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A VARIANT OF JUNG'S THEOREM

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ABSTRACT. A variant of the old Jung's Theorem is given in such a way that it can provide a better estimate of the size of an n-dimensional sphere which can enclose a bounded subset of the n-dimensional Euclidean space. This estimate is obtained for certain simplices by considering a barycentrally centered n-dimensional sphere and then it is extended for general bounded domains. In both cases the improvement of the estimate of the size of the n-dimensional sphere can be achieved without any additional computational cost.

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1. Introduction.

In 1901 Jung gave an answer to the question of best possible estimate of the size of an n-dimensional sphere in the n-dimensional Euclidean space $\mathbb{E}^n$ which can enclose a subset of $\mathbb{E}^n$ of a given diameter, (i.e. the maximal distance of any two points of the set). A given spherical surface S_r^{n-1} of a n-dimensional sphere of radius r in $\mathbb{E}^n$ encloses a subset $\mathcal{M}$ of $\mathbb{E}^n$, provided $\mathcal{M}$ is contained in the sphere with this surface; while $\mathcal{M}$ is enclosable by a given S_r^{n-1} whenever $\mathcal{M}$ is subset of a sphere whose surface is congruent with S_r^{n-1}. We shall give here a variant of the following theorem that provides a better estimate of the length of the radius r of S_r^{n-1}, without any additional computational cost.

THEOREM 1. (JUNG, 1901) [9,10,6,8]. Let δ be the diameter of a bounded subset $\mathcal{M}$ of $\mathbb{E}^n$ (containing more than a single point). Then, (1) there exists a unique spherical surface S_r^{n-1} of smallest radius r enclosing $\mathcal{M}$, and (2) $r \leqslant [n/2(n+1)]^{1/2}\, \delta$.

First, H.W.E. Jung established these results in his dissertation in 1901; but a complete and elegant proof was given by L.M. Blumenthal and G.E. Wahlin forty years later [6], (for a more recent proof applying Helly's Theorem see [8, p. 140]. The above problem is old, and first appeared in 1860 by J.J. Sylvester [17] who proposed the problem of drawing the smallest circle enclosing a given finite set of points in the plane, (see [6] for some historical remarks). For some implementations or applications of Theorem 1 see [2-5,7,12-15].

2. Results.

L.M. Blumenthal and G.E. Wahlin asserted in [6,p. 776] that since $r = [n/2(n+1)]^{1/2}\,\delta$ is the radius of the sphere circumscribing an equilateral n-simplex in $\mathbb{E}^n$ of edge δ, the inequality for r in Theorem 1 cannot be sharpened. Although it is not possible to sharpen the above inequalityfor r, as we show in the sequel we can obtain a better estimate for the radius of S_r^{n-1}. So, suppose that $\sigma^m = \{u^0,u^1,..,u^m\}$ is an m-simplex in $\mathbb{E}^n$, $m \leqslant n$, (i.e. the convex hull of $m+1$ affinely independent points $u^i, i = 0,...,m$ in $\mathbb{E}^n$) and let $K = (m+1)^{-1}\sum_{i=0}^{m} u^i$ be its barycenter. Define now the barycentric radius β of σ^m as $\beta = \max_i \|K - u^i\|$.

THEOREM 2. Any $m-$ simpler $\sigma^m = \{u^0,u^1,...,u^m\}$ in $\mathbb{E}^m$, $m \leqslant n$, is enclosable by the spherical surface S_β^{n-1} with

$$\beta = \frac{1}{m+1}\max_i\,\{m\sum_{\substack{j=0\\ j\neq i}}^{m}\|u^i - u^j\|^2 - \sum_{\substack{p=0\\ p\neq i}}^{m-1}\ \sum_{\substack{q=p+1\\ q\neq i}}^{m}\|u^p - u^q\|^2\}^{1/2}.$$

