Generalizations of the Intermediate Value Theorem for Approximating Fixed Points and Zeros of Continuous Functions

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Abstract. Generalizations of the traditional intermediate value theorem are presented. The obtained generalized theorems are particular useful for the existence of solutions of systems of nonlinear equations in several variables as well as for the existence of fixed points of continuous functions. Based on the corresponding criteria for the existence of a solution emanated by the intermediate value theorems, generalized bisection methods for approximating fixed points and zeros of continuous functions are given. These bisection methods require only algebraic signs of the function values and are of major importance for tackling problems with imprecise (not exactly known) information.

Keywords: Bolzano theorem · Bolzano-Poincaré-Miranda theorem · Intermediate value theorems · Existence theorems · Bisection methods · Fixed points · Nonlinear equations

1 Introduction

A system of n nonlinear equations in n real unknowns,

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0, \end{aligned} \tag{1}$$

may be represented in the real n-dimensional vector space $\mathbb{R}^n$ as follows:

$$F_n(x) = \theta^n, \tag{2}$$

where $F_n = (f_1, f_2, \ldots, f_n)\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping and $\theta^n = (0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$. The problem of solving the Eq. (2) is to find a *zero* $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \mathcal{D}$ for which $F_n(x^\star) = \theta^n$. Similarly, the problem of finding a *fixed point* of F_n in $\mathcal{D} \subset \mathbb{R}^n$ is to find a point $x^\star \in \mathcal{D}$ which satisfies the equation $F_n(x^\star) = x^\star$. Obviously, the problem of finding a fixed point is equivalent to the problem of solving the Eq. (2) by considering the mapping

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$\varPhi_n = I_n - F_n$ (where I_n indicates the identity mapping) instead of F_n and solving the equation $\varPhi_n(x) = \theta^n$, instead of the Eq. (2).

Many problems require solution of systems of equations for which Newton's method and the related class of algorithms [15] fail due to nonexistence of derivatives or poorly behaved partial derivatives. Also, Newton's method as well as Newton's-like methods often converge to a solution x^* of $F_n(x) = \theta^n$ almost independently of the initial guess, while $F_n(x) = \theta^n$ may have several solutions, all of which are desired for the application [28]. Because of this reason, generalized bisection methods have been investigated. According to these methods one establishes the existence of at least one solution of the Eq. (2) in a given domain using a specific criterion for the existence of a solution. These kind of criteria can be obtained using the conditions of various "existence theorems" (intermediate value theorems). Once we have obtained a domain for which the criterion of the existence is fulfilled, we are able to obtain upper and lower bounds for solution values. To this end, by computing a sequence of bounded domains of decreasing diameters, we are able to obtain a region with arbitrarily small diameter that contains at least one solution of the Eq. (2).

These methods require only algebraic signs of the function values. The algebraic sign is the smallest amount of information (one bit of information) necessary for the purpose needed. Thus, the methods that require only algebraic signs are of major importance for tackling problems with imprecise (not exactly known) information. This kind of problems occurs in various scientific fields including mathematics, economics, engineering, computer science, biomedical informatics, medicine and bioengineering, among others. This is so, because, in a large variety of applications, precise function values are either impossible or time consuming and computationally expensive to obtain. One such application is provided in [28]. This application concerns the computation of all the periodic orbits (stable and unstable) of any period and accuracy which occur, among others, in the study of beam dynamics in circular particle accelerators like the Large Hadron Collider (LHC) machine at the European Organization for Nuclear Research (CERN). In this application, the method which is presented in [24] and is implemented in [25] is used. Furthermore, these methods are particularly useful for tackling various problems where the corresponding functions take very large and/or very small values.

2 Background Material

Notation 1. We denote by ϑA the boundary of a set A, by cl$A$ its closure, by int$A$ its interior, by card$\{A\}$ its cardinality (i.e., the number of elements in the set A) and by co$A$ its convex hull (i.e., the set of all finite convex combinations of elements of A).

Notation 2. We shall frequently use the index sets $N^n = \{0, 1, \ldots, n\}$, $N^n_{\neg 0} = \{1, 2, \ldots, n\}$ and $N^n_{\neg i} = \{0, 1, \ldots, i-1, i+1, \ldots, n\}$. Furthermore, for a given set $I = \{i, j, \ldots, \ell\} \subset N^n$ we denote by $N^n_{\neg I}$ or equivalently by $N^n_{\neg ij\cdots\ell}$ the set $\{k \in N^n \mid k \notin I\}$.

Definition 1. For any positive integer n, and for any set of points $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ in some linear space which are affinely independent (i.e., the vectors $\{\upsilon^1 - \upsilon^0, \upsilon^2 - \upsilon^0, \ldots, \upsilon^n - \upsilon^0\}$ are linearly independent) the convex hull $\mathrm{co}\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is called the *n-simplex with vertices $\upsilon^0, \upsilon^1, \ldots, \upsilon^n$*. For each subset of $(m+1)$ elements $\{\omega^0, \omega^1, \ldots, \omega^m\} \subset \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$, the m-simplex $[\omega^0, \omega^1, \ldots, \omega^m]$ is called an *m-face* of $[\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. In particular, 0-faces are vertices and 1-faces are edges. The m-faces are also called *facets* of the n-simplex. An m-face of the n-simplex is called the *carrier* of a point p if p lies on this m-face and not on any sub-face of this m-face.

