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From multi-layered resonance tori to period-doubled ergodic tori

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ABSTRACT

The Letter presents a number of new bifurcation structures that can be observed when a multi-dimensional period-doubling system is subjected to a periodic forcing. We show how multi-layered tori arise through transverse period-doubling bifurcations of the resonant saddle and node cycles, and how these multi-layered tori transform into period-doubled ergodic tori through sets of saddle–node bifurcations.

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1. Introduction

The periodically forced Rössler system has been investigated by many authors, in recent years often with the aim of studying synchronization of the internally generated chaotic dynamics with the external forcing [1–5]. In the chaotic regime, the power spectrum of the Rössler oscillator displays a clearly distinguishable maximum and, depending on the parameters of the Rössler system and the forcing amplitude, chaotic phase synchronization may be achieved over a smaller or larger range of forcing frequencies.

Vadivasova et al. [6] have obtained a relatively detailed chart of the distribution of stable modes in the two-dimensional parameter plane spanned by the forcing frequency and one of the parameters of the Rössler system. An interesting aspect of this chart is the unusual structure of period-doubling bifurcation curves that are observed to emanate directly from the edges of the resonance tongue. This clearly distinguishes the structure from the classic swallow-tail structure [7,8] that can be used to describe the substructure of the resonance tongues for many low-dimensional systems. Vadivasova et al. [6] also determined a couple of bifurcation curves in which the ergodic torus that exists outside of the resonance tongue doubles its period.

Observing the same unusual bifurcation structure both in a two-dimensional map and in the forced Rössler system, Kuznetsov et al. [9,10] determined the scaling properties that characterize a period-doubling cascade that unfolds along the edge of a resonance tongue (denoted as cyclic or C-type criticality). More recently, Kuznetsov et al. [11] have determined the scaling relations for the terminal points of the torus doubling bifurcation curves for a related problem in which a low-dimensional period-doubling system is driven by quasiperiodic forcing.

The purpose of the present Letter is to examine the bifurcation structure associated with the interaction between the processes of period-doubling and synchronization in greater detail. In particular we show how complex structures of multi-layered resonance tori arise through period-doubling bifurcations of resonant node and saddle cycles in a direction transverse to the torus manifold. We also show how these multi-layered tori transform into period-doubled ergodic tori through a set of saddle–node bifurcations delineating the edge of the synchronization regime.

Torus doubling was first investigated by Arneodo et al. [12] and by Kaneko [13] who described the phenomenon both for three- and four-dimensional maps and for time-continuous systems. More recently, Sekikawa et al. [14] have demonstrated the formation of period-doubled tori in an electronic oscillator system and Postnov et al. [15] have used a double Poincaré-section technique to illustrate torus doubling in a van der Pol oscillator driven by a strong chaotic forcing. However, to our knowledge, a description of how these ergodic tori interact with the resonance structure in the synchronization tongues has not been presented.

We have previously observed the formation of multi-layered tori both for a variety of two-dimensional maps [16,17] and for a model of a pulse modulated power electronic DC/DC con-

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verter [18]. In a recent Letter [19] we have described the transition from double-layered resonance torus to period-doubled ergodic torus through a series of local and global bifurcations that occur at relatively low forcing amplitudes. The present Letter presents a more generic mechanism that involves sets of saddle–node bifurcations.

2. Main bifurcation structure for the periodically forced Rössler system

Let us consider the periodically forced Rössler system:

$$\dot{x} = -y - z + A\sin(\omega t); \qquad \dot{y} = x + ay;$$
$$\dot{z} = b + z(x - c), \tag{1}$$

where x, y and z are the dynamical variables of the Rössler oscillator and $A\sin(\omega t)$ represents the externally applied forcing. The nonlinearity parameter c and the forcing frequency ω are used as bifurcation parameters while the parameters a and b and the forcing amplitude A are kept constant at the values $a = 0.2$, $b = 0.2$ and $A = 0.1$. The unforced Rössler system ($A = 0$) undergoes a Hopf bifurcation at $c = 0.4$ and for increasing values of c, the system hereafter exhibits a Feigenbaum cascade of period-doubling bifurcations to chaos. When an external forcing is applied in the regime of periodic oscillations, the Rössler system displays regions of quasiperiodic dynamics interrupted by an infinite set of resonance zones where the internally generated periodic oscillations synchronize with the external forcing. Our aim is to examine the structures that arise from the interplay between these two processes in multi-dimensional systems.

