ΘΕΩΡΗΜΑ

Αν f είναι μια συνεχής συνάρτηση σε ένα διάστημα Δ και α είναι ένα σημείο του Δ, τότε η συνάρτηση

$$F(x)=\int_{\alpha}^{x} f(t)dt,\ x\in\Delta,$$

είναι μια παράγουσα της f στο Δ. Δηλαδή ισχύει:

$$\left(\int_{a}^{x} f(t)dt\right)' = f(x), \text{ για κάθε } x\in\Delta.$$

ΘΕΩΡΗΜΑ (Θεμελιώδες θεώρημα του ολοκληρωτικού λογισμού)

Έστω f μια συνεχής συνάρτηση σ' ένα διάστημα $[\alpha,\beta]$. Αν G είναι μια παράγουσα της f στο $[\alpha,\beta]$, τότε

$$\int_{\alpha}^{\beta} f(t)dt = G(\beta)-G(\alpha)$$

Ασκήσεις

Θεμελιώδες θεώρημα του ολοκληρωτικού λογισμού

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

find dy/dx if $\quad y = \int_a^x (t^3 + 1)\,dt \quad \frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$

$$y = \int_1^{x^2} \cos t\,dt \quad y = \int_1^u \cos t\,dt \quad \text{and} \quad u = x^2.$$

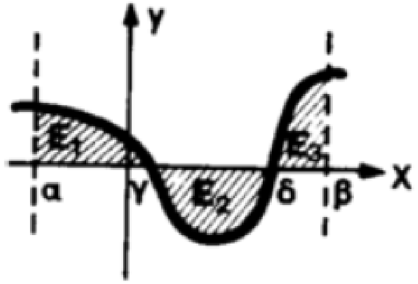
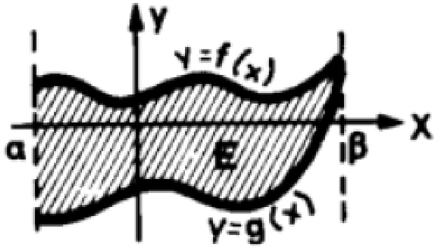
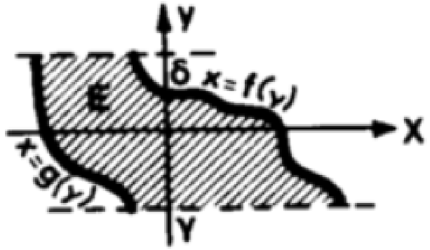
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \left(\frac{d}{du}\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx}\\ &= \cos u\cdot\frac{du}{dx}\\ &= \cos(x^2)\cdot 2x\\ &= 2x\cos x^2\end{aligned}$$

$$y = \int_{1+3x^2}^4 \frac{1}{2+t}\,dt$$

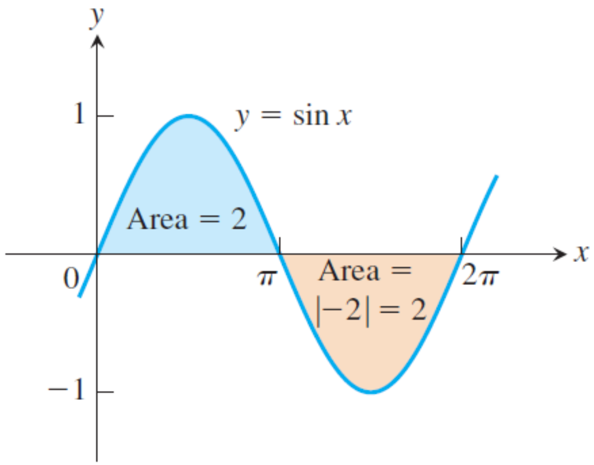
$$\begin{aligned}\frac{d}{dx}\int_{1+3x^2}^4 \frac{1}{2+t}\,dt &= \frac{d}{dx}\left(-\int_4^{1+3x^2}\frac{1}{2+t}\,dt\right)\\ &= -\frac{d}{dx}\int_4^{1+3x^2}\frac{1}{2+t}\,dt\\ &= -\frac{1}{2+(1+3x^2)}\cdot\frac{d}{dx}(1+3x^2)\\ &= -\frac{2x}{1+x^2}\end{aligned}$$