Notation 3. We denote the n-simplex with set of vertices $V = \{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ by $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$. Also, we denote the $(n-1)$-simplex that determines the i-th $(n-1)$-face of σ^n by $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$. Furthermore, for a given index set $I = \{i, j, \ldots, \ell\} \subset N^n$ with cardinality $\mathrm{card}\{I\} = \kappa$, we denote by $\sigma^n_{\neg I}$ or equivalently by $\sigma^n_{\neg ij\cdots\ell}$ the $(n-\kappa)$-face of σ^n with vertices $\upsilon^m, m \in N^n_{\neg I}$.

Definition 2 [23,26]. The *diameter* of an m-simplex σ^m in $\mathbb{R}^n$, $m \leqslant n$, denoted by $\mathrm{diam}(\sigma^m)$, is defined to be the length of the longest edge (1-face) of σ^m while the *microdiameter*, $\mu\mathrm{diam}(\sigma^m)$, of σ^m is defined to be the length of the shortest edge of σ^m.

Definition 3. Let $\sigma^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^m]$ be an m-simplex in $\mathbb{R}^n$, $m \leqslant n$. Then the *barycenter* of σ^m denoted by K is the point $K = (m+1)^{-1} \sum_{i=0}^{m} \upsilon^i$ in $\mathbb{R}^n$.

Remark 1. By convexity it is obvious that the barycenter of any m-simplex σ^m in $\mathbb{R}^n$ is a point in the relative interior of σ^m.

Definition 4. An n-simplex is *oriented* if an order has been assigned to its vertices. If $\langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is an orientation of $\{\upsilon^0, \upsilon^1, \ldots, \upsilon^n\}$ this is regarded as being the same as any orientation obtained from it by an even permutation of the vertices and as the opposite of any orientation obtained by an odd permutation of the vertices. We shall denote oriented n-simplices by $\sigma^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$, and we shall write, for example, $\langle \upsilon^0, \upsilon^1, \upsilon^2, \ldots, \upsilon^n \rangle = -\langle \upsilon^1, \upsilon^0, \upsilon^2, \ldots, \upsilon^n \rangle = \langle \upsilon^2, \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$. The *boundary* $\vartheta\sigma^n$ of an oriented n-simplex $\sigma^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is given by $\vartheta\sigma^n = \sum_{i=0}^{n} (-1)^i \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n \rangle$. The oriented $(n-1)$-simplex $\langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n \rangle$ will be called the ith *face* of σ^n.

Definition 5. An n-dimensional *polyhedron Π^n* is a union of a finite number of oriented n-simplices σ^n_i, $i = 1, 2, \ldots, k$ such that the σ^n_i have pairwise-disjoint interiors. We write $\Pi^n = \sum_{i=1}^{k} \sigma^n_i$ and $\vartheta\Pi^n = \sum_{i=1}^{k} \vartheta\sigma^n_i$.

Definition 6. Let $\psi \in \mathbb{R}$, then the *sign (or signum) function*, denoted by sgn, maps ψ to the set $\{-1, 0, 1\}$ as follows:

$$\mathrm{sgn}\,\psi = \begin{cases} -1, & \text{if} \quad \psi < 0, \\ 0, & \text{if} \quad \psi = 0, \\ 1, & \text{if} \quad \psi > 0. \end{cases} \tag{3}$$

Furthermore, for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ the *sign* of a, denoted $\operatorname{sgn} a$, is defined as $\operatorname{sgn} a = (\operatorname{sgn} a_1, \operatorname{sgn} a_2, \ldots, \operatorname{sgn} a_n)$.

3 Bolzano Intermediate Value Theorem

The fundamental and pioneering Bolzano's theorem states the following [2,7]:

Theorem 1 (Bolzano's theorem). *If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if it holds that $f(a)f(b) < 0$, then there is at least one $x \in (a,b)$ such that $f(x) = 0$.*

This theorem is also called *intermediate value theorem* since it can be easily formulated as follows:

Theorem 2 (Bolzano's intermediate value theorem). *If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if y_0 is a real number such that:*

$$\min\{f(a), f(b)\} < y_0 < \max\{f(a), f(b)\},$$

then there is at least one $x_0 \in (a,b)$ such that $f(x_0) = y_0$.

Remark 2. Obviously, Theorem 2 can be deduced from Theorem 1 by considering the function $g(x) = f(x) - y_0$.

Remark 3. The first proofs of the above theorem, given independently by Bolzano in 1817 [2] and Cauchy in 1821 [4], were crucial in the procedure of *arithmetization of analysis*, which was a research program in the foundations of mathematics during the second half of the 19th century.

Based on the hypotheses of Theorem 1, a simple and very useful criterion for the existence of a zero of a continuous mapping $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ in some interval (a,b) is the following *Bolzano's existence criterion*:

$$f(a)\, f(b) < 0, \tag{4}$$

or equivalently:

$$\operatorname{sgn} f(a) \operatorname{sgn} f(b) = -1, \tag{5}$$

where sgn denotes the sign function (3).

Remark 4. The Bolzano existence criterion is well-known and widely used and it can be generalized to higher dimensions, see [27,30] (cf. Sects. 4 and 5). Note that when the condition (4) (or the condition (5)) is not fulfilled, then in the interval (a,b) either no zero exists or there are zeros for which the sum of their multiplicities is an even number (e.g., two simple zeros, one double and two simple zeros, one triple and one simple zeros etc.).