Fig. 1 provides an overview of the main bifurcations associated with the first four period-doubling transitions. Below the first period-doubling curve PD_1^S, the 1:1 resonance zone is delineated to the left and right by the saddle–node bifurcation curves SN_1^L and SN_1^R, respectively. In this area, the system displays a stable node and a saddle solution, both situated on the closed invariant curve that represents the two-dimensional resonance torus. Along the lower curve PD_1^S, the 1:1 node solution undergoes its first period-doubling bifurcation, while the corresponding saddle solution doubles its period along the upper curve PD_1^U. At the edge of the tongue, the two solutions merge and, as our calculations show, the system displays both an eigenvalue (Floquet multiplier) of $+1$ and an eigenvalue of -1. The two eigenvalues correspond to different directions in phase space. Hence, we conclude that the period-doubling bifurcation occurs in a direction transverse to the torus manifold.

A pair of new saddle–node bifurcation curves, SN_2^L and SN_2^R, emanate from the period-doubling curve close to the two edges of the tongue. They delineate the synchronization region for the stable and unstable 2:2 solutions and are, therefore, tangents to the next period-doubling curve. The stable 2:2 solution undergoes a second period doubling along the lower branch PD_2^S and the saddle 2:2 solution period doubles along the upper PD_2^U. These period doublings again take place in a direction transverse to the closed invariant curve, and again a pair of new saddle–node bifurcation curves is born to delineate the synchronization range for the 4:4 solutions.

As the value of parameter c increases, the same process is found to repeat itself until the system undergoes a transition to chaos. Close to the tongue edges, corresponding node and saddle solutions period double almost simultaneously. However, as we move deeper into the resonance zone, the period-doubling of the node cycles occurs earlier and earlier in comparison with the period doubling of the corresponding saddle cycles. For this reason, the accumulation points for the two cascades are different and the structure of the multi-layered tori produced in the

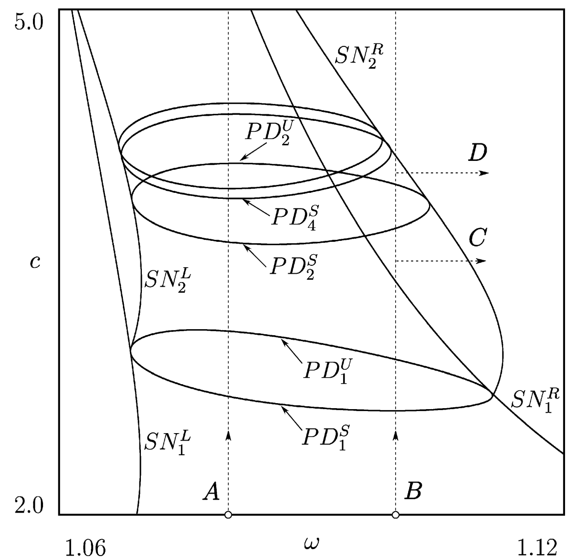

Fig. 1. Bifurcation structure associated with the first four period-doubling bifurcations of the 1:1 resonance cycles in the periodically forced Rössler oscillator. At the edges of the resonance tongue, defined by the saddle–node bifurcation curves SN_i^L and SN_i^R, $i = 1, 2$, period doubling of corresponding node and saddle cycles occurs simultaneously. Inside the zone, the stable cycles are found to period double at lower values of c than the saddle cycles. Note that each period-doubling gives rise to a new pair of saddle–node bifurcation curves to delineate the resonance zone for the period-doubled cycles. Arrows A, B, C and D denote scanning directions to be examined in the following figures.

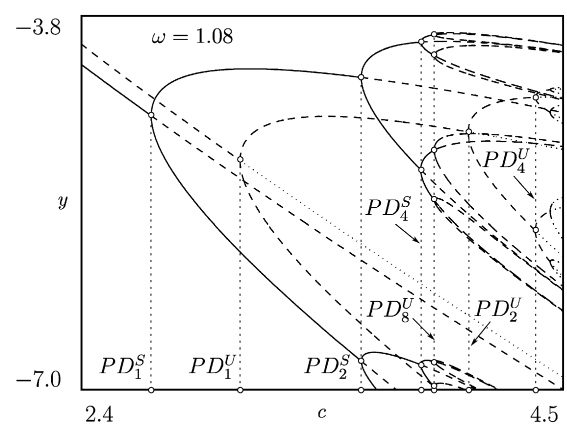

Fig. 2. One-dimensional bifurcation diagram along the direction A in Fig. 1. Full curves represent stable node solutions, dashed curves are saddle solutions, and dotted curves are doubly unstable saddle solutions. The period-doubling cascade for the stable node cycles accumulates in a transition to chaos approximately at $c_\infty = 3.95381$.

transition will depend on the forcing frequency ω. A similar bifurcation structure, involving interconnected cascades of period-doubling bifurcations for symmetric and antisymmetric orbits, has been discussed for two coupled Rössler systems [20]. However, the observed bifurcations were not related to the formation of multi-layered tori.

