Lagrange's equations of motion

Solving **Newton's equations** of motion for a system of N particles

$$\boxed{m_i \ddot{r}_i = F_i^e + \sum_j F_{ij}} \qquad (1)$$

where F_i^e stands for the external force and F_{ij} is the internal force on the ith particle due to the jth particle, we get the position of every particle as a function of time t, $r_i(t)$.

Usually things are more complicated: there exist **constraints** that limit the motion of the system. E.g.

- A point mass moving on the surface of a sphere
- A rigid body (the distances between the particles are constants)
- Gas molecules within a container are constrained by the walls of the vessel to move only inside the container.

Constraints introduce two types of difficulties:

1. The coordinates r_i are no longer independent, since they are connected by the equations of the constraint. Thus the equations (1) are not all independent.

2. The forces of the constraint are also unknowns.

We overcome these difficulties by:

1. Introducing suitable generalized coordinates

2. Using Lagrange's equations of motion

Types of constraints

➤ **Holonomic constraints:**

$$f(r_1, r_2, r_3, \ldots, r_{3N}, t)=0 \qquad (2)$$

- **rheonomous,** if t appears explicitly in (2)
- **scleronomous** if the constraint does not depend explicitly on time

Examples:

- Motion of a particle on a sphere of radius R:

$$x^2 + y^2 + z^2 - R^2 = 0$$

- Two particles on the plane moving as a rigid body (their distance d remains constant)

$$(x_1-x_2)^2 + (y_1-y_2)^2 - d^2 = 0$$

➤ **Nonholonomic constraints**

Example:

A particle placed in the interior of a sphere:

$$x^2 + y^2 + z^2 < R^2$$

Generalized coordinates (q_i): The independent parameters needed for defining the position of a particle or a system of particles.

Configuration space: The space whose coordinates are the generalized coordinates q_i.

Degrees of freedom: the number of generalized coordinates.

Generalized velocities: The time derivatives of **generalized** coordinates $\dot{q}_i$

A system of N particles, free from constraints, has 3N independent coordinates or degrees of freedom. **If there exist k holonomic constraints** of the form (2), then we may use these equations to eliminate k of the 3N coordinates. So we are left with 3N-k independent coordinates, so **the system has 3N-k degrees of freedom.**

This elimination can be done by introducing **3N-k new independent variables $q_1, q_2, \ldots, q_{3N-k}$** in terms of which the **old coordinates $r_1, r_2, \ldots, r_N$** are expressed by the equations:

$$r_i = r_i\,(q_1, q_2, \ldots, q_{3N-k}, t) \quad \text{for } i=1, 2, \ldots, 3N \tag{3}$$

In general we can solve (3) with respect to the new generalized coordinates and get:

$$q_i = q_i\,(r_1, r_2, \ldots, r_{3N}, t) \quad \text{for } i=1, 2, \ldots, 3N\text{-}k \tag{4}$$

The equations expressing nonholonomic constraints cannot be used to eliminate the dependent coordinates. So nonholonomic constraints do not reduce the number of degrees of freedom.

A **virtual infinitesimal displacement** of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates δr_i, **consistent with the forces and constraints imposed on the system at a given time t.** The displacement is called virtual and denoted by (δ) to distinguish it from an actual displacement of the system, denoted by (d) occurring in a time interval dt, during which the forces and constrains may be changing.

Example

A mass point moving free of constraints is a system with 3 degrees of freedom. As generalized coordinates can be used the cartesian coordinates x, y, z or the spherical coordinates r, θ, φ.

If the point is moving on a sphere of radius R then we have 1 holonomic constraint of the form:

$$x^2 + y^2 + z^2 - R^2 = 0$$

This constraint is scleronomous if R is constant and rheonomous if R changes in time R(t). So **the system has 2 degrees of freedom**. We can choose as generalized coordinates the **angles θ, φ**.