Proof. Let $u^i = (u_1^i, u_2^i,...,u_n^i)$ be a vertex of σ^m and let K be the barycenter of σ^m, then

$$\|K - u^i\|^2 = \frac{1}{(m+1)^2}\sum_{t=1}^{n}\{\sum_{\substack{j=0\\ j\neq i}}^{m} u_t^j - m u_t^i\}^2,$$

or

$$\|K - u^i\|^2 = \frac{1}{(m+1)^2}\sum_{t=1}^{n}\{[\sum_{\substack{j=0\\ j\neq i}}^{m} u_t^j]^2 + m^2\,[u_t^i]^2 - 2m u_t^i \sum_{\substack{j=0\\ j\neq i}}^{m} u_t^j\},$$

or

$$||K - u^i||^2 = \frac{1}{(m+1)^2} \sum_{t=1}^{n} \{ \sum_{\substack{j=0 \\ j\neq i}}^{m} [u_t^j]^2 + 2 \sum_{\substack{p=0 \\ p\neq i}}^{m-1} [\sum_{\substack{q=p+1 \\ q\neq i}}^{m} u_t^p u_t^q] + m^2 [u_t^i]^2 - 2mu_t^i \sum_{\substack{j=0 \\ j\neq i}}^{m} u_t^j \},$$

or, after some algebraic manipulations,

$$||K - u^i||^2 = \frac{1}{(m+1)^2} \{ m \sum_{\substack{j=0 \\ j\neq i}}^{m} ||u^i - u^j||^2 - \sum_{\substack{p=0 \\ p\neq i}}^{m-1} \sum_{\substack{q=p+1 \\ q\neq i}}^{m} ||u^p - u^q||^2 \}.$$

By convexity, it is clear that σ^m is a subset of m-dimensional sphere with center the barycenter K of σ^m and radius $\beta = \max_i ||K - u^i||$, so σ^m is enclosable by the spherical surface S_β^{m-1} . □

For an alternative proof of the above theorem see [18]. Note that the computation of the barycentric radius does not require any additional computational cost and it can be easily achieved during the computation of the longest edge δ that it is required for the Jung's estimate. Now, applying the above theorem we can see that the barycentric radius and Jung's estimate are identical in the case of equilateral simplices.

COROLLARY 1. Let σ^m be an equilateral m-simplex in $\mathbb{E}^n$ with diameter δ, then $\beta = [m/2(m+1)]^{1/2}\delta$.

Proof. Applying Theorem 2 we obtain,

$$\beta = \frac{1}{m+1} \{m^2\delta^2 - \binom{m}{2} \delta^2\}^{1/2} = [m/2(m+1)]^{1/2}\, \delta. \ \square$$

The barycentric radius β of an n-simplex in $\mathbb{E}^n$ with diameter δ is not always greater than $[n/2(n+1)]^{1/2}$ δ. (i.e. the barycentric radius of any equilateral n-simplex with diameter δ). Consider, for example, the 2-simplex σ^2 in $\mathbb{E}^2$ with set of vetrices $\{K, u^1, u^2\}$, where K is the barycenter of an equilateral 2-simplex τ^2 in $\mathbb{E}^2$ with diameter δ and u^1, u^2 vertices of τ^2. Then, the lengths of the edges of σ^2 are δ, $3^{-1/2}$ δ and $3^{-1/2}$ δ, so σ^2 also has diameter δ. Now, using Theorem 2 we find that the barycentric radius of σ^2 is equal to

$[7/27]^{1/2}\,\delta$, which is smaller than the barycentric radius of τ^2, which is equal to $3^{-1/2}\,\delta$. But from Theorem 2, we see that σ^2 is enlosable by S^1_β with $\beta = [7/27]^{1/2}\,\delta$. Nevertheless, the barycentric radius β of an n-simplex σ^n in $\mathbb{E}^n$ is never smaller than the smallest radius r of the sphere of which the surface encloses σ^n (see Lemma 2 of [6]). With this in mind, using Theorem 2 we can easily modify Lemma 3 of [6,p.774] to the following lemma.