Fig. 2 shows a one-dimensional bifurcation diagram along the line A in Fig. 1. Here, $\omega = 1.08$. Full curves represent stable node solutions, dashed curves are saddle solutions, and dotted curves are doubly unstable saddle solutions. In accordance with the notation used in Fig. 1, a superscript S denotes a bifurcation of a stable node, and a superscript U indicates that the bifurcation occurs for a saddle cycle. Subscripts 1, 2, 4, etc., denote period-doubling bifurcations of period-1, period-2, etc., solutions. For the stable node cycles the transition to chaos occurs at $c \approx 3.95381$.

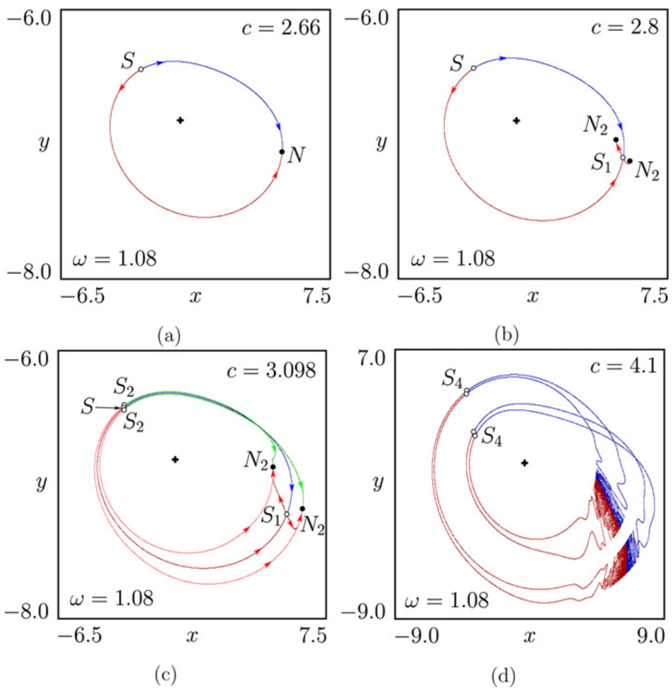

Fig. 3. Phase portraits of the resonance torus along the scan line A in Fig. 1. (a) Original 1 : 1 resonance torus with its node N and saddle cycle S. (b) The node cycle has undergone a period-doubling bifurcation transversely to the torus manifold. (c) Formation of a double-layered torus after the saddle cycle has also undergone a transverse period doubling transition. (d) Multi-layered chaotic structure after the saddle cycle has undergone a second period doubling and the original node cycle a complete transverse period-doubling cascade to chaos.

3. Formation of double-layered resonance tori and torus doubling

Below the period-doubling curve PD_1^S the system displays an ordinary 1 : 1 resonance torus with a stable period-1 node N and corresponding saddle cycle S (see Fig. 3(a)). As the system crosses the bifurcation curve PD_1^S, the 1 : 1 node undergoes a period-doubling bifurcation. As illustrated by the phase portrait in Fig. 3(b), this period-doubling takes place in a direction transverse to the torus manifold. This implies that whereas the original saddle cycle S is stable transversely to the torus manifold and unstable along this manifold, the saddle S_1, arising in the period-doubling bifurcation, is stable in the direction along the torus manifold and unstable in the transverse direction. N_2 denotes the points of the 2 : 2 resonance node. As c is further increased, the saddle cycle S undergoes a first period-doubling bifurcation when the system crosses the bifurcation curve PD_1^U. As the result, a multi-layered torus structure softly arises from the 1 : 1 resonance torus. Note how the now repelling 1 : 1 resonance torus is surrounded by the stable period-2 resonance torus (Fig. 3(c)).