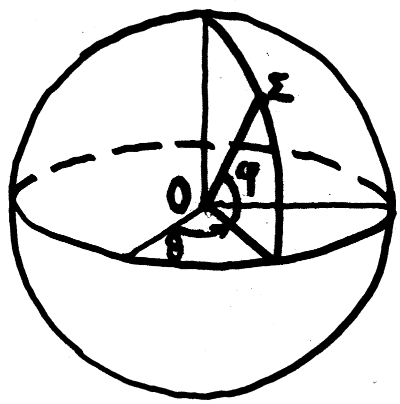
The transformations between the generalized coordinates and the cartesian coordinates are:

$$x = R\cos\varphi\cos\theta,$$
$$y = R\cos\varphi\sin\theta,$$
$$z = R\sin\varphi$$

and

$$\vartheta = \arctan\left(\frac{y}{x}\right)$$

$$\varphi = \arctan\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$



Principle of virtual work

We consider the system of N particles with k holonomic constraints, to be **in equilibrium** i.e. the total force on each particle vanishes

$$F_i = 0 \tag{5}$$

Then the virtual work of these forces also vanishes

$$\sum_i F_i \cdot \delta r_i = 0 \tag{6}$$

We decompose the total force F_i into the applied force F_i^a and the force of the constraint R_i

$$F_i = F_i^a + R_i \tag{7}$$

Then eq.(6) becomes:

$$\sum_i F_i^a \cdot \delta r_i + \sum_i R_i \cdot \delta r_i = 0 \tag{8}$$

We consider systems where **the net virtual work of the forces of constraints is zero (or the forces of constraints are perpendicular to the constraint's surface):**

$$\sum_i R_i \cdot \delta r_i = 0 \tag{9}$$

Principle of virtual work:

The necessary and sufficient condition for a system to be in equilibrium is that the virtual work of the applied forces vanishes for all virtual displacements:

$$\boxed{\sum_i F_i^a \cdot \delta r_i = 0} \tag{10}$$

D' Alembert's principle and Lagrange's equations

In the general case that the system **is not at equilibrium**, Newton's equations of motion give:

$$F_i = \dot{p}_i \Leftrightarrow F_i - \dot{p}_i = 0 \Leftrightarrow F_i^a + R_i - \dot{p}_i = 0 \Leftrightarrow \sum_i (F_i^a - \dot{p}_i)\cdot \delta r_i + \sum_i R_i \cdot \delta r_i = 0$$

Since the virtual work of the forces of constraints is zero (eq.(9)) we get the **D' Alembert's principle:**

$$\boxed{\sum_i (F_i^a - m_i \ddot{r}_i)\cdot \delta r_i = 0} \tag{11}$$

The D' Alembert's principle does not contain the forces of constraints, and the superscript (a) can be dropped. Since δr_i are not independent we will transform eq.(11) into an equation involving the independent virtual displacements δq_i.

From eq.(3)

$$r_i = r_i (q_1, q_2, \ldots, q_{3N-k}, t) \quad \text{for } i=1, 2, \ldots, 3N$$

we get

$$u_i = \frac{dr_i}{dt} = \dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \tag{12}$$

and

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j \tag{13}$$

So in terms of the generalized coordinates the virtual work of F_i becomes:

$$\sum_i F_i \cdot \delta r_i = \sum_{i,j} F_i \cdot \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_i Q_j \cdot \delta q_j \tag{14}$$

where Q_j are called the **generalized forces** defined as

$$\boxed{Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}} \tag{15}$$

From eq.(12), since r_i and $(\partial r_i/\partial t)$ do not depend on $\dot{q}_k$ we get

$$\frac{\partial u_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \tag{16}$$

We also have (using eq.(12)).

