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Partial Differential Equations

Riemann–Hilbert formulation for the KdV equation on a finite interval

Iasonas Hitzazis, Dimitri Tsoubelis

Department of Mathematics, University of Patras, 26500 Patras, Greece

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Abstract

The initial-boundary value problem for the KdV equation on a finite interval is analyzed in terms of a singular Riemann–Hilbert problem for a matrix-valued function in the complex k-plane which depends explicitly on the space–time variables. For an appropriate set of initial and boundary data, we derive the k-dependent "spectral functions" which guarantee the uniqueness of Riemann–Hilbert problem's solution. The latter determines a solution of the initial-boundary value problem for KdV equation, for which an integral representation is given. ***To cite this article: I. Hitzazis, D. Tsoubelis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

L'équation KdV sur un intervalle borné par la méthode de Riemann–Hilbert. Nous étudions un problème aux limites pour l'équation KdV sur un intervalle borné : l'étude est faite en termes d'un problème de Riemann–Hilbert singulier dans le k-plan complexe pour une fonction matricielle qui dépend de façon explicite de variables d'espace–temps. Pour un ensemble particulier de données de Cauchy ainsi que de valeurs aux limites, nous donnons les « fonctions spectrales » qui rendent la solution du problème de Riemann–Hilbert unique. A partir de cette solution on obtient une expression intégrale de la solution du problème aux limites pour l'équation KdV. ***Pour citer cet article : I. Hitzazis, D. Tsoubelis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

La méthode « application spectrale inverse » a été modifiée récemment afin de devenir plus efficace à traiter des problèmes aux limites pour des équations aux dérivées partielles non-linéaires intégrables. L'élément fondamental de cette modification est l'analyse spectrale simultanée des deux équations linéaires de la paire de Lax associée.

Cette méthode, développée initialement pour traiter le problème aux limites des ondes gravitationnelles en relativité générale [5,10], a été appliquée à plusieurs problèmes aux limites donnant des résultats remarquables. En particulier, le problème aux limites sur un intervalle borné pour l'équation non-linéaire de Schrödinger a été étudié dans [8], pour l'équation sine-Gordon dans [4] et pour l'équation modifiée de Korteweg–de Vries dans [2,3].

E-mail addresses: hitzazis@math.upatras.gr (I. Hitzazis), tsoubeli@math.upatras.gr (D. Tsoubelis).

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Dans cette Note, nous appliquons la méthode rigoureuse développée dans [4,6,7,9], et [12] à l'étude des solutions $q(x,t)$ de l'équation KdV dans le domaine $\Omega := \{(x,t) \in \mathbb{R}^2\colon\ 0 < x < L,\ 0 < t < T\}$, $L < \infty$, $T \leqslant \infty$, avec données de Cauchy $q(x,0) = q_0(x)$ et valeurs aux limites $(\partial_x^j q)(0,t) = g_j(t)$, $(\partial_x^j q)(L,t) = f_j(t)$, $j = 0, 1, 2$.

En supposant qu'une solution $q(x,t)$ de ce problème aux limites existe, on utilise la paire de Lax pour définir les fonctions spectrales $\{a(k), b(k), A(k), B(k), A_L(k), B_L(k)\}$ qui déterminent les conditions aux limites d'un problème de Riemann–Hilbert. Nous démontrons que, (i) si elle existe, la solution $M(x,t,k)$ du problème de Riemann–Hilbert est unique (Theorème 1.2) et (ii) à partir de $M(x,t,k)$ on peut construire la solution $q(x,t)$ du problème aux limites de départ pour l'équation KdV.

1. Introduction

The well known inverse scattering technique for solving the initial value problem for nonlinear partial differential equations (PDEs) has been recently modified in a manner that has rendered it very effective in dealing with problems in which the spatial independent variable is restricted to a semi-infinite or finite interval of the real line. The key element of this new approach to solving initial and boundary value problems (IBVPs) consists of using both members of the associated Lax pair simultaneously, in the analysis of the spectral problem defined by them.

First developed in the context of the characteristic initial value problem for colliding gravitational waves in general relativity [5,10], the new method has been applied on several IBVPs with impressive results. In particular, the finite interval problems that have already been solved include the nonlinear Schrödinger [8], the sine-Gordon [4], and the modified Korteweg–de Vries (mKdV) [2,3] equations.