Ασκήσεις

ΕΦΑΡΜΟΓΩΝ ΟΛΟΚΛΗΡΩΜΑΤΩΝ

Εφαρμογή	Σχήμα	Τύπος
Εμβαδό χωρίου που περικλείεται μεταξύ της y=f(x) του άξονα των x και των ευθειών x=α και x=β		$E=E_1+E_2+E_3= =\int_{\alpha}^{\beta}\lvert f(x)\rvert dx=\int_{\alpha}^{\gamma} f(x)dx- -\int_{\gamma}^{\delta} f(x)dx+\int_{\delta}^{\beta} f(x)dx$
Εμβαδό χωρίου Ε που περικλείεται μεταξύ των y=f(x) και y=g(x) και των ευθειών x=α και x=β		$E=\int_{\alpha}^{\beta}\big(f(x)-g(x)\big)dx$
Εμβαδό χωρίου Ε που περικλείεται μεταξύ των x=f(y) και x=g(y) και των ευθειών y=γ και y=δ		$E=\int_{\gamma}^{\delta}\big(f(y)-g(y)\big)dy$

Ασκήσεις

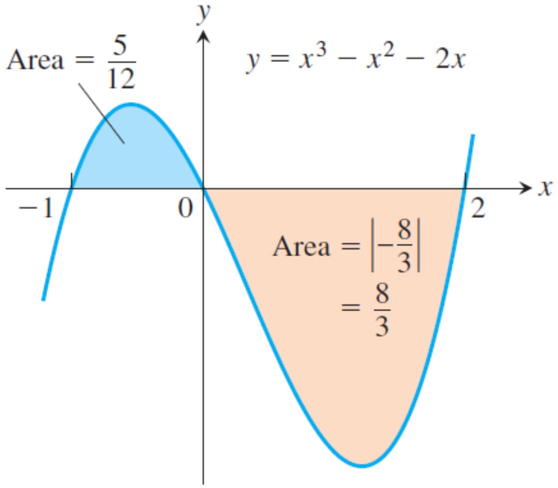


The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$



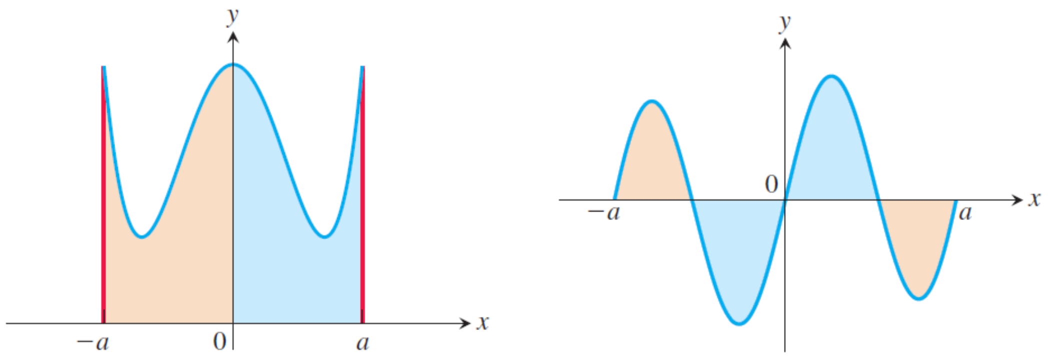
The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis

$$\int_{-1}^{0} (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

$$\text{area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

Ασκήσεις



(a) (b)