LEMMA 1. Let σ^n be an n-simplex in $\mathbb{E}^n$ with diameter and barycenter radius δ and β, respectively. If S^{n-1}_r is an $(n-1)$ dimensional spherical surface of smallest radius r enclosing σ^n, then

$$r \leqslant \min\{\beta, [n/2(n+1)]^{1/2}\,\delta\}.$$

LEMMA 2. [6,p. 772]. If each set of $n+1$ points of a subset $\mathscr{M}$ of $\mathbb{E}^n$ is enclosable by a S^{n-1}_r of given radius r, then $\mathscr{M}$ is itself enclosable by this S^{n-1}_r.

Using Lemma 2, the result of Lemma 1 can be extended to any bounded subset of $\mathbb{E}^n$. We can easily see from the proof of Theorem 2 that β can also be computed in the case of which the points $\{u^0, u^1, \ldots, u^m\}$ are not affinely independent. Suppose now that $\mathscr{M}$ is a bounded subset of $\mathbb{E}^n$, with diameter δ containing more than $n+1$ points, then the barycentric radius β of $\mathscr{M}$ is the supremum of barycentric radii between every set of $n+1$ points of $\mathscr{M}$; if $\mathscr{M}$ contains l points $\{u^0, \ldots, u^l\}$, $l \leqslant n+1$, then the barycentric radius β of $\mathscr{M}$ is given by

$$\beta = \frac{1}{l+1} \max_{0\leqslant i\leqslant l} \{\, l \sum_{\substack{j=0\\ j\neq i}}^{l} \|u^i - u^j\|^2 - \sum_{\substack{p=0\\ p\neq i}}^{l-1} \sum_{\substack{q=p+1\\ q\neq i}}^{l} \|u^p - u^q\|^2 \}^{1/2}$$

Now, since $\mathscr{M}$ is bounded and $\beta \leqslant n(n+1)^{-1}\,\delta$, the set of barycentric radii of $\mathscr{M}$ is a bounded set and β is a positive (finite) number. Taking into consideration the proof of [6, pp. 775-776] of Theorem 1 using Lemma 1 and Lemma 2, the following theorem can be easily proven.

THEOREM 3. Let δ and β be the diameter and the barycentric radius, respectively, of a bounded subset $\mathscr{M}$ of $\mathbb{E}^n$ (containing more than a single point). Then (1) there exists a unique spherical surface S^{n-1}_r of smallest radius r enclosing $\mathscr{M}$, and (2) $r \leqslant \min\{\beta, [n/2(n+1)]^{1/2}\,\delta\}$.

REMARKS. (1) An analogue of Theorem 3 in more general spaces will be of interest. Note that an analogue of Jung's Theorem in an n-dimensional Minkowski space has been given by F. Bohnenblust [4,p. 306] who proved that the radius r of any region of diameter δ is $r \leqslant n(n+1)^{-1}\delta$.

(2) The barycentric radius β of a n-simplex σ^n in $\mathbb{E}^n$ can be used to estimate error bounds for approximate fixed points or approximate roots of mappings in $\mathbb{E}^n$, by approximating a fixed point or a root by the barycenter of σ^n. Note that the computation of β requires only the lengths of the edges of σ^n, which are also required to compute the diameter δ of σ^n.

(3) Moreover, since the distance of the barycenter K of an n-simplex $\sigma^n = \{u^0,u^1,\ldots,u^n\}$ in $\mathbb{E}^n$ from the barycenter K_i of the i-th face $\sigma_i^n = \{u^0,u^1,\ldots,u^{i-1}, u^{i+1},\ldots,u^n\}$ of σ^n is equal to $||K - u^i||/n$ [18], then using Theorem 2 we can easily compute the value of $\gamma = \min_i ||K - K_i||/\delta$, (where δ is the diameter of σ^n). The value γ can be used to estimate the thickness θ of σ^n [19,11,16,1], (i.e. $\theta = \min_i \{\min_{x\in \sigma_i^{n-1}} ||K - x||\}/\delta$) which is important to piecewise linear approximations of smooth mappings and, generally, to simplicial and continuation methods for approximating fixed points or roots of systems of nonlinear equations.

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