In the present Note, we take advantage of the insights and rigorous methodology developed in [4,6,7,9], and [12], in order to analyze the solutions $q(x,t)$ of the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0$$

in the space–time region $\Omega := \{(x,t) \in \mathbb{R}^2\colon\ 0 < x < L,\ 0 < t < T\}$, where $L < \infty$ and $T \leqslant \infty$.

The KdV equation forms the integrability condition of the Lax pair

$$\Psi_x + \mathrm{i}k[\sigma_3, \Psi] = Q\Psi, \qquad \Psi_t + 4\mathrm{i}k^3[\sigma_3, \Psi] = \hat{Q}\Psi,$$

where $\Psi(x,t,k)$ is a 2×2-matrix valued function, $[\sigma_3, \Psi] := \sigma_3\Psi - \Psi\sigma_3$,

$$Q(x,t,k) := \frac{q}{2k}(\sigma_2 - \mathrm{i}\sigma_3), \qquad \hat{Q}(x,t,k) := 2kq\sigma_2 + q_x\sigma_1 + \frac{1}{2k}\big(q_{xx} + 2q^2\big)(\mathrm{i}\sigma_3 - \sigma_2),$$

and $\{\sigma_j\}_{j=1}^3$ are the Pauli matrices.

The new method for solving an IBVP starts from the assumption that a solution of the given problem exists and is represented by the function $q(x,t)$ appearing in the Lax pair. In the present case, this means that $q(x,t)$ satisfies the KdV equation in Ω and extends to the boundary $\partial\Omega$ so that $q(x,0)$ and an appropriate subset of $\{\partial_x^j q(0,t), \partial_x^j q(L,t)\}_{j=0}^2$ take up preassigned values. One then shows that the eigenfunctions of the corresponding Lax pair determine the boundary data for a specific Riemann–Hilbert (RH) problem.

Since the above procedure of constructing the RH problem is quite long and does not differ significantly from the one presented in [2] with regard to the mKdV equation, we restrict ourselves to giving only the end result of the construction. This allows us to devote the main part of this Note to analysing the solution of the RH problem so constructed and its relation to the solution of the IBVP for the KdV equation.

The solution $M(x,t,k)$ of the RH problem is a 2×2-matrix valued function with the following properties:

(i) It is a sectionally holomorphic function of the spectral parameter k.
(ii) Its discontinuities along the curves defining the sections of the complex k-plane are determined by a hexad of *spectral functions*, $\{a(k), b(k), A(k), B(k), A_L(k), B_L(k)\}$, which are defined implicitly by a set of smooth functions $q_0(x)$, $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$.

We show that the solution of the above RH problem, if it exists, is unique. Moreover, it determines a solution $q(x,t)$ of the KdV equation which satisfies an appropriate set of initial and boundary conditions.

An integral representation of this solution of the IBVP for the KdV equation is given in Theorem 1.2 which constitutes the main result of this paper. An outline of its proof is presented in Section 2. Details of our analysis can be found in [11].

Definition 1.1. Assume that $q_0(x)$, $0 \leqslant x \leqslant L$, $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$, $0 \leqslant t \leqslant T$, are given, smooth functions such that $(\partial_x^j q_0)(0) = g_j(0)$, $(\partial_x^j q_0)(L) = f_j(0)$, $j = 0, 1, 2$, and let $\phi(x,k)$, $\Phi(t,k)$, $\varphi(t,k)$ be the 2-vector functions defined, respectively, by the solutions of the initial value problems

$$\partial_x \phi(x,k) + 2\mathrm{i}k\,\mathrm{Diag}[1,0]\phi(x,k) = Q_0(x,k)\phi(x,k), \quad \phi(L,k) = [0,1]^T,\ k \in \mathbb{C}^\star$$
$$\partial_t \Phi(t,k) + 8\mathrm{i}k^3\,\mathrm{Diag}[1,0]\Phi(t,k) = \hat{Q}_0(t,k)\Phi(t,k), \quad \Phi(0,k) = [0,1]^T,\ k \in \mathbb{C}^\star$$
$$\partial_t \varphi(t,k) + 8\mathrm{i}k^3\,\mathrm{Diag}[1,0]\varphi(t,k) = \hat{Q}_L(t,k)\varphi(t,k), \quad \varphi(0,k) = [0,1]^T,\ k \in \mathbb{C}^\star$$

where $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$, and $Q_0(x,k)$, $\hat{Q}_0(t,k)$, $\hat{Q}_L(t,k)$ are the matrices obtained when $q(x,0)$, $\{(\partial_x^j q)(0,t)\}_{j=0}^2$, $\{(\partial_x^j q)(L,t)\}_{j=0}^2$ are replaced by $q_0(x)$, $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$ in $Q(x,0,k)$, $\hat{Q}(0,t,k)$, and $\hat{Q}(L,t,k)$, respectively.