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

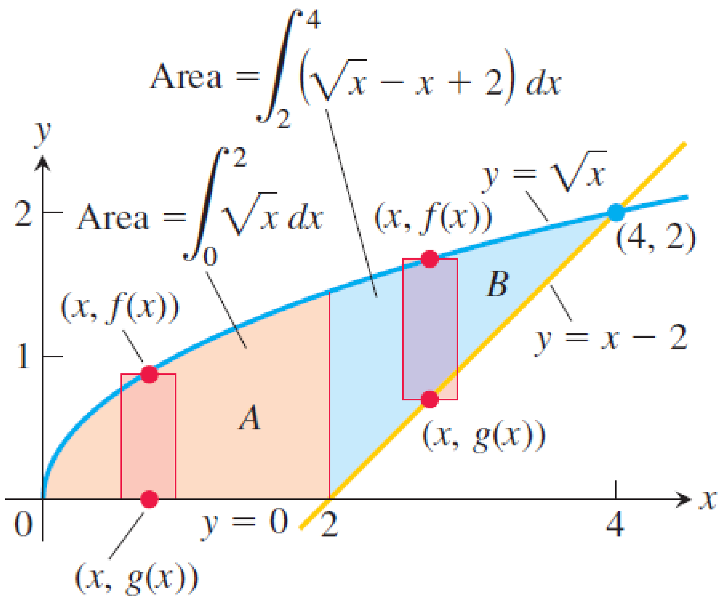
$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$$

$f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$.

$$\begin{aligned}\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx \\ &= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 \\ &= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}\end{aligned}$$

Ασκήσεις

area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here

$\sqrt{x} = x - 2$	Equate $f(x)$ and $g(x)$.
$x = (x-2)^2 = x^2 - 4x + 4$	Square both sides.
$x^2 - 5x + 4 = 0$	Rewrite.
$(x-1)(x-4) = 0$	Factor.
$x = 1, \quad x = 4.$	Solve.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

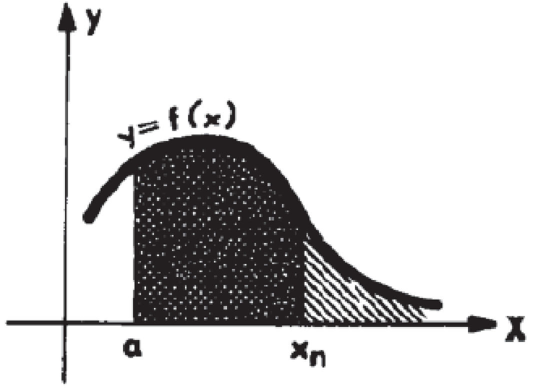
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Μη-γνήσια ολοκληρώματα

Α΄ είδους



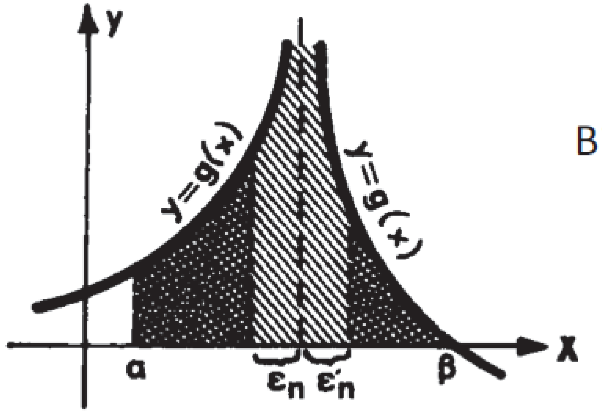
$$\int_{\alpha}^{\infty} f(x)dx = \lim_{M\to\infty} \int_{\alpha}^{M} f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\alpha} f(x)dx + \int_{\alpha}^{\infty} f(x)dx$$

υπολογίζονται ξεχωριστά

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{M\to\infty} \int_{-M}^{M} f(x)dx$$

πρωτεύουσα τιμή



Β΄ είδους

Αν $\lim_{x\to c} f(x) = \pm\infty$

$$\int_{\alpha}^{c} f(x)dx = \lim_{\varepsilon\to 0} \int_{\alpha}^{c-\varepsilon} f(x)dx$$

