\rho) > 0, \ \forall \rho > 0 \)

e.g. \( f(\rho) = \sinh \rho \) (Hyperbolic space)
Rotationally symmetric manifolds

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\[ K_M(\rho) = -\frac{f''(\rho)}{f(\rho)} \]

\( M \) asymptotically hyperbolic \( \Leftrightarrow K_M \to -1, \lim_{\rho \to \infty} \frac{f(\rho)}{e^\rho} = 1 \)
New results

$M, N$ rotationally symmetric, asymptotically hyperbolic, with curvatures $\leq -a^2$

**Theorem (Existence)**

$\phi: \partial M \rightarrow \partial N$ $C^3$ s.t. $\phi' \neq 0$ $\Rightarrow$

$\exists u: M \rightarrow N$ harmonic s.t. $u|_{\partial M} = \phi$

**Theorem (Uniqueness)**

$u, v: M \rightarrow N$ proper harmonic s.t. $u|_{\partial M} = v|_{\partial M} = \phi$, $\phi' \neq 0$

$\Rightarrow u = v$
Existence

\( \Phi: M \to N \) initial extension of \( \phi: \partial M \to \partial N \)

\( u_R: B_R \to N \) harmonic s.t. \( u_R|_{\partial B_R} = \Phi|_{\partial B_R} \)

It is enough to show that \( d_R = d(u_R, \Phi) < C \)
Existence

$\Phi: M \to N$ initial extension of $\phi: \partial M \to \partial N$

$u_R: B_R \to N$ harmonic s.t. $u_R|_{\partial B_R} = \Phi|_{\partial B_R}$

It is enough to show that $d_R = d(u_R, \Phi) < C$

Typically we need to construct an extension $\Phi$ close to be harmonic. We can do that in the hyperbolic case:

$$\Phi(\rho, \eta) = \left( \rho - \log \phi'(\eta), \phi(\eta) \right)$$
Lemma

\[ -K_M(\rho) = \frac{f''(\rho)}{f(\rho)} \rightarrow 1 \Rightarrow \frac{f'(\rho)}{f(\rho)} \rightarrow 1 \]

Lemma \( \Rightarrow e(\Phi) \rightarrow 2 \) and \( \|\tau(\Phi)\| \rightarrow 0 \),

where \( \Phi \) as in the hyperbolic case
Proof

Choose $R_0 > 0$ big s.t. $||\tau(\Phi)|| < \epsilon$ out of $B_{R_0}$
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Let $x_R$ where $d_R(x)$ attains maximum.
One can assume that $x_R \notin B_{R_0}$. ...
Proof

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Lemma

$\exists C > 0$ s.t. $\Delta_M d_R \geq -2\|\tau(\Phi)\| + C \tanh \frac{a d_R}{2}$, where $-a^2$ upper bound of curvatures
Proof

Choose $R_0 > 0$ big s.t. $\|\tau(\Phi)\| < \varepsilon$ out of $B_{R_0}$

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Maximum principle $\Rightarrow \tanh \frac{a d_R(x_R)}{2} < c < 1 \Rightarrow d_R(x_R) < C$
Uniqueness

Proof.

\[ u|_{\partial M} = v|_{\partial M} = \phi, \quad \Delta_M d^2(u, v) \geq 0 \] (apply maximum principle)

It is enough to show that \( d(u, v)(x) \to 0 \).
Proof.

\[ u|_{\partial M} = v|_{\partial M} = \phi, \Delta_M d^2(u, v) \geq 0 \text{ (apply maximum principle)} \]

It is enough to show that \( d(u, v)(x) \to 0 \)

Use disk model:

\[ g = \frac{4}{(1-\rho^2)^2} \left( d\rho^2 + h^2(\rho) d\eta^2 \right), \text{ where } h(1) = 1 \]
Uniqueness

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Lemma

\[ u = (r, \theta) \text{ harmonic} \Rightarrow \left. \frac{\partial r}{\partial \rho}\right|_{\rho=1} = \phi' \text{ and } \left. \frac{\partial \theta}{\partial \rho}\right|_{\rho=1} = 0 \]

(we use harmonic map equations and that \( \lim_{\rho \to 1} r = 1 \))
Uniqueness

Proof.

\[ u \big|_{\partial M} = v \big|_{\partial M} = \phi, \quad \Delta_M d^2(u, v) \geq 0 \text{ (apply maximum principle)} \]

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\[ u = (r, \theta) \text{ harmonic } \Rightarrow \frac{\partial r}{\partial \rho} \big|_{\rho=1} = \phi' \text{ and } \frac{\partial \theta}{\partial \rho} \big|_{\rho=1} = 0 \]

(we use harmonic map equations and that \( \lim_{\rho \to 1} r = 1 \))

We use lemma to compare \( u \) and \( v \) with \( \Phi = (\bar{r}, \bar{\theta}) \),

where \( \bar{r} = 1 - (1 - \rho)\phi' \) and \( \bar{\theta} = \phi \)

Use triangular inequality to show that \( d(u, \Phi) \to 0 \),

same with \( d(v, \Phi) \).
Open Problems

- What if $K_M(\rho) \to -1$ fails?
- How about when $f = f(\rho, \eta)$?
Summary

- Dirichlet problem for harmonic maps
- Known results
- New results
- Open problems