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The relative invariant covering dimension $r$-dim

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Abstract

In [1] two relative covering dimensions defined and studied.


In [2] and [3] we studied these dimensions and we gave some properties including subspace, sum, partition, compactification, and product theorems. Also, we gave answers for the questions which are given in [1].


[3] D.N. Georgiou and A.C. Megaritis, *On the relative dimensions* \( \dim \) and \( \dim^* \) *II*, Questions and Answers in General Topology, **29**
In [4] we gave and studied a new relative covering dimension, denoted by r-dim.


Finally, in [5] we gave an algorithm of polynomial order for computing the dimension r-dim of a pair $(Q, X)$, where $Q$ is a subset of a finite space $X$, using matrix algebra.


The main results of the papers [4] and [5] are presented and discussed.
The cardinality of a set $X$ is denoted by $|X|$ and the first infinite cardinal is denoted by $\omega$. We also consider two symbols, “$-1$” and “$\infty$”, for which we suppose that:

(i) $-1 < n < \infty$ for every $n \in \omega$.

(ii) $\infty + n = n + \infty = \infty$ and $-1 + n = n + (-1) = n$ for every $n \in \omega \cup \{-1, \infty\}$.

The “relative dimensions” or “positional dimensions” are functions whose domains are classes of subsets. By a class of subsets we mean a class consisting of pairs $(Q, X)$, where $Q$ is a subset of a space $X$.

Let $X$ be a space. A family $r$ of subsets of $X$ is said to be a refinement of a family $c$ of subsets of $X$ if each element of $r$ is contained in an element of $c$. 

Preliminaries
Define the order of a family $r$ of subsets of a space $X$ as follows:

(a) $\text{ord}(r) = -1$ if and only if $r = \{\emptyset\}$.

(b) $\text{ord}(r) = n$, where $n \in \omega$, if and only if the intersection of any $n + 2$ distinct elements of $r$ is empty and there exist $n + 1$ distinct elements of $r$, whose intersection is not empty.

(c) $\text{ord}(r) = \infty$, if and only if for every $n \in \omega$ there exist $n$ distinct elements of $r$, whose intersection is not empty.
**Definition 1.** We denote by $r$-$\dim$ the (unique) function that has as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$ satisfying the following condition

$$r$-$\dim(Q, X) \leq n, \text{ where } n \in \{-1\} \cup \omega$$

if and only if for every finite family $c$ of open subsets of $X$ such that $Q \subseteq \bigcup\{U : U \in c\}$ there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup\{V : V \in r\}$ and $\text{ord}(r) \leq n$.

**Proposition 1.** Let $Q$ be a subset of a space $X$. The following statements are true.

1. $\dim(Q) \leq r$-$\dim(Q, X)$. Moreover, if the subset $Q$ of $X$ is open, then $\dim(Q) = r$-$\dim(Q, X)$.
2. If the subset $Q$ of $X$ is closed, then $r$-$\dim(Q, X) \leq \dim(X)$. 
Examples.

(1) Let $X$ be the space of the real numbers and $Q = \{0\}$. Then, $\text{r-dim}(Q, X) = 0$ and $\text{dim}(X) = 1$.

(2) Let $(X, \tau)$ be the topological space, where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Let $Q = \{a, c\}$. Then, $\text{dim}(Q) = \text{dim}(X) = 0$ and $\text{r-dim}(Q, X) = 1$.

(3) Let $X = [-1, 1]$ and $Q = \{-1, 1\}$. The family consisting of all sets of the form $[-1, b)$ for $b > 0$, $(a, 1]$ for $a < 0$, and $(a, b)$ is a basis for some topology in $X$. We observe that $\text{dim}(Q) = 0$ and $\text{r-dim}(Q, X) = \text{dim}(X) = 1$. 
Proposition 2. For every subset $Q$ of a space $X$ the following conditions are equivalent:

1. $\text{r-dim}(Q, X) \leq n$.