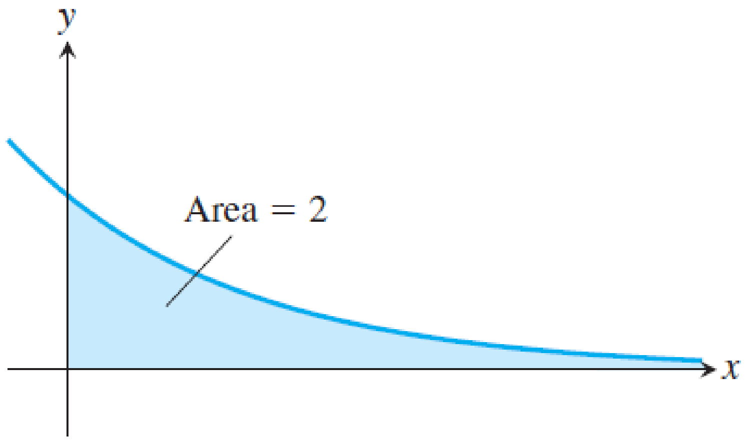
$$\int_{\alpha}^{\beta} f(x)dx = \int_{\alpha}^{c} f(x)dx + \int_{c}^{\beta} f(x)dx$$

υπολογίζονται ξεχωριστά

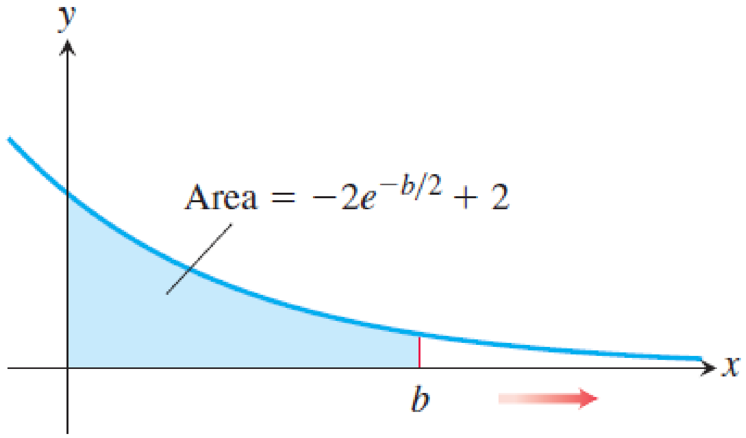
$$\int_{\alpha}^{\beta} f(x)dx = \lim_{\varepsilon\to 0} \left\{ \int_{\alpha}^{c-\varepsilon} f(x)dx + \int_{c-\varepsilon}^{\beta} f(x)dx \right\}$$

πρωτεύουσα τιμή

Ασκήσεις



(a)



(b)

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} \left(-2e^{-b/2} + 2\right) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$

Ασκήσεις

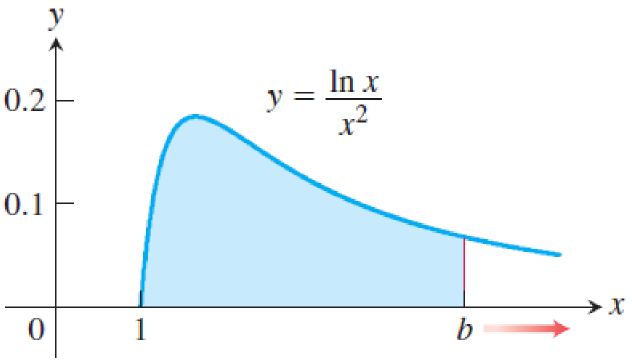


FIGURE 8.14 The area under this cur is an improper integral (Example 1).