The well-know and widely applied *bisection method* is based on the Bolzano existence criterion in order to approximate a zero of a continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ in a given interval (a, b). A simplified version described in [24] is the following:

$$x^{p+1} = x^p + c \operatorname{sgn} f(x^p) \,/\, 2^{p+1}, \quad p = 0, 1, \dots , \tag{6}$$

where $x^0 = a$ and $c = \operatorname{sgn} f(a)\,(b - a)$. Instead of the iterative formula (6) we can also use the following [24]:

$$x^{p+1} = x^p - \hat{c} \operatorname{sgn} f(x^p) \,/\, 2^{p+1}, \quad p = 0, 1, \dots, \tag{7}$$

where $x^0 = b$ and $\hat{c} = \operatorname{sgn} f(b)\,(b - a)$.

The sequences (6) and (7) converge with certainty to a zero $r \in (a, b)$ if for some x^p it holds that:

$$\operatorname{sgn} f(x^0) \operatorname{sgn} f(x^p) = -1, \quad \text{for} \quad p = 1, 2, \dots .$$

Furthermore, the number of iterations ν required to obtain an approximate zero r^* such that $|r - r^*| \leqslant \varepsilon$ for some $\varepsilon \in (0, 1)$ is given by:

$$\nu = \left\lceil \log_2 (b - a)\, \varepsilon^{-1} \right\rceil , \tag{8}$$

where $\lceil x \rceil = \operatorname{ceil}(x)$ denotes the ceiling function that maps a real number x to the least integer greater than or equal to x.

Remark 5. The reasons for choosing the iterative schemes (6) and (7) are that:

1. They converge with certainty within the given interval (a, b).
2. They are globally convergent methods in the sense that they converge to a zero from remote initial guesses.
3. Using the relation (8) we may predetermine the number of iterations that are required for the attainment of an approximate zero to a given accuracy.
4. They have a great advantage since they are worst-case optimal. That is, they possess asymptotically the best possible rate of convergence in the worst case [20]. This means that they are guaranteed to converge within the predefined number of iterations, and, moreover, no other method has this important property.
5. They require only the algebraic signs of the function values to be computed, as is evident from (6) and (7); thus they can be applied to problems with imprecise function values.

For applications of the iterative schemes (6) and (7) we refer the interested reader, among others, to [5, 8, 9, 34, 35].

4 Bolzano-Poincaré-Miranda Intermediate Value Theorem

A straightforward generalization of Bolzano's intermediate value theorem to continuous mappings of an n-cube (parallelotope) into $\mathbb{R}^n$ was proposed (without proof) by Poincaré in 1883 and 1884 in his work on the *three body problem* [16,17]. This theorem, now known as Bolzano-Poincaré-Miranda theorem, states that [13,22,27]:

Theorem 3 (Bolzano - Poincaré - Miranda theorem). *Suppose that* $P = \{x \in \mathbb{R}^n \mid |x_i| < L,$ *for* $1 \leqslant i \leqslant n\}$ *and let the mapping* $F_n = (f_1, f_2, \ldots, f_n)\colon P \to \mathbb{R}^n$ *be continuous on the closure of* P *such that* $F_n(x) \neq \theta^n = (0, 0, \ldots, 0)$ *for* x *on the boundary of* P*, and*

(a) $f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geqslant 0,$ *for* $1 \leqslant i \leqslant n,$
(b) $f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \leqslant 0,$ *for* $1 \leqslant i \leqslant n.$

Then, there is at least one $x \in P$ *such that* $F_n(x) = \theta^n$*.*

Theorem 3 it has come to be known as "Miranda's theorem" since in 1940 Miranda [13] proved that it is equivalent to the traditional Brouwer fixed point theorem [3]. It is worthy to mention that the Bolzano-Poincaré-Miranda theorem is closely related to important theorems in analysis and topology and constitutes an invaluable tool for verified solutions of numerical problems by means of interval arithmetic. For a short proof and a generalization of the Bolzano-Poincaré-Miranda theorem using topological degree theory we refer the interested reader to [27]. In addition, for generalizations with respect to an arbitrary basis of $\mathbb{R}^n$ that eliminate the dependence of the Bolzano-Poincaré-Miranda theorem on the standard basis of $\mathbb{R}^n$ see [6,27]. For various interesting relations between the theorems of Bolzano-Poincaré-Miranda, Borsuk, Kantorovich and Smale with respect to the existence of a solution of a system of nonlinear equations, we refer the interested reader to [1].

The conditions of the Bolzano-Poincaré-Miranda theorem give an invaluable existence criterion for a solution of the Eq. (2) where $F_n = (f_1, f_2, \ldots, f_n)\colon P \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Remark 6. Similarly to Bolzano's criterion, the Bolzano - Poincaré - Miranda criterion requires only the algebraic sings of the function values to be computed on the boundary of the n-cube P. On the other hand, for general continuous functions, in contrary to Bolzano's criterion, the hypotheses (a) and (b) are not always fulfilled or it is impossible to be verified for a given n-cube P.

Next, the characteristic polyhedron criterion and the characteristic bisection method are briefly presented. These approaches, in contrary to Bolzano - Poincaré - Miranda criterion require only the algebraic sings of the function values to be computed on the vertices of the considered polyhedron.