Then the scalar functions $a(k)$, $b(k)$, $A(k)$, $B(k)$, $A_L(k)$ and $B_L(k)$ defined by

$$\big[b(k), a(k)\big]^T := \phi(0,k), \quad \big[-\mathrm{e}^{-8\mathrm{i}k^3T}B(k), \overline{A(\bar{k})}\big]^T := \Phi(T,k), \quad \big[-\mathrm{e}^{-8\mathrm{i}k^3T}B_L(k), \overline{A_L(\bar{k})}\big]^T := \varphi(T,k)$$

will be referred to as *spectral functions*.

The functions $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$ will be said to form an *admissible set* with respect to $q_0(x)$ if there exists a function $c(k)$, analytic in $\mathbb{C}^\star$, such that

(i) $c(k) = \frac{\mathrm{i}\sigma}{k} + \mathrm{O}(1)$, $k \to 0$, $\sigma \in \mathbb{R}$,

(ii) $c(k) = \mathrm{O}(\frac{1}{k})$, $k \to \infty$, $\mathrm{Im}\,k > 0$, $c(k) = \mathrm{O}(\frac{1+\mathrm{e}^{2\mathrm{i}kL}}{k^2})$, $k \to \infty$, $\mathrm{Im}\,k < 0$, and the following condition, where $\tilde{a} = \tilde{a}(k)$ and $\tilde{b} = \tilde{b}(k)$ denote $\overline{a(\bar{k})}$ and $\overline{b(\bar{k})}$, respectively, is satisfied:

$$G(k) := \big(aA_L + \tilde{b}\mathrm{e}^{2\mathrm{i}kL}B_L\big)B - \big(bA_L + \tilde{a}\mathrm{e}^{2\mathrm{i}kL}B_L\big)A = \mathrm{e}^{8\mathrm{i}k^3T}c(k), \quad k \in \mathbb{C}^\star.$$

The latter will referred to as the *global relation* and becomes $G(k) = 0$, $k \in D_1 \cup D_3 \cup D_5$, when $T = \infty$. In that case, we further assume that the boundary functions belong to the Schwarz class $S(\mathbb{R}^+)$. Moreover, the spectral functions $A(k)$, $B(k)$, $A_L(k)$ and $B_L(k)$ are defined in a slightly different manner (cf. [9,11]).

In order to describe clearly our main result, it is necessary to define the following regions of the complex k-plane and the curves forming their boundaries:

- The wedge like sections $D_j := \{k \in \mathbb{C}\colon (j-1)\pi/3 < \arg k < j\pi/3\} \setminus \{0\}$, $j = 1, 2, \ldots, 6$. Their boundaries are made up of the rays $L_j := \{k \in \mathbb{C}\colon \arg k = (j-1)\pi/3\}$.
- The annular regions $S_0 := \{k \in \mathbb{C}\colon 0 < |k| < R_1\}$, $S_1 := \{k \in \mathbb{C}\colon R_1 < |k| < R_2\}$ and $S_2 := \{k \in \mathbb{C}\colon |k| > R_2\}$, centered at the origin. Their boundaries are made up of the circles $C_1 := \{k \in \mathbb{C}\colon |k| = R_1\}$ and $C_2 := \{k \in \mathbb{C}\colon |k| = R_2\}$.
- The subregions $\Omega_{j\alpha} := D_j \cap S_\alpha$, which will be distinguished as positive or negative, respectively, depending on the sign of $(-1)^{j+\alpha-1}$. The union of the $\pm$ve $\Omega_{j\alpha}$'s will be denoted by $\Omega_\pm$, respectively. Their boundaries define the oriented curve $\Sigma := \partial\Omega_+ = \partial\Omega_-$, along which the subregions $\Omega_{j\alpha}$ meet. The latter can be broken down to eight parts, Σ_j, defined as follows: $\Sigma_1 := (L_1 \cup L_4) \setminus \bar{S}_1$, $\Sigma_2 := (L_2 \cup L_3) \setminus \bar{S}_1$, $\Sigma_3 := (C_1 \cup C_2) \cap (D_1 \cup D_3)$, and $\Sigma_4 := (C_1 \cup C_2) \cap D_2$. In their turn, Σ_5, Σ_6, and Σ_7 are the reflections with respect to the real axis of Σ_2, Σ_3, and Σ_4, respectively. Finally, Σ_8 is the remaining part of Σ.