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

l'Hôpital's Rule

Ασκήσεις

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

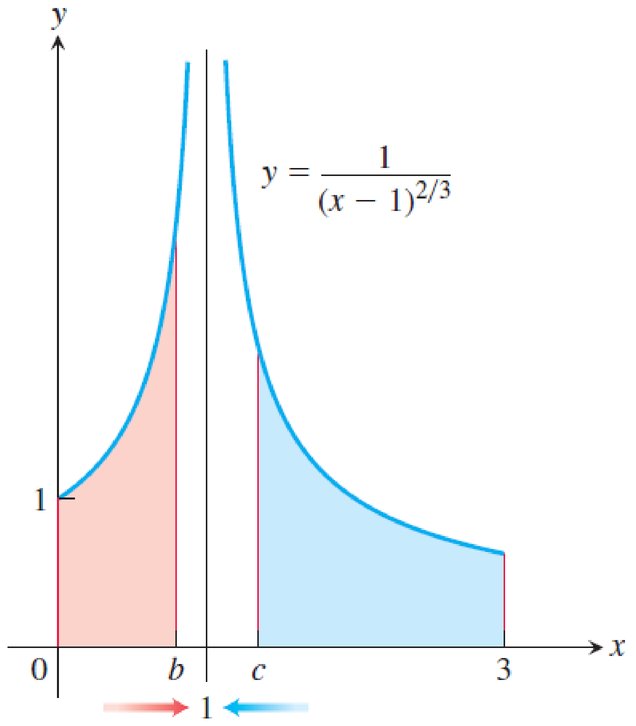


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b$$

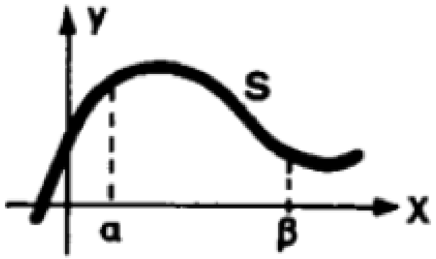
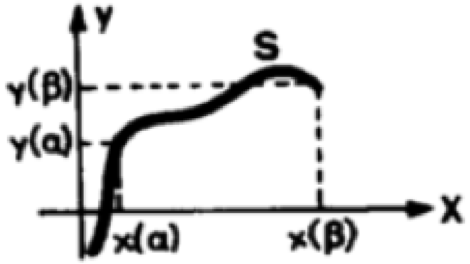
$$= \lim_{b\to 1^-} \left[3(b-1)^{1/3} + 3\right] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c\to 1^+} \left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}$$

Εφαρμογή	Σχήμα	Τύπος
Μήκος καμπύλης y=f(x) από x=α μέχρι x=β		$s=\int_{\alpha}^{\beta}\sqrt{1+(y')^2}\,dx$ όπου $y'=\dfrac{dy}{dx}$
Μήκος καμπύλης x=x(t), y=y(t) από t=α μέχρι t=β		$s=\int_{\alpha}^{\beta}\sqrt{\dot{x}^2+\dot{y}^2}\,dt$ όπου $\dot{x}=\dfrac{dx}{dt}$, $\dot{y}=\dfrac{dy}{dt}$

Ασκήσεις

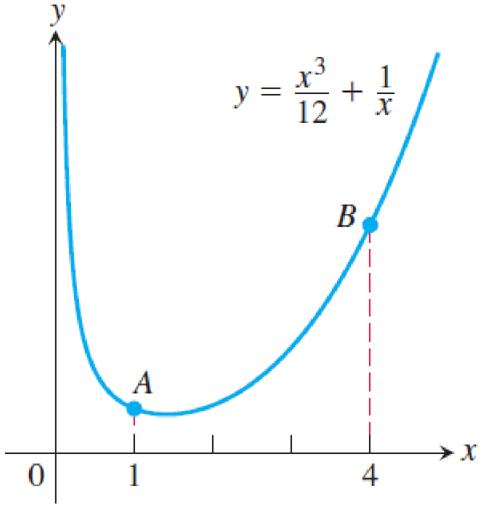


FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6.$$

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