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 r_i}{\partial q_j\,\partial q_k}\dot{q}_k + \frac{\partial^2 r_i}{\partial q_j\,\partial t} = \frac{\partial \dot{r}_i}{\partial q_j} = \frac{\partial u_i}{\partial q_j} \tag{17}$$

The second part of D' Alembert's principle (eq.(11)) using eq.(13) becomes:

$$\sum_i m_i\,\ddot{r}_i \cdot \delta r_i = \sum_{i,j} m_i\,\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\delta q_j \tag{18}$$

So we get

$$\sum_i m_i\,\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left\{\frac{d}{dt}\left(m_i\,\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i\,\dot{r}_i\,\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right\} \tag{19}$$

Using eqs.(16) and (17), eq.(19) becomes

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m_i u_i \cdot \frac{\partial u_i}{\partial \dot{q}_j} \right) - m_i u_i \frac{\partial u_i}{\partial q_j} \right\} \tag{20}$$

so the second term of eq.(11) can be written as

$$\sum_i m_i \ddot{r}_i \cdot \delta r_i = \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \tfrac{1}{2} m_i u_i^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m_i u_i^2 \right) \right\} \delta q_j \tag{21}$$

Identifying $\sum_i \frac{1}{2} m_i u_i^2$ with the **kinetic energy T** of the system, D' Alembert's principle becomes:

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0 \tag{22}$$

Since the virtual displacements are independent eq.(22) holds when

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad j = 1,2,\ldots,3N - k} \tag{23}$$

When the forces are derived from a scalar potential function V

$$F_i = - \nabla_i V \tag{24}$$

the generalized forces can be written as:

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = - \sum_i \nabla_i V \cdot \frac{\partial r_i}{\partial q_j} = - \frac{\partial V}{\partial q_j} \tag{25}$$

Since V does not depend on the generalized velocities **eq.(23)** becomes

$$\boxed{\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}=0} \tag{26}$$

Defining the **Lagrangian function L**

$$L = T - V \tag{27}$$

we get the **Lagrange's equations**:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0} \tag{28}$$

Advantages of Lagrange's equations:

- They have the same form for any set of generalized functions
- The unknown forces of the constraints do not appear

Hamilton's Principle

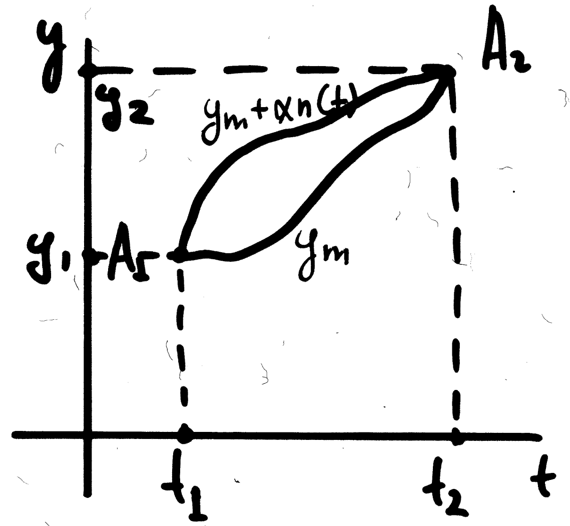
We will obtain **Lagrange's equations** from a principle that considers the entire motion of the system between times t_1 and t_2, and small virtual variations of the entire motion from the actual motion.

- First we consider the problem:

We have a function $f(y, \dot{y}, t)$ defined on a path $y = y(t)$ between two values t_1 and t_2. We wish to find a particular path $y(t)$ such that the line integral

$$J = \int_{t_1}^{t_2} f(y, \dot{y}, t)\, dt \qquad (29)$$

has a stationary value relative to paths differing infinitesimally from the correct function $y(t)$



We consider paths for which:

$$y(t_1) = y_1 \qquad y(t_2) = y_2$$

So all paths pass from points A_1, A_2

Let y_m be the path we seek. Then a possible set of varied paths is given by:

$$y(t) = y_m(t) + \alpha \cdot \eta(t) \qquad (30)$$

where α an infinitesimal parameter and $\eta(t)$ any function that vanishes at t_1, t_2

For any parametric family of curves of eq. (30) the integral J (eq. (29)) is also a function of a:

$$J(a) = \int_{t_1}^{t_2} f(y(t,a), \dot{y}(t,a), t)\, dt \qquad (31)$$

Since J has a stationary value for the path $y = y_m(t)$ (which coresponds to $a=0$) we have:

$$\left(\frac{dJ}{da}\right)_{a=0} = 0 \qquad (32)$$

Differentiating $J(a)$ (eq. (31)) we get:

$$\frac{dJ}{da} = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial a}\right) dt \qquad (33)$$