Theorem 1.2. *Suppose that $q_0(x)$, $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$ are functions satisfying the conditions of Definition* 1.1 *and assume that the following RH problem is solvable*:

To find a 2×2 matrix valued function $M(x,t,k)$ which is sectionally holomorphic in $\Omega_+ \cup \Omega_-$ and satisfies the following condition.

The limiting values of the restrictions M_+, M_- of M to Ω_+ and Ω_-, respectively, as k approaches the discontinuity contour Σ are related by

$$M_-(x,t,k) = M_+(x,t,k)\mathrm{e}^{-\mathrm{i}\theta(x,t,k)\sigma_3} J_m(k)\mathrm{e}^{\mathrm{i}\theta(x,t,k)\sigma_3}, \quad k \in \Sigma_m,\ m = 1,2,\dots,8,$$

where $\theta(x,t,k) := kx + 4k^3t$,

$$J_1 = \begin{bmatrix} 1 & -\tilde{r}(k) \\ r(k) & 1-|r(k)|^2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} D(k) & \Delta(k) \\ \Gamma(k) & R(k) \end{bmatrix}, \quad J_3 = \begin{bmatrix} \alpha(k) & -\beta(k) \\ 0 & \frac{1}{\alpha(k)} \end{bmatrix}, \quad J_4 = \begin{bmatrix} \frac{\tilde{A}(k)}{d(k)} & b(k) \\ \frac{\tilde{B}(k)}{d(k)} & a(k) \end{bmatrix},$$

$J_{m+3}(k) = \sigma_3 J_m^*(\bar{k})\sigma_3$, $m = 2,3,4$, $J_8(k) = I$, *with*

$$\alpha(k) := a(k)A_L(k) + \mathrm{e}^{2\mathrm{i}kL}\tilde{b}(k)B_L(k), \quad \beta(k) := b(k)A_L(k) + \mathrm{e}^{2\mathrm{i}kL}\tilde{a}(k)B_L(k),$$
$$d(k) := a(k)\tilde{A}(k) - b(k)\tilde{B}(k), \quad \delta(k) := \alpha(k)\tilde{A}(k) - \beta(k)\tilde{B}(k), \quad \Delta(k) := \alpha(k)b(k) - a(k)\beta(k),$$
$$\Gamma(k) := \tilde{B}(k)/\alpha(k)d(k), \quad r(k) := \tilde{\beta}(k)/\alpha(k), \quad R(k) := a(k)/\alpha(k), \quad D(k) := \delta(k)/d(k).$$

Assume further that,

- *There exist positive numbers R_1 and R_2, with $R_2 > R_1$, such that the zeros of $\alpha(k)$, $d(k)$, $a(k)$, $A(k)$, and $A_L(k)$ are contained in the annulus S_1.*
- *There are real functions $\mu_j(x,t)$, $j = 1,2$ such that, as $k \to 0$ for $k \in D_1 \cup D_3$, D_2, $D_4 \cup D_6$, D_5,*

$$M \sim \frac{\mathrm{i}\mu_1(x,t)}{k}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \frac{\mathrm{i}\mu_2(x,t)}{k}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \frac{\mathrm{i}\mu_1(x,t)}{k}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \frac{\mathrm{i}\mu_2(x,t)}{k}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

- *For $k \in \mathbb{C} \setminus \Sigma$, $M = I + \mathrm{O}(\frac{1}{k})$, $M_{12} = \mathrm{O}(\frac{1}{k^2})$, $k \to \infty$.*

Then:

(i) *$M(x,t,k)$ is unique*;
(ii) *The function $q(x,t)$ defined by $q(x,t) := -2\mathrm{i}\partial_x \lim_{k\to\infty}[kM_{22}(x,t,k)]$ satisfies the KdV equation in Ω, the initial condition $q(x,0) = q_0(x)$, $0 \leqslant x \leqslant L$, and the boundary conditions $(\partial_x^j q)(0,t) = g_j(t)$, $(\partial_x^j q)(L,t) = f_j(t)$, $j = 0,1,2$, $0 \leqslant t \leqslant T$*;
(iii) *The solution of the original IBVP admits the following integral representation*:

$$\begin{aligned} q(x,t) = {} & \frac{1}{\pi}\int_{\partial D_5} \big\{\big[1-\tilde{R}(k)\big]\partial_x M_{22}^+ + \Gamma(k)\mathrm{e}^{-2\mathrm{i}kx-8\mathrm{i}k^3t}(\partial_x - 2\mathrm{i}k)M_{21}^+\big\}\,\mathrm{d}k \\ & + \frac{1}{\pi}\int_{\mathbb{R}} \big\{r(k)\mathrm{e}^{-2\mathrm{i}kx-8\mathrm{i}k^3t}(\partial_x - 2\mathrm{i}k)M_{21}^+ + \big|r(k)\big|^2\partial_x M_{22}^+\big\}\,\mathrm{d}k \\ & + \frac{1}{\pi}\int_{\partial D_2} \big\{\big[1-R(k)\big]\partial_x M_{22}^+ - \Delta(k)\mathrm{e}^{-2\mathrm{i}kx-8\mathrm{i}k^3t}(\partial_x - 2\mathrm{i}k)M_{21}^+\big\}\,\mathrm{d}k \\ & + \frac{1}{\pi}\int_{D_{13}\cap C_{12}} \left\{\beta(k)\partial_x M_{21}^+ + \left[1-\frac{1}{\alpha(k)}\right]\partial_x M_{22}^+\right\}\mathrm{d}k \\ & + \frac{1}{\pi}\int_{D_2\cap C_{12}} \big\{\big[1-a(k)\big]\partial_x M_{22}^+ - b(k)\partial_x M_{21}^+\big\}\,\mathrm{d}k \\ & + \frac{1}{\pi}\int_{D_{46}\cap C_{12}} \left[1-\frac{1}{\tilde{\alpha}(k)}\right]\partial_x M_{22}^+\,\mathrm{d}k + \frac{1}{\pi}\int_{D_5\cap C_{12}} \left\{\frac{B(k)}{\tilde{d}(k)}\partial_x M_{21}^+ + \big[1-\tilde{a}(k)\big]\partial_x M_{22}^+\right\}\mathrm{d}k, \end{aligned}$$

where $D_{13} := D_1 \cup D_3$, $D_{46} := D_4 \cup D_6$, and $C_{12} := C_1 \cup C_2$.

Remark 1.3. The assumption that the zeros of the spectral functions mentioned are contained in a bounded domain and do not touch the rays L_j restricts the generality of this theorem. However, the case under consideration remains generic.

Remark 1.4. The solvability of the RH problem can also be proven, indirectly, at least. Indeed, the existence of a solution $q(x,t)$ to the IBVP for the KdV equation has been established using PDE techniques [1]. Then, the simultaneous spectral analysis of the Lax pair associated to $q(x,t)$, in the spirit of [2], [3], leads to a function $M(x,t,k)$ having the properties described in Theorem 1.2 [11].

2. Sketch of proof of the theorem

(i) Assuming that $M(x,t,k)$ and $M'(x,t,k)$ are solutions of the RH problem, then $M'M^{-1}$ is an entire function of $k\in\mathbb{C}$ which tends to I as $k\to\infty$ [7,11]. Then Liouville's theorem implies $M'\equiv M$.

(ii) The proof that $q(x,t)$ solves the KdV equation is a straightforward application of the so-called dressing method and follows the proof for the whole line problem.

The proof that $q(x,t)$ satisfies the initial condition $q(x,0)=q_0(x)$ is based on the fact that the RH problem for $M(x,0,k)$ can be mapped to that for a sectionally holomorphic matrix, $M^{(x)}(x,k)$. The latter is associated to the inversion of the spectral map $\{q_0(x)\}\mapsto\{a(k),b(k)\}$ described in Definition 1.1. More specifically, $M^{(x)}(x,k)=M(x,0,k)P^{(x)}(x,k)$, where the matrix $P^{(x)}(x,k)$ is given by

$$P^{(x)}=\begin{bmatrix}\frac{1}{R(k)} & \mathrm{e}^{-2\mathrm{i}kx}\Delta(k)\\ 0 & R(k)\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ -\mathrm{e}^{2\mathrm{i}kx}\frac{\Gamma(k)}{R(k)} & 1\end{bmatrix},\quad \begin{bmatrix}\tilde{R}(k) & 0\\ \mathrm{e}^{2\mathrm{i}kx}\tilde{\Delta}(k) & \frac{1}{\tilde{R}(k)}\end{bmatrix},\quad \begin{bmatrix}1 & -\mathrm{e}^{-2\mathrm{i}kx}\frac{\tilde{\Gamma}(k)}{\tilde{R}(k)}\\ 0 & 1\end{bmatrix}$$