2. For every finite family $\{U_1, U_2, \ldots, U_m\}$ of open subsets of $X$ with $Q \subseteq \bigcup_{i=1}^{m} U_i$ there exists a family $\{V_1, V_2, \ldots, V_m\}$ of open subsets of $X$ such that $V_i \subseteq U_i$ for $i = 1, \ldots, m$, $Q \subseteq \bigcup_{i=1}^{m} V_i$, and $\text{ord}(\{V_1, V_2, \ldots, V_m\}) \leq n$.

3. For every family $\{U_1, U_2, \ldots, U_{n+2}\}$ of open subsets of $X$ with $Q \subseteq \bigcup_{i=1}^{n+2} U_i$ there exists a family $\{V_1, V_2, \ldots, V_{n+2}\}$ of open subsets of $X$ such that $V_i \subseteq U_i$, $i = 1, \ldots, n + 2$, $Q \subseteq \bigcup_{i=1}^{n+2} V_i$ and $\bigcap_{i=1}^{n+2} V_i = \emptyset$. 
Subspace theorems

**Proposition 3.** Let $K$ and $Q$ be two subspaces of a space $X$ with $K \subseteq Q$. If $K$ is a closed subspace of $X$ or $Q \setminus K$ is an open subspace of $X$, then

$$r\text{-dim}(K, X) \leq r\text{-dim}(Q, X).$$

**Proposition 4.** Let $Y$ be a subspace of a space $X$ and $Q \subseteq Y$. Then,

$$r\text{-dim}(Q, Y) \leq r\text{-dim}(Q, X).$$

**Proposition 5.** For every closed subspace $Q$ of a normal space $X$ we have

$$r\text{-dim}(Q, X) = r\text{-dim}(Q, \beta X).$$
Sum theorems

**Proposition 6.** Let $Q$ be a subspace of a space $X$. If $X = X_1 \cup X_2$, where $Q \subseteq X_1 \cap X_2$, $r\text{-dim}(Q, X_1) \leq n$, and $r\text{-dim}(Q, X_2) \leq n$, then $r\text{-dim}(Q, X) \leq n$.

**Proposition 7.** Let $Q$ be a subspace of a space $X$. For every subset $A$ of $X$ such that $Q \subseteq A$ we have

$$r\text{-dim}(Q, X) \leq \max\{r\text{-dim}(Q, A), r\text{-dim}(Q, (X \setminus A) \cup Q)\}.$$  

**Proposition 8.** Let $Q$ be a subspace of a space $X$. If $X = X_1 \cup X_2$, where $Q \subseteq X_1 \cap X_2$, then

$$r\text{-dim}(Q, X) = \max\{r\text{-dim}(Q, X_1), r\text{-dim}(Q, X_2)\}.$$  

**Proposition 9.** Let $Q_1$ and $Q_2$ be two subsets of a space $X$. Then,

$$r\text{-dim}(Q_1 \cup Q_2, X) \leq r\text{-dim}(Q_1, X) + r\text{-dim}(Q_2, X) + 1.$$  

Partition and Product theorems

**Definition 2.** Let $A$ and $B$ be two disjoint subsets of a space $X$. We say that a subset $L$ of $X$ is a partition between $A$ and $B$ if there exist two open subsets $U$ and $W$ of $X$ such that

1. $A \subseteq U$, $B \subseteq W$,
2. $U \cap W = \emptyset$, and
3. $X \setminus L = U \cup W$. 
Proposition 10. Let $Q$ be a normal subspace of a space $X$. If for every family $\{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\}$ of $n + 1$ pairs of disjoint subsets of $X$, where $A_i$’s are closed in $X$ and $B_i$’s are closed in $Q$, there exist partitions $L_i$ between $A_i$ and $B_i$ such that