As illustrated in Fig. 2, with further increase of the value of parameter c, one can observe a cascade of period-doubling bifurcations transverse to the 2 : 2 resonance torus, leading finally to a transition to chaos. Starting with the first period-doubling bifurcation of the 1 : 1 node cycle near $c = 2.70551$, the cascade of period-doubling bifurcations for the node cycles accumulates approximately at $c_\infty = 3.95381$. Fig. 3(d) shows the phase portrait in the region of chaotic dynamics for the original node cycle and after the second period-doubling bifurcation of the saddle cycle S_2.

Let us now examine the transition that occurs as we move out of the resonance tongue in the direction B (see Fig. 1), i.e., as we increase the parameter c from 2.4 to 4.3 while maintaining the forcing frequency constant at $\omega = 1.1$. This transition is shown in Fig. 4.

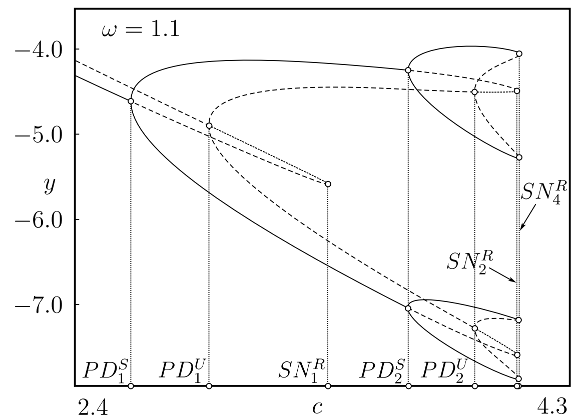

Fig. 4. One-dimensional bifurcation diagrams along the scan B in Fig. 1. After the period-doubling transitions at PD_1^S and PD_1^U, the 1 : 1 node and saddle cycles disappear in a saddle–node bifurcation on SN_1^R. The 2 : 2 node and saddle cycles undergo new period doublings at PD_2^S and PD_2^U, respectively, and then disappear in the saddle–node bifurcation SN_2^R. Finally, the 4 : 4 node and saddle cycles disappear in a saddle–node bifurcation at SN_4^R. Note how each level in the period-doubling cascade requires a saddle–node bifurcation of its own to demarcate the resonance zone.

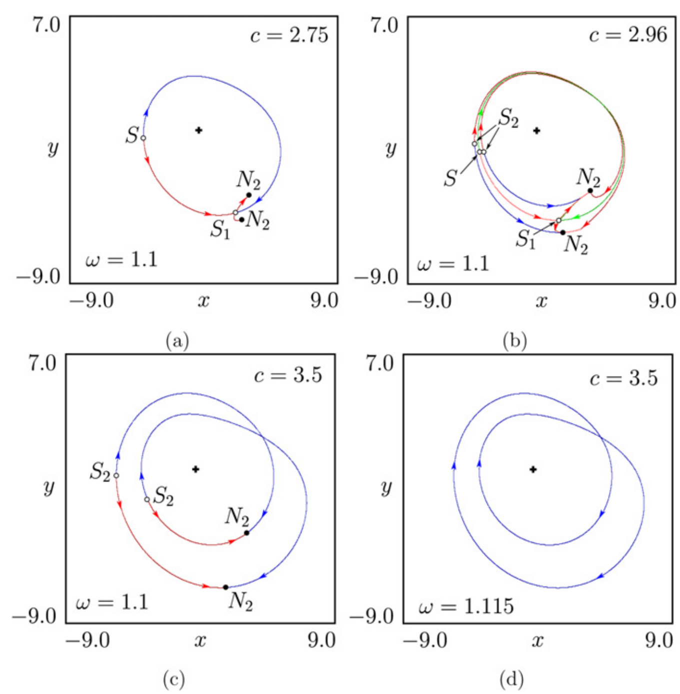

Fig. 5. Transition from double-layered resonance torus to period-doubled ergodic torus. (a) Phase portrait after the period-doubling bifurcation of the stable node transversely to the 1 : 1 resonance torus. As the parameter c is increased the saddle period-1 cycle S undergoes a period-doubling bifurcation. This leads to the formation of the period-2 resonance torus with a double-layered structure (b). (c) Phase portrait after the saddle–node bifurcation in which the saddle cycles S and S_1 merge and disappear. (d) Period-2 ergodic torus that arises when we leave the resonance tongue along the direction C in Fig. 1.