for $k\in D_{13}\setminus\bar{S}_1$, $D_2\setminus\bar{S}_1$, $D_{46}\setminus\bar{S}_1$, and $D_5\setminus\bar{S}_1$, respectively. Splitting it into its diagonal and off-diagonal parts, $P^{(x)}(x,k)=P^{(x)}_{\mathrm{diag}}(x,k)+P^{(x)}_{\mathrm{off}}(x,k)$, we find that $P^{(x)}_{\mathrm{diag}}(x,k)=I+\mathrm{O}(1/k)$ as $k\to\infty$ and $P^{(x)}_{\mathrm{off}}(x,k)\to 0$ exponentially as $k\to\infty$. This guarantees that the asymptotic expression for $M(x,0,k)$, which defines $q(x,0)$, is identical to the asymptotic expression of $M^{(x)}(x,k)$, which defines $q_0(x)$.

The proof that $(\partial_x^j q)(0,t)=g_j(t)$ and $(\partial_x^j q)(L,t)=f_j(t)$, $j=0,1,2$, is based on analogous considerations regarding a pair of matrices, $M^{(t,0)}(t,k)$ and $M^{(t,L)}(t,k)$, which are associated to the inversion of the spectral maps $\{g_j(t)\}_{j=0}^2\mapsto\{A(k),B(k)\}$ and $\{f_j(t)\}_{j=0}^2\mapsto\{A_L(k),B_L(k)\}$ introduced in Definition 1.1. More specifically, $M^{(t,0)}(t,k):=M(0,t,k)P^{(t,0)}(t,k)$ and $M^{(t,L)}(t,k):=M(L,t,k)P^{(t,L)}(t,k)$, where

$$P^{(t,0)}=\begin{bmatrix}\frac{\alpha(k)}{A(k)} & G(k)\mathrm{e}^{-8\mathrm{i}k^3t}\\ 0 & \frac{A(k)}{\alpha(k)}\end{bmatrix},\quad \begin{bmatrix}d(k) & -\frac{b(k)}{\tilde{A}(k)}\mathrm{e}^{-8\mathrm{i}k^3t}\\ 0 & \frac{1}{d(k)}\end{bmatrix},\quad \begin{bmatrix}\frac{\tilde{A}(k)}{\tilde{\alpha}(k)} & 0\\ \tilde{G}(k)\mathrm{e}^{8\mathrm{i}k^3t} & \frac{\tilde{\alpha}(k)}{\tilde{A}(k)}\end{bmatrix},$$
$$\begin{bmatrix}\frac{1}{\tilde{d}(k)} & 0\\ -\frac{\tilde{b}(k)}{A(k)}\mathrm{e}^{8\mathrm{i}k^3t} & \tilde{d}(k)\end{bmatrix},$$
$$P^{(t,L)}=\begin{bmatrix}1 & 0\\ \frac{\tilde{b}(k)}{\alpha(k)A_L(k)}\mathrm{e}^{2\mathrm{i}\theta(L,t,k)} & 1\end{bmatrix},\quad \begin{bmatrix}\tilde{A}_L(k) & 0\\ -\frac{\tilde{G}(k)}{d(k)}\mathrm{e}^{2\mathrm{i}\theta(L,t,k)} & \frac{1}{\tilde{A}_L(k)}\end{bmatrix},\quad \begin{bmatrix}1 & \frac{b(k)}{\tilde{\alpha}(k)\tilde{A}_L(k)}\mathrm{e}^{-2\mathrm{i}\theta(L,t,k)}\\ 0 & 1\end{bmatrix},$$
$$\begin{bmatrix}\frac{1}{A_L(k)} & -\frac{G(k)}{\tilde{d}(k)}\mathrm{e}^{-2\mathrm{i}\theta(L,t,k)}\\ 0 & A_L(k)\end{bmatrix},$$

for $k\in D_{13}\setminus\bar{S}_1$, $D_2\setminus\bar{S}_1$, $D_{46}\setminus\bar{S}_1$, and $D_5\setminus\bar{S}_1$, respectively. Finally, for $k\in S_1$ we have $P^{(x)}(x,k)=P^{(t,0)}(t,k)=P^{(t,L)}(t,k)=I$.

(iii) The integral representation of the solution to the IBVP is derived from the column-wise consideration of the RH formulation, using the Cauchy integral formula and the defining relation $q(x,t):=-2\mathrm{i}\partial_x\lim_{k\to\infty}[kM_{22}(x,t,k)]$.

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