$$Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset,$$

then $r\text{-dim}(Q, X) \leq n$. 
Proposition 11. Let $Q$ be a closed subspace of a normal space $X$ satisfying $r\text{-dim}(Q, X) \leq n$. Then, for every family
\[
\{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\}
\]
of $n + 1$ pairs of disjoint closed subsets of $X$ there exist partitions $L_i$ between $A_i$ and $B_i$ such that
\[
Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset.
\]
Proposition 12. Let $Q^X$ be a closed subspace of a compact Hausdorff space $X$ and $Q^Y$ a closed subspace of a compact Hausdorff space $Y$. Then,

$$r\text{-dim}(Q^X \times Q^Y, X \times Y) \leq r\text{-dim}(Q^X, X) + r\text{-dim}(Q^Y, Y).$$
An algorithm of polynomial order for computing the dimension $r \text{-dim}(Q, X)$, where $Q$ is a subset of a finite space $X$.

Let $X = \{x_1, \ldots, x_n\}$ be a finite space of $n$ elements. In what follows we denote by $U_i$ the smallest open set of $X$ containing the point $x_i$, $i = 1, \ldots, n$.

The $n \times n$ matrix $T = (t_{ij})$, where 

$$t_{ij} = \begin{cases} 
1, & \text{if } x_i \in U_j \\
0, & \text{otherwise}
\end{cases}$$

is called the incidence matrix of $X$.

We denote by $c_1, \ldots, c_n$ the $n$ columns of the matrix $T$. 

The relative invariant covering dimension $r \text{-dim}$
The relative invariant covering dimension $r$-dim

Let

$$c_i = \begin{pmatrix}
    c_{1i} \\
    c_{2i} \\
    \vdots \\
    c_{ni}
\end{pmatrix}$$

and

$$c_j = \begin{pmatrix}
    c_{1j} \\
    c_{2j} \\
    \vdots \\
    c_{nj}
\end{pmatrix}$$

be two $n \times 1$ matrices.

Then, by $\max c_i$ we denote the maximum

$$\max\{c_{1i}, c_{2i}, \ldots, c_{ni}\}.$$

Also, we write $c_i \leq c_j$ if only if $c_{ki} \leq c_{kj}$ for each $k = 1, \ldots, n$. 

An algorithm of polynomial order for computing the dimension $r$-dim($Q, X$)
Let $Q \subseteq X$. We denote by $\mathbf{1}_Q$ the $n \times 1$ matrix

\[
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix},
\]

where

\[
a_i = \begin{cases} 
1, & \text{if } x_i \in Q \\ 
0, & \text{otherwise.}
\end{cases}
\]
Proposition 13. Let $X = \{x_1, \ldots, x_n\}$ be a finite space and $Q \subseteq X$. Then, $r\text{-dim}(Q, X) \leq k$, where $k \in \omega$, if and only if there exists a family $\{U_{j_1}, \ldots, U_{j_m}\}$ such that

$$\{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq U_{j_1} \cup \ldots \cup U_{j_m}$$

and

$$\text{ord}(\{U_{j_1}, \ldots, U_{j_m}\}) \leq k.$$

Proposition 14. Let $X = \{x_1, \ldots, x_n\}$ be a finite space and $Q \subseteq X$. Then,

$$r\text{-dim}(Q, X) \leq |Q| - 1.$$
Let $X = \{x_1, \ldots, x_n\}$ be a finite space of $n$ elements, $Q = \{x_{\lambda_1}, \ldots, x_{\lambda_l}\} \subseteq X$, and $T = (t_{ij})$ the $n \times n$ incidence matrix of $X$.

Our intended algorithm contains $l - 1$ steps:

**Step 1.** Read the $l$ columns $c_{\lambda_1}, \ldots, c_{\lambda_l}$ of the matrix $T$. If some column is equal to $1_Q$, then print

$$ r \text{-dim}(Q, X) = 0. $$

Otherwise go to the Step 2.
Step 2. Find the sums

\[ c_{\lambda_{j_{11}}} + c_{\lambda_{j_{21}}} + \ldots + c_{\lambda_{j_{(l-1)1}}} \]

for each \( \{j_{11}, j_{21}, \ldots, j_{(l-1)1}\} \subseteq \{1, \ldots, l\} \).