At the starting point the system displays a period-1 resonance torus. As the value of parameter c is increased, the stable node N undergoes a period-doubling bifurcation transverse the 1 : 1 resonance torus when the system crosses the curve PD_1^S. This is illustrated in the phase portrait in Fig. 5(a). When the system crosses the bifurcation curve PD_1^U the saddle period-1 cycle S undergoes a period-doubling bifurcation. As a result, a double-layered torus structure softly arises from the original resonance torus. Fig. 5(b) presents the phase portrait for the double-layered torus. As illustrated in Figs. 1 and 4, with further increase of the parameter c the saddle cycles S and S_1 merge and disappear in a saddle–node

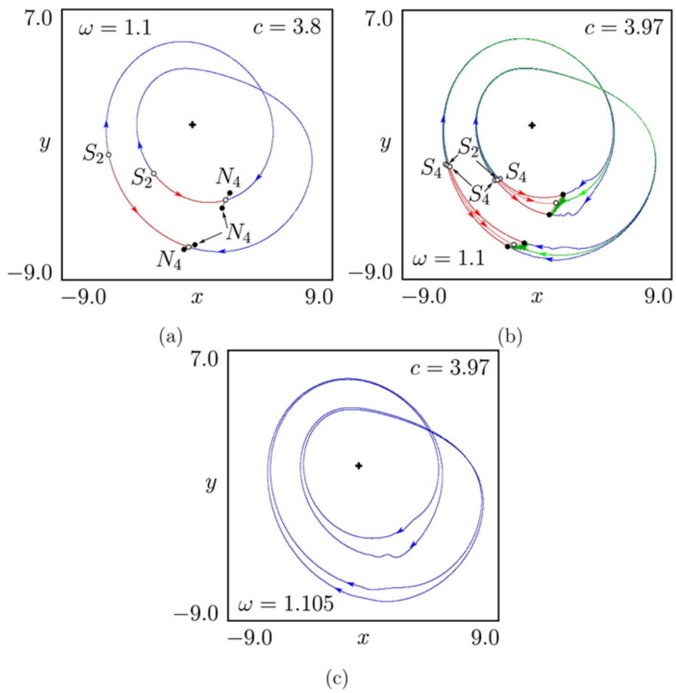

Fig. 6. Transition from a period-4 resonance torus to period-4 ergodic torus. (a) Period-doubling bifurcation of the stable period-2 cycle transversely to the 2:2 resonance torus. (b) Period-4 resonance torus after the period-doubling bifurcation of the saddle period-2 cycle S_2. S_4 and N_4 are the points of the period-4 saddle and stable cycles, respectively. (c) Period-4 ergodic torus. This torus appears when the system leaves the resonance tongue along the direction D in Fig. 1.

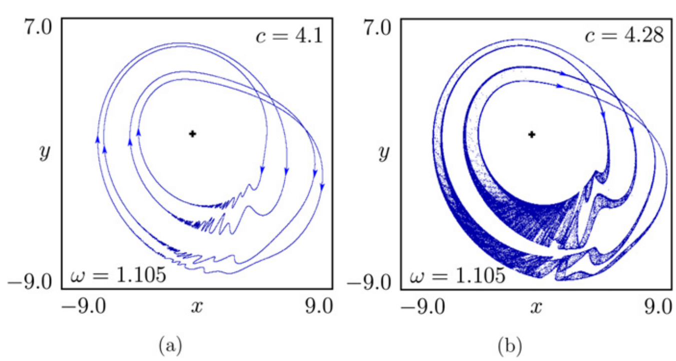

Fig. 7. (a) Folded structure for the period-4 ergodic torus. (b) Phase portrait for the chaotic dynamics.

bifurcation on the bifurcation curve SN_1^R, and the repelling layer of the original period-1 resonance torus disappears (Fig. 5(c)). Finally, when we leave the resonance tongue through the second saddle–node bifurcation curve SN_2^R along the direction C in Fig. 1, the period-2 resonance torus transforms into the period-2 ergodic torus through a saddle–node bifurcation. Fig. 5(d) shows the phase portrait for the period-2 ergodic torus after the saddle–node bifurcation.

When crossing the period-doubling bifurcation curves PD_2^S and PD_2^U with increasing parameter c along the direction B (see Figs. 1 and 4), we again observe the doubling of the resonance torus. Fig. 6(a) presents the phase portrait of system (1) after the second period-doubling bifurcation for the stable period-2 node N_2 transverse to the period-2 resonance torus. Fig. 6(b) shows the phase portrait of the system (1) after the second period-doubling bifurcation of the saddle period-2 cycle S_2 in which the double-layered period-4 resonance torus appears. When we leave the resonance tongue along the direction D in Fig. 1, the period-4 resonance torus transforms into a period-4 ergodic torus in a similar manner (Fig. 6(c)).