There are various generalized bisection methods that require the computation of the topological degree in order to localize a solution of the Eq. (2) (see, e.g., [11,23]). We shall allow us to briefly discuss a few basic concepts regarding topological degree theory. To this end, suppose that a function $F_n = (f_1, f_2, \ldots, f_n)\colon \mathrm{cl}\mathcal{D}_n \subset \mathbb{R}^n \to \mathbb{R}^n$ is defined and twice continuously differentiable in an open and bounded domain $\mathcal{D}_n$ of $\mathbb{R}^n$ with boundary $\vartheta\,\mathcal{D}_n$. Suppose further that the solutions of the equation $F_n(x) = p$, where $p \in \mathbb{R}^n$ is a given vector, are not located on $\vartheta\,\mathcal{D}_n$, and that they are simple, i.e., the determinant, $\det J_{F_n}$, of the Jacobian matrix of F_n at these solutions is non-zero.

Definition 7. The *topological degree of F_n at p relative to $\mathcal{D}_n$* is denoted by $\deg[F_n, \mathcal{D}_n, p]$ and is defined by the following sum:

$$\deg[F_n, \mathcal{D}_n, p] = \sum_{x \in F_n^{-1}(p) \cap \mathcal{D}_n} \operatorname{sgn} \det J_{F_n}(x), \tag{9}$$

where sgn denotes the sign function (3).

Remark 7. The topological degree can be generalized when the function is only continuous [15]. Furthermore, if $\mathcal{D}_n = \mathcal{D}_n^1 \cup \mathcal{D}_n^2$ where $\mathcal{D}_n^1$ and $\mathcal{D}_n^2$ have disjoint interiors and $F_n(x) \neq \theta^n$ for all $x \in \vartheta\mathcal{D}_n^1 \cup \vartheta\mathcal{D}_n^2$, then the topological degree is additive, i.e.:

$$\deg[F_n, \mathcal{D}_n, \theta^n] = \deg[F_n, \mathcal{D}_n^1, \theta^n] + \deg[F_n, \mathcal{D}_n^2, \theta^n]. \tag{10}$$

The topological degree is invariant under changes of the vector p in the sense that, if $q \in \mathbb{R}^n$ is any vector, then it holds that [15]:

$$\deg[F_n, \mathcal{D}_n, p] \equiv \deg[F_n - q, \mathcal{D}_n, p - q],$$

where $F_n - q$ denotes the mapping $F_n(x) - q$, $x \in \mathcal{D}_n$. Thus, for simplicity reason, we consider the case where the topological degree is defined at the origin $\theta^n = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$.

The topological degree $\deg[F_n, \mathcal{D}_n, \theta^n]$ can be represented by the Kronecker integral which is defined as follows:

$$\deg[F_n, \mathcal{D}_n, \theta^n] = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int\!\!\int_{\vartheta\mathcal{D}_n}\!\cdots\!\int \frac{\sum_{i=1}^n A_i dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n}{\left({f_1}^2 + {f_2}^2 + \cdots + {f_n}^2\right)^{n/2}}, \tag{11}$$

where Γ denotes the gamma function and A_i define the following determinants:

$$A_i = (-1)^{n(i-1)} \det \begin{bmatrix} F_n & \dfrac{\partial F_n}{\partial x_1} & \cdots & \dfrac{\partial F_n}{\partial x_{i-1}} & \dfrac{\partial F_n}{\partial x_{i+1}} & \cdots & \dfrac{\partial F_n}{\partial x_n} \end{bmatrix},$$

where $\frac{\partial F_n}{\partial x_k} = \left(\frac{\partial f_1}{\partial x_k}, \frac{\partial f_2}{\partial x_k}, \ldots, \frac{\partial f_n}{\partial x_k}\right)$ is the kth column of the determinant $\det J_{F_n}$ of the Jacobian matrix J_{F_n}.

The important Kronecker's theorem [15] states that the equation $F_n(x) = \theta^n$ has at least one zero in $\mathcal{D}_n$ if $\deg[F_n, \mathcal{D}_n, \theta^n] \neq 0$. To this end, several methods for the computation of the topological degree have been proposed in the past few years (see, e.g., [11,22]). One such method is the fundamental and pioneering Stenger's method [22] that in some classes of functions is an almost optimal complexity algorithm (see, e.g., [14,20,22]). The accurate computation of topological degree using Stenger's or other related methods [11], is based on suitable assumptions, including appropriate representation of the boundary of $\mathcal{D}_n$. Specifically, if the boundary of $\mathcal{D}_n$ can be "sufficiently refined" then Stenger's method gives the value of the topological degree.

Definition 8 [11,22,33]. Let Π^n be an n-polyhedron. Let $F_n = (f_1, f_2, \ldots, f_n)\colon \Pi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuous with $\theta^n \notin F_n(\vartheta \Pi^n)$. If $n = 1$, $\vartheta \Pi^1$ is said to be *sufficiently refined relative to* $\operatorname{sgn} F_1$, if $0 \notin F_1(\vartheta \Pi^1)$. If $n > 1$, $\vartheta \Pi^n$ is said to be *sufficiently refined relative to* $\operatorname{sgn} F_n$, if $\vartheta \Pi^n$ has been subdivided so that it may be written as a union of a finite number of $(n-1)$-dimensional regions $Q_1^{n-1}, Q_2^{n-1}, \ldots, Q_m^{n-1}$, each consisting of a union of a finite number of $(n-1)$-simplices with pairwise disjoint $(n-1)$-dimensional interiors and having the following properties:

(a) the interiors of the Q_i^{n-1} are pairwise disjoint and each Q_i^{n-1} is connected;
(b) for each region Q_i^{n-1}, there exists at least one component of F_n, (for example f_{r_i}), that does not vanish on it;
(c) if $f_{r_i} \neq 0$ on Q_i^{n-1}, then ϑQ_i^{n-1} is sufficiently refined relative to $\operatorname{sgn} F_{n-1}^{r_i}$ where $F_{n-1}^{r_i} = \big(f_1, f_2, \ldots, f_{r_i - 1}, f_{r_i + 1}, \ldots, f_n\big)$.