If there exists \( \{j_{11}^0, j_{21}^0, \ldots, j_{(l-1)1}^0\} \subseteq \{1, \ldots, l\} \) such that

\[ c_{\lambda_{j_{11}^0}} + c_{\lambda_{j_{21}^0}} + \ldots + c_{\lambda_{j_{(l-1)1}^0}} \geq 1_Q, \]

then go to the Step 3.

Otherwise print

\[ \text{r-dim}(Q, X) = \max(c_{\lambda_1} + c_{\lambda_2} + \ldots + c_{\lambda_l}) - 1. \]
Step 3. Find the sums

\[ c_{\lambda j_{12}} + c_{\lambda j_{22}} + \ldots + c_{\lambda j_{(l-2)2}} \]

for each \( \{j_{12}, j_{22}, \ldots, j_{(l-2)2}\} \subseteq \{j_{11}^0, j_{21}^0, \ldots, j_{(l-1)1}^0\} \).

If there exists \( \{j_{12}^0, j_{22}^0, \ldots, j_{(l-2)2}^0\} \subseteq \{j_{11}^0, j_{21}^0, \ldots, j_{(l-1)1}^0\} \) such that

\[ c_{\lambda j_{12}^0} + c_{\lambda j_{22}^0} + \ldots + c_{\lambda j_{(l-2)2}^0} \geq 1_Q, \]

then go to the Step 4.

Otherwise print

\[ \text{r-dim}(Q, X) = \max(c_{\lambda j_{11}^0} + c_{\lambda j_{21}^0} + \ldots + c_{\lambda j_{(l-1)1}^0}) - 1. \]

..........
Step $l - 2$. Find the sums

$$c\lambda j_1(l-3) + c\lambda j_2(l-3) + c\lambda j_3(l-3)$$

for each $\{j_1(l-3), j_2(l-3), j_3(l-3)\} \subseteq \{j_1^0(l-4), j_2^0(l-4), j_3^0(l-4), j_4^0(l-4)\}$.

If there exists

$\{j_1^0(l-3), j_2^0(l-3), j_3^0(l-3)\} \subseteq \{j_1^0(l-4), j_2^0(l-4), j_3^0(l-4), j_4^0(l-4)\}$ such that

$$c\lambda j_1^0 + c\lambda j_2^0 + c\lambda j_3^0 \geq 1_Q,$$

then go to the Step $l - 1$.

Otherwise print

$$r\text{-dim}(Q, X) = \max(c\lambda j_1^0 + c\lambda j_2^0 + c\lambda j_3^0 + c\lambda j_4^0) - 1.$$
Step \( l - 1 \). Find the sums

\[ c\lambda j_1(l-2) + c\lambda j_2(l-2) \]

for each \( \{j_1(l-2), j_2(l-2)\} \subseteq \{j_1^0(l-3), j_2^0(l-3), j_3^0(l-3)\} \).

If there exists \( \{j_1^0(l-2), j_2^0(l-2)\} \subseteq \{j_1^0(l-3), j_2^0(l-3), j_3^0(l-3)\} \) such that

\[ c\lambda j_1^0(l-2) + c\lambda j_2^0(l-2) \geq 1, \]

then print

\[ r\text{-dim}(Q, X) = \max(c\lambda j_1^0(l-2) + c\lambda j_2^0(l-2)) - 1. \]

Otherwise print

\[ r\text{-dim}(Q, X) = \max(c\lambda j_1^0(l-3) + c\lambda j_2^0(l-3) + c\lambda j_3^0(l-3)) - 1. \]
Proposition 15. An upper bound on the number of iterations of the algorithm for computation of the dimension $r$-dim of a pair $(Q, X)$, where $Q$ is a subset of a finite space $X$, is the number

$$\frac{1}{2}|Q|^2 + \frac{3}{2}|Q| - 3.$$