With further increase the value of the c, the invariant set of the ergodic period-4 torus first starts to fold. Further change of the parameter c leads to the appearance of the chaotic oscillations. This transition is illustrated for $\omega = 1.105$ in the Fig. 7.

4. Conclusions

The Letter established a more complete picture of the bifurcation phenomena that occur when a multi-dimensional period-doubling system is subjected to an external forcing and showed how the recently discovered phenomena of multi-layered resonance tori is linked to the phenomenon of period-doubled ergodic tori.

We first demonstrated how multi-layered resonance tori are formed through cascades of period doubling bifurcations of the resonant saddle and node cycles transversely to the torus manifold. Close to the edge of synchronization zone, where the bifurcations occur more and less simultaneously for the node and saddle cycles, one can follow a relatively high number of interconnected period doublings. In the interior of the resonance zone, where period doubling of the node cycle proceeds faster then period doubling of the saddle cycle, one can observe multi-layered chaotic structures produced through a complete period-doubling cascade of the original node cycle.

Each pair of period-doubling bifurcations generates a new set of saddle–node bifurcation curves along the sides of resonance tongue. As the period-doubling process proceeds, the edges of the tongue, therefore, accumulate more and more saddle–node bifurcation curves, each delineating the boundaries for the resonance modes at a particular level of the period-doubling cascade. Similarly to what one observes in the case of phase multistability [21] it appears that there is no specific ordering of the saddle–node bifurcation curves, and the ordering differs between the two sides of the tongue. This ordering influences the detailed transition by which a resonance torus is transformed into an ergodic torus. At low forcing amplitudes, one can observe that additional local and global bifurcation are involved in this transition [19].

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References

[1] E.F. Stone, Phys. Lett. A 163 (1992) 367.
[2] M.G. Rosenblum, A. Pikovsky, J. Kurths, Phys. Rev. Lett. 76 (1996) 1804.
[3] A.S. Pikovsky, M.G. Rosenblum, G.V. Osipov, J. Kurths, Physica D 104 (1997) 219.
[4] E. Rosa Jr., E. Ott, M.H. Hess, Phys. Rev. Lett. 80 (1998) 1642.
[5] E. Rosa Jr., W.B. Pardo, C.M. Ticos, J.A. Walkenstein, M. Monti, Int. J. Bifur. Chaos 10 (2000) 2551.
[6] T.E. Vadivasova, A.G. Balanov, O.V. Sosnovtseva, D.E. Postnov, E. Mosekilde, Phys. Lett. A 253 (1999) 66.
[7] J.P. Carcasses, C. Mira, M. Bosch, C. Simo, J.C. Tatjer, Int. J. Bifur. Chaos 1 (1991) 183–196.
[8] L. Glass, R. Perez, Phys. Rev. Lett. 48 (1982) 1772.
[9] A.P. Kuznetsov, S.P. Kuznetsov, I.R. Sataev, Physica D 109 (1997) 91.
[10] S.P. Kuznetsov, I.R. Sataev, Phys. Rev. E 64 (2001) 046214.
[11] S.P. Kuznetsov, U. Feudel, A. Pikovsky, Phys. Rev. E 57 (1998) 1585.
[12] A. Arneodo, P.H. Coullet, E.A. Spiegel, Phys. Lett. 94A (1983) 1.
[13] K. Kaneko, Prog. Theor. Phys. 69 (1983) 1806.
[14] M. Sekikawa, T. Miyoshy, N. Inaba, IEEE Trans. Circuits Syst. I 48 (2001) 28.

[15] D.E. Postnov, A.G. Balanov, O.V. Sosnovtseva, E. Mosekilde, Phys. Lett. A 283 (2001) 195.
[16] Zh.T. Zhusubaliyev, E. Mosekilde, Phys. Lett. A 373 (2009) 946.
[17] Zh.T. Zhusubaliyev, E. Mosekilde, Physica D 238 (2009) 589.
[18] Zh.T. Zhusubaliyev, E. Mosekilde, Phys. Lett. A 351 (2006) 167.
[19] J.L. Laugesen, E. Mosekilde, Zh.T. Zhusubaliyev, Physica D, submitted for publication.
[20] J. Rasmussen, E. Mosekilde, C. Reick, Math. Comput. Simulation 40 (1996) 247.
[21] E. Mosekilde, D.E. Postnov, O.V. Sosnovtseva, Prog. Theor. Phys. Suppl. 150 (2003) 147.