Integrating by parts, the second integral becomes:

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial a}\, dt = \int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial^2 y}{\partial t\, \partial a} \cdot dt = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial a}\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\frac{\partial y}{\partial a}\, dt \qquad (34)$$

From eq. (30) we get

$$\frac{\partial y}{\partial a} = n(t)$$

Since $n(t)$ vanishes at t_1, t_2 ($n(t_1) = n(t_2) = 0$) eq. (34) gives:

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial a}\, dt = -\int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] n(t) \cdot dt \qquad (35)$$

So eq.(33) becomes:

$$\frac{dJ}{da} = \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] n(t)\, dt \qquad (36)$$

For $a=0$ the above equation gives (using eq.(32)):

$$\int_{t_1}^{t_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]_{\alpha=0} n(t)\, dt = 0 \qquad (37)$$

where y is actually $y_m(t)$ for which J has a stationary value.

Since eq.(37) holds for all arbitrary functions $n(t)$ we get

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \qquad (38)$$

So, the function $y_m(t)$ for which the line integral J has a stationary value solves eq. (38). The assertion that J is stationary for y_m can be also written as:

$$\delta J = \delta \int_{t_1}^{t_2} f(y, \dot{y}, t)\, dt = 0 \qquad (39)$$

The above theorem is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$

$$J = \int_1^2 f(y_1, y_2, \dots, y_n, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_n, t)\, dt \qquad (40)$$

Then $\delta J = 0$ is valid when the so-called Euler-Lagrange Differential Equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0 \qquad (41)$$

hold.

<u>Hamilton's principle</u>: The motion of a dynamical system from time t_1 to time t_2 is such that the line integral:

$$I = \int_{t_1}^{t_2} L\, dt \qquad (42)$$

where $L = T - V$, has a stationary value. In other words

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)\, dt = 0 \qquad (43)$$

I. is called the <u>ACTION</u>

Remark: For a particular set of equations of motion there is no unique choise of Lagrangian

If $L(q,\dot{q},t)$ is an appropriate Lagrangian and $F(q,t)$ is any differentiable function of the generalized coordinates and time then:

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}F(q,t) \qquad (44)$$

is a Lagrangian also resulting in the same equations of motion since:

$$\delta\int_1^2 L'\,dt = \delta\int_1^2 L\,dt + \delta\int_1^2 \frac{d}{dt}F(q,t)\,dt =$$

$$= \delta\int_1^2 L\,dt + \underbrace{\delta F(q,t)\Big|_1^2}_{\emptyset} = \delta\int_1^2 L\,dt$$

We note that $\delta F(q,t)\Big|_1^2 = 0$ since all the curves $q_i(t)$ pass from points A_1 and A_2 (denoted by 1 and 2)

Conservation theorems

A> Momentum

- Canonical momentum or conjugate momentum p_i:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (45)$$

- If the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be **cyclic** or **ignorable**

$\Rightarrow$ The generalized momentum conjugate to a cyclic coordinate is conserved

The Lagrange equation of motion (eq.(28))

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

reduces, for a cyclic coordinate q_i, to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \xRightarrow{(45)} \frac{dp_i}{dt} = 0 \Longrightarrow \underline{p_i = \text{constant}} \quad (46)$$

B> Energy

The total time derivative of the Lagrangian L is:

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t} \qquad (47)$$

From Lagrange's equations (eq. (28)) we have

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \qquad (48)$$

so eq. (47) becomes:

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t} =$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t} \Longrightarrow$$

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0 \qquad (49)$$

$\Rightarrow$ If the Lagrangian is not an explicit function of time, the **energy function** $h(q,\dot{q},t)$

$$h(q,\dot{q},t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \qquad (50)$$

is conserved. The function $h(q,\dot{q},t)$ is also referred to as **Jacobi's integral**

Example: Motion of one particle

•) Cartesian coordinates

The generalized forces (eq. (15)) are F_x, F_y, F_z

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

So eq.(23):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=Q_i$$

give Newton's equations:

$$\frac{d}{dt}(m\dot{x})=F_x \qquad \frac{d}{dt}(m\dot{y})=F_y \qquad \frac{d}{dt}(m\dot{z})=F_z$$

•) Plane polar coordinates

The equations of transformation give:

$$\left.\begin{array}{l} x=r\cos\vartheta \\ y=r\sin\vartheta \end{array}\right\} \Longrightarrow \begin{array}{l} \dot{x}=\dot{r}\cos\vartheta-r\dot{\vartheta}\sin\vartheta \\ \dot{y}=\dot{r}\sin\vartheta+r\dot{\vartheta}\cos\vartheta \end{array}$$

so

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2) \Longrightarrow T=\frac{1}{2}m(\dot{r}^2+(r\dot{\vartheta})^2)$$

Generalized forces:

$$Q_r=\vec{F}\cdot\frac{\partial\vec{r}}{\partial r}=\vec{F}\cdot\frac{\vec{r}}{r}=F_r$$

$$Q_\vartheta=\vec{F}\cdot\frac{\partial\vec{r}}{\partial\vartheta}=\vec{F}\cdot r\vec{n}=rF_\vartheta \qquad \text{where } \vec{r}\cdot\vec{n}=0 \text{ or } \vec{r}\perp\vec{n}$$

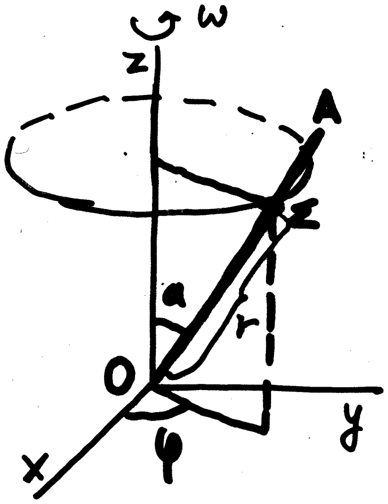
So eq.(23) give:

(r) $$m\ddot{r}-mr\dot{\vartheta}^2=F_r$$

(ϑ) $$\frac{d}{dt}\underbrace{(mr^2\dot{\vartheta})}_{\text{angular momentum}}=r\cdot F_\vartheta$$

$$\left\{\begin{array}{l} \frac{\partial T}{\partial r}=mr\dot{\vartheta}^2 \;/\; \frac{\partial T}{\partial \dot{r}}=m\dot{r} \;/\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)=m\ddot{r} \\ \\ \frac{\partial T}{\partial \vartheta}=0 \;/\; \frac{\partial T}{\partial \dot{\vartheta}}=mr^2\dot{\vartheta} \end{array}\right.$$

Example



Motion of particle Σ on the line OA which revolves around Oz with constant angular velocity ω

$\alpha = \text{constant} \quad \varphi = \omega t$

2 holonomic constraints:

$$\tan\alpha = \frac{\sqrt{x^2+y^2}}{z}, \quad \tan\omega t = \frac{y}{x}$$

We have 1 generalized coordinate: r (1 degree of freedom)

$$x = r\cdot\sin a\cdot\cos\omega t, \quad y = r\cdot\sin a\cdot\sin\omega t, \quad z = r\cdot\cos a$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \Rightarrow T = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2\sin^2 a)$$

$$V = mgz = mgr\cos a$$

Lagrangian: $L = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2\sin^2 a) - mgr\cdot\cos a$

Equation of motion: $m\ddot{r} - m\omega^2 r\sin^2 a + mg\cos a = 0$

Jacobi's integral: $h = \frac{\partial L}{\partial \dot{r}}\dot{r} - L = \text{constant} \Rightarrow$

$$h = \frac{1}{2}m\dot{r}^2 + mgr\cos a - \frac{1}{2}mr^2\omega^2\sin^2 a = \text{constant}$$

<u>Remark:</u> h is not the energy E of the system

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2\sin^2\alpha) + mgr\cos\alpha$$