As we have already mentioned previously, once we have obtained a domain for which the value of the topological degree relative to this domain is nonzero, we are able to obtain upper and lower bounds for solution values. To this end, by computing a sequence of bounded domains with nonzero values of topological degree and decreasing diameters, we are able to obtain a region with arbitrarily small diameter that contains at least one solution of the Eq. (2). However, although the nonzero value of topological degree plays an important role in the existence of a solution of the Eq. (2), the computation of this value is a time-consuming procedure. The bisection method, on the other hand, which is briefly described below, avoids all calculations concerning the topological degree by implementing the concept of the *characteristic* n*-polyhedron criterion* for the existence of a solution of the Eq. (2) within a given bounded domain. This criterion is based on the construction of a *characteristic* n*-polyhedron* [24,25,33]. To define a characteristic n-polyhedron (n-dimensional convex polyhedron) we construct the n*-complete* $2^n \times n$ *matrix* $\mathcal{M}_n$ whose rows are formed by all possible combinations of -1 and 1. To this end we compute the n*-binary* $2^n \times n$ *matrix* $\mathcal{M}_n^* = \big[e_{ij}^*\big]_{i,j=1}^{2^n,n}$ where e_{ij}^* is the jth digit of the n-digit binary representation of the number $(i-1)$ counting the left-most digit first. Then the elements of $\mathcal{M}_n = \big[e_{ij}\big]_{i,j=1}^{2^n,n}$ are given by $e_{ij} = 2e_{ij}^* - 1$.

Suppose now that $\Pi^n = \langle V_1, V_2, \ldots, V_{2^n} \rangle$ is an oriented (i.e., an orientation has been assigned to its vertices) n-dimensional convex polyhedron with 2^n vertices, $V_i \in \mathbb{R}^n$, and let $F_n = (f_1, f_2, \ldots, f_n)\colon \Pi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. Then,

Definition 9. The $2^n \times n$ matrix $\mathcal{S}(F_n;\, \varPi^n)$ whose entries in the k-th row are the corresponding coordinates of the vector:

$$\operatorname{sgn} F_n(V_k) = \big(\operatorname{sgn} f_1(V_k), \operatorname{sgn} f_2(V_k), \dots, \operatorname{sgn} f_n(V_k)\big), \tag{12}$$

will be called *matrix of signs associated with* F_n *and* $\varPi^n$, where $\operatorname{sgn}\psi$ defines the sign function (3).

Definition 10. An n-polyhedron $\varPi^n$ is called *characteristic n-polyhedron relative to* F_n, iff the matrix $\mathcal{S}\,(F_n;\, \varPi^n)$ is identical with the matrix $\mathcal{M}_n$, after some permutation of its rows.

Definition 11. A polyhedron which is a convex hull of 2^{n-1} vertices of a characteristic n-polyhedron $\varPi^n$ relative to F_n, will be called *r-side* of $\varPi^n$ and will be noted by P_r, $r = 1, 2, \dots, n$ iff for all its vertices V_k, $k = 1, 2, \dots, 2^{n-1}$ the corresponding vectors $\operatorname{sgn} F_n(V_k)$ have their r-th coordinate equal to each other. Moreover, if this common r-th element is -1 or 1 then the P_r will be called *negative* or *positive r-side* correspondingly.

Lemma 1 [33]**.** *In each characteristic n-polyhedron relative to F_n there are n positive and n negative sides. Moreover, each side P_r of a characteristic n-polyhedron $\varPi^n$ relative to $F_n = (f_1, f_2, \dots, f_n)\colon \varPi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ is itself a characteristic $(n-1)$-polyhedron relative to $F^r_{n-1} = (f_1, f_2, \dots, f_{r-1}, f_{r+1}, \dots, f_n) :$ $P_r \to \mathbb{R}^{n-1}$.*

If the boundary $\vartheta\varPi^n$ of a characteristic polyhedron $\varPi^n$ can be sufficiently refined then there is (at least) one zero within $\varPi^n$. More specifically, the following theorem holds:

Theorem 4 [33]**.** *Let $\mathcal{V} = \langle V_i\rangle_{i=1}^{2^n}$ and $\mathcal{P} = \{P_i\}_{i=1}^{2n}$ be the ordered set of vertices and the set of the sides, respectively, of a characteristic n-polyhedron $\varPi^n$ relative to continuous $F_n\colon \varPi^n \subset \mathbb{R}^n \to \mathbb{R}^n$ for which $\theta^n \notin F_n(\vartheta\varPi^n)$. Suppose that $S = \{S_{i,j}\}_{i=1,\,j=1}^{2n,\ j_i}$ is a finite set of $(n-1)$-dimensional oriented simplices which lie on $\vartheta\varPi^n$ with the following properties:*

(a) $\vartheta\varPi^n = \sum_{i=1}^{2n} \sum_{j=1}^{j_i} S_{i,j}$,
(b) the interiors of the members of S are disjoint,
(c) these simplices make $\vartheta\varPi^n$ sufficiently refined relative to $\operatorname{sgn}(F_n)$*, and*
(d) the vertices of each simplex $S_{i,j}$ are a subset of vertices of P_i.

Then, it holds that $\deg[F_n, \varPi^n, \theta^n] = \pm 1$.

Remark 8. The above result implies the existence of at least one solution of the Eq. (2) within $\varPi^n$. For more details on how to construct a characteristic n-polyhedron and locate a desired solution see [24,25,28]. The characteristic polyhedron can be considered as a translation of the Poincaré-Miranda hypercube [22,27].

Next, we describe a generalized bisection method. This method combined with the above mentioned criterion, produces a sequence of characteristic polyhedra of decreasing size always containing the desired solution. We call it *Characteristic Bisection*. This version of bisection does not require the computation of the topological degree at each step, as others do [11,23]. It can be applied to problems with imprecise function values, since it depends only on their signs. The method simply amounts to constructing another refined characteristic polyhedron, by bisecting a known one, say Π^n. To do this, we compute the midpoint M of the longest edge $\langle V_i, V_j \rangle$, of Π^n (where the distances are measured in Euclidean norms). Then we obtain another characteristic polyhedron, Π^n_*, by comparing the sign, $\mathrm{sgn} F_n(M)$, of $F_n(M)$ with that of $F_n(V_i)$ and $F_n(V_j)$ and substituting M for that vertex for which the signs are identical [24,25,28]. Then we select the longest edge of Π^n_* and continue the above process. If the assumptions of Theorem 4 are satisfied, one of the $\mathrm{sgn} F_n(V_i)$, $\mathrm{sgn} F_n(V_j)$ coincides with $\mathrm{sgn} F_n(M)$, otherwise, we continue with another edge.

Theorem 5 [33]**.** *Let Π^n be a characteristic n-polyhedron whose longest edge length is $\Delta(\Pi^n)$. Then, the minimum number ζ of bisections of the edges of Π^n required to obtain a characteristic polyhedron Π^n_* whose longest edge length satisfies $\Delta(\Pi^n_*) \leqslant \varepsilon$, for some accuracy $\varepsilon \in (0, 1)$, is given by*

$$\zeta = \left\lceil \log_2 \left(\Delta(\Pi^n)\, \varepsilon^{-1} \right) \right\rceil . \tag{13}$$

Remark 9. Notice that ζ is independent of the dimension n and that the bisection algorithm has the same number of iterations as the bisection in one-dimension which is optimal and possesses asymptotically the best rate of convergence [19].

5 Intermediate Value Theorem for Simplices

In [30] the intermediate value theorem for simplices is proposed. The obtained proof is based on the Knaster-Kuratowski-Mazurkiewicz covering principle [12] (cf. Lemma 2 below). Also, in [31] two short proofs of this theorem are given which are based on Sperner covering principles (cf. Lemmas 3 and 4 below).

Lemma 2 (Knaster-Kuratowski-Mazurkiewicz (KKM Lemma)). *Let C_i, $i \in N^n = \{0, 1, \ldots, n\}$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) For each $\emptyset \neq I \subset N^n$ *it holds that* $\bigcap_{i \in I} \sigma^n_{\neg i} \subset \bigcup_{j \in N^n_{\neg I}} C_j$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Remark 10. It is worthy to mention that, the three fundamental and pioneering classical results, namely, the Brouwer fixed point theorem [3], the Sperner lemma [21], and the KKM lemma [12] are mutually equivalent in the sense that each one can be deduced from another. Furthermore, Scarf proposed a method for approximating a fixed point of a continuous function from a unit simplex into itself [18]. This approach is considered as the first constructive proof to Brouwer's fixed point theorem. Scarf's method is based on a simplicial subdivision (triangulation) of the given simplex and it uses a labeling of the vertices of the simplicial subdivision.

Definition 12. A system (family) of subsets of a set A whose union is A is called a *covering* of A. The *order* of a finite system of sets is the greatest integer k for which the system has k elements with nonempty intersection. A system of sets is said to be *simple* if every two elements of the system are distinct. A covering is called an ε-*covering* if the finite system of sets of this covering are of diameter less than $\varepsilon > 0$.

A similar to KKM covering principle was proposed by Sperner [21]:

Lemma 3 (Sperner covering principle). *Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) $\sigma^n_{\neg i} \cap C_i = \emptyset, \quad \forall\, i \in N^n$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

A similar result is the following:

Lemma 4 (Sperner covering principle). *Let C_i, $i \in N^n$ be a family of $(n+1)$ closed subsets of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ satisfying the following hypotheses:*

(a) $\sigma^n = \bigcup_{i \in N^n} C_i$ *and*
(b) $\sigma^n_{\neg i} \subset C_i\,, \quad \forall\, i \in N^n$.

Then, it holds that $\bigcap_{i \in N^n} C_i \neq \emptyset$.

Next, we give the intermediate value theorem for simplices [30].

Theorem 6 (Intermediate value theorem for simplices [30]). *Assume that $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ is an n-simplex in $\mathbb{R}^n$. Let $F_n = (f_1, f_2, \ldots, f_n)\colon \sigma^n \to \mathbb{R}^n$ be a continuous function such that $f_j(\upsilon^i) \neq 0$, $\forall\, j \in N^n_{\neg 0} = \{1, 2, \ldots, n\}$, $i \in N^n = \{0, 1, \ldots, n\}$ and $\theta^n = (0, 0, \ldots, 0) \notin F_n(\vartheta \sigma^n)$ (i.e. F_n does not vanish on the boundary $\vartheta \sigma^n$ of σ^n). Assume that the vertices υ^i, $i \in N^n$ are reordered such that the following hypotheses are fulfilled:*

$$(a) \quad \mathrm{sgn} f_j(\upsilon^j)\, \mathrm{sgn} f_j(x) = -1, \quad \forall\, x \in \sigma^n_{\neg j}\,, \;\; j \in N^n_{\neg 0}\,, \tag{14}$$

$$(b) \quad \mathrm{sgn} F_n(\upsilon^0) \neq \mathrm{sgn} F_n(x), \quad \forall\, x \in \sigma^n_{\neg 0}\,, \tag{15}$$

where $\mathrm{sgn} F_n(x) = \big(\mathrm{sgn} f_1(x), \mathrm{sgn} f_2(x), \ldots, \mathrm{sgn} f_n(x)\big)$ *and $\sigma^n_{\neg i}$ denotes the face opposite to vertex υ^i. Then, there is at least one point $x \in \operatorname{int} \sigma^n$ such that $F_n(x) = \theta^n$.*

Remark 11. The only computable information required by the hypotheses (14) and (15) of Theorem 6 is the algebraic sign of the function values on the boundary of the n-simplex σ^n. Thus, Theorem 6 is applicable whenever the signs of the function values are computed correctly. Theorem 6 has been applied for the localization and approximation of fixed points and zeros of continuous mappings using a simplicial subdivision of a simplex [31].

Next, we present a generalized method of bisection for simplices.

Definition 13 [10]**.** Let $\sigma_0^m = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^m \rangle$ be an oriented m-simplex in $\mathbb{R}^n$, $m \leqslant n$, suppose that $\langle \upsilon^i, \upsilon^j \rangle$ is the longest edge of σ_0^m and let $\Upsilon = (\upsilon^i + \upsilon^j)/2$ be the midpoint of $\langle \upsilon^i, \upsilon^j \rangle$. Then the *bisection* of σ_0^m is the order pair of m-simplices $\langle \sigma_{10}^m, \sigma_{11}^m \rangle$ where:

$$\begin{aligned} \sigma_{10}^m &= \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \Upsilon, \upsilon^{i+1}, \ldots, \upsilon^j, \ldots, \upsilon^m \rangle, \\ \sigma_{11}^m &= \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^i, \ldots, \upsilon^{j-1}, \Upsilon, \upsilon^{j+1}, \ldots, \upsilon^m \rangle. \end{aligned}$$

The m-simplices σ_{10}^m and σ_{11}^m will be called *lower simplex* and *upper simplex* respectively corresponding to σ_0^m while both σ_{10}^m and σ_{11}^m will be called *elements of the bisection* of σ_0^m. Suppose that $\sigma_0^n = \langle \upsilon^0, \upsilon^1, \ldots, \upsilon^n \rangle$ is an oriented n-simplex in $\mathbb{R}^n$ which includes at least one solution of the Eq. (2). Suppose further that $\langle \sigma_{10}^n, \sigma_{11}^n \rangle$ is the bisection of σ_0^n and that there is at least one root of the system (2) in some of its elements. Then this element will be called *selected n-simplex produced after one bisection of* σ_0^n and it will be denoted by σ_1^n. Moreover if there is at least one solution of the system (2) in both elements, then the selected n-simplex will be the lower simplex corresponding to σ_0^n. Suppose now that the bisection is applied with σ_1^n replacing σ_0^n giving thus the σ_2^n. Suppose further that this process continues for p iterations. Then we call σ_p^n *the selected n-simplex produced after p iterations of the bisection of* σ_0^n .

Theorem 7 [23]**.** *Suppose that $\sigma^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^m]$ is an m-simplex in $\mathbb{R}^n$, $m \leqslant n$. Let K be the barycenter of σ^m and let K_i be the barycenter of the i-th face $\sigma_{\neg i}^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^m]$ of σ^m then the following relationships hold for all $0 \leqslant i \leqslant m$,*

(a) The points υ^i, K and K_i are collinear points,

(b) $\|K - \upsilon^i\|_2 = \dfrac{m}{m+1} \left(\dfrac{1}{m} \sum\limits_{\substack{j=0 \\ j \neq i}}^{m} \|\upsilon^i - \upsilon^j\|_2^2 - \dfrac{1}{m^2} \sum\limits_{\substack{p=0 \\ p \neq i}}^{m-1} \sum\limits_{\substack{q=p+1 \\ q \neq i}}^{m} \|\upsilon^p - \upsilon^q\|_2^2 \right)^{1/2},$

(c) $\|K - K_i\|_2 = m^{-1} \|K - \upsilon^i\|_2$.

Definition 14 [26]**.** The *barycentric radius* $\beta(\sigma^m)$ of an m-simplex σ^m in $\mathbb{R}^n$ is the radius of the smallest ball centered at the barycenter of σ^m and containing the simplex. The barycentric radius $\beta(A)$ of a subset A of $\mathbb{R}^n$ is the supremum of the barycentric radii of simplices with vertices in A.

Remark 12. The length of the barycentric radius $\beta(\sigma^m)$ of an m-simplex σ^m in $\mathbb{R}^n$, $m \leqslant n$, is $\max_i \|K - \upsilon^i\|_2$.

Theorem 8 [26]**.** *Any m-simplex $\sigma^m = [\upsilon^0, \upsilon^1, \ldots, \upsilon^m]$ in $\mathbb{R}^n$, $m \leqslant n$ is enclosable by the spherical surface S_β^{m-1} with radius $\beta(\sigma^m)$ given by:*

$$\beta(\sigma^m) = \frac{1}{m+1} \max_i \left(m \sum_{\substack{j=0 \\ j \neq i}}^{m} \|\upsilon^i - \upsilon^j\|_2^2 - \sum_{\substack{p=0 \\ p \neq i}}^{m-1} \sum_{\substack{q=p+1 \\ q \neq i}}^{m} \|\upsilon^p - \upsilon^q\|_2^2 \right)^{1/2}.$$

Remark 13. The barycentric radius $\beta(\sigma^n)$ of a n-simplex σ^n in $\mathbb{R}^n$ can be used to estimate error bounds for approximate fixed points or approximate roots of mappings in $\mathbb{R}^n$, by approximating a fixed point or a root by the barycenter of σ^n. Note that the computation of $\beta(\sigma^n)$ requires only the lengths of the edges of σ^n, which are also required in order to compute the diameter $\mathrm{diam}(\sigma^n)$ of σ^n. Furthermore, since the distance of the barycenter K of an n-simplex $\sigma^n = [\upsilon^0, \upsilon^1, \ldots, \upsilon^n]$ in $\mathbb{R}^n$ from the barycenter K_i of the i-th face $\sigma^n_{\neg i} = [\upsilon^0, \upsilon^1, \ldots, \upsilon^{i-1}, \upsilon^{i+1}, \ldots, \upsilon^n]$ of σ^n is equal to $\|K - \upsilon^i\|_2/n$ [23,26], then using Theorem 8 we can easily compute the value of $\gamma(\sigma^n) = \min_i \|K - K_i\|_2/\mathrm{diam}(\sigma^n)$. The value $\gamma(\sigma^n)$ can be used to estimate the thickness $\theta(\sigma^n)$ of σ^n, that is:

$$\theta(\sigma^n) = \min_i \left\{ \min_{x \in \sigma^n_{\neg i}} \big\|K - x\big\|_2 \right\} / \mathrm{diam}(\sigma^n).$$

In general, the thickness $\theta(\sigma^n)$ is important to piecewise linear approximations of smooth mappings and, in general, to simplicial and continuation methods for approximating fixed points or roots of systems of nonlinear equations.

Theorem 9 [10]**.** *Suppose that σ_0^m is an m-simplex in $\mathbb{R}^n$ and let σ_p^m be any m-simplex produced after p bisections of σ_0^m. Then*

$$\mathrm{diam}(\sigma_p^m) \leqslant \left(\sqrt{3}/2\right)^{\lfloor p/m \rfloor} \mathrm{diam}(\sigma_0^m), \tag{16}$$

where $\mathrm{diam}(\sigma_p^m)$ *and* $\mathrm{diam}(\sigma_0^m)$ *are the diameters of σ_p^m and σ_0^m respectively and $\lfloor p/m \rfloor$ is the largest integer less than or equal to p/m.*

Theorem 10 [23,29]**.** *Suppose that σ_0^m, σ_p^m,* $\mathrm{diam}(\sigma_0^m)$ *and* $\mathrm{diam}(\sigma_p^m)$ *are as in Theorem 9 and let K_p^m be the barycenter of σ_p^m. Then for any point T in σ_p^m the following relationship is valid*

$$\|T - K_p^m\|_2 \leqslant \frac{m}{m+1} \left(\sqrt{3}/2\right)^{\lfloor p/m \rfloor} \mathrm{diam}(\sigma_0^m). \tag{17}$$

Definition 15. Let σ^n be an n-simplex in $\mathbb{R}^n$ and let $\mathrm{diam}(\sigma^n)$ and $\mu\mathrm{diam}(\sigma^n)$ be the diameter and the microdiameter of σ^n respectively. Suppose that r is a solution of the Eq. (2) in σ^n. Then we define the barycenter K^n of σ^n to be an *approximation* of r and the quantity

$$\varepsilon(\sigma^n) = \frac{n}{n+1}\left(\big(\mathrm{diam}(\sigma^n)\big)^2 - \frac{n-1}{2n}\big(\mu\mathrm{diam}(\sigma^n)\big)^2\right)^{1/2}, \tag{18}$$

to be an *error estimate* for K^n.

Theorem 11 [23,29]**.** *Suppose that σ_p^n is the selected n-simplex produced after p bisections of an n-simplex σ_0^n in $\mathbb{R}^n$. Let r be a solution of the Eq. (2) which is included in σ_p^n and that K_p^n and $\varepsilon(\sigma_p^n)$ are the approximation of r and the error estimate for K_p^n respectively. Then the following hold:*

(a) $\varepsilon(\sigma_p^n) \leqslant \dfrac{n}{n+1}\left(\sqrt{3}/2\right)^{\lfloor p/n \rfloor} \mathrm{diam}(\sigma_0^n)$,

(b) $\varepsilon(\sigma_p^n) \leqslant \left(\sqrt{3}/2\right)^{\lfloor p/n \rfloor} \varepsilon(\sigma_0^n)$,

(c) $\lim\limits_{p\to\infty} \varepsilon_p = 0$,

(d) $\lim\limits_{p\to\infty} K_p^n = r$.

6 Synopsis

The paper presents, among some new results, an overview on generalizations of the intermediate value theorem for approximating fixed points and zeros of continuous functions. The presented generalized theorems are particular useful for the existence of solutions of systems of nonlinear equations in several variables as well as for the existence of fixed points of continuous functions. Based on the corresponding criteria for the existence of a solution emanated by the intermediate value theorems, generalized bisection methods for approximating fixed points and zeros of continuous functions are given. These bisection methods require only algebraic signs of the function values and are of major importance for tackling problems with imprecise (not exactly